NAME ______ MASTER ______

SYDNEY GRAMMAR SCHOOL

2014 Annual Examination

# FORM V
# MATHEMATICS EXTENSION 1

Monday 1st September 2014

**General Instructions**
- Writing time — 2 Hours
- Write using black or blue pen.
- Board-approved calculators and templates may be used.
- A list of standard integrals is provided at the end of the examination paper.

**Total — 100 Marks**
- All questions may be attempted.

**Section I – 9 Marks**
- Questions 1–9 are of equal value.
- Record your solutions to the multiple choice on the sheet provided.

**Section II – 91 Marks**
- Questions 10–16 are of equal value.
- All necessary working should be shown.
- Start each question in a new booklet.

**Collection**
- Write your name, class and master on each booklet and on your multiple choice answer sheet.
- Hand in the booklets in a single well-ordered pile.
- Hand in a booklet for each question in Section II, even if it has not been attempted.
- If you use a second booklet for a question, place it inside the first.
- Place your multiple choice answer sheet inside the answer booklet for Question Ten.
- Write your name and master on this question paper and submit it with your answers.

| | | | |
|---|---|---|---|
| 5A: BDD | 5B: MLS | 5C: LYL | 5D: LRP |
| 5E: PKH | 5F: BR | 5G: SG | |

**Checklist**
- SGS booklets — 7 per boy
- Multiple choice answer sheet
- Candidature — 131 boys

Examiner
BDD



## SECTION I - Multiple Choice

**Answers for this section should be recorded on the separate answer sheet handed out with this examination paper.**

### QUESTION ONE

Which of the following is a correct expression for a primitive of $e^{5x}$?

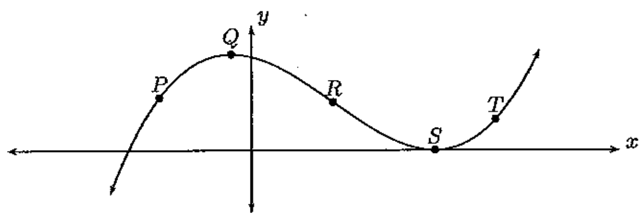
(A) $\frac{1}{5}e^{5x}$

(B) $5e^{5x}$

(C) $\frac{1}{6}e^{6x}$

(D) $\frac{1}{5x+1}e^{5x+1}$

### QUESTION TWO




The graph of a function is sketched above. In this graph, $Q$ and $S$ are stationary points and $R$ is a point of inflexion.

At which of the marked points is the second derivative $y'' > 0$?

(A) $P$ and $Q$

(B) $S$ and $T$

(C) $R$, $S$ and $T$

(D) $P$ and $T$

Exam continues next page ...

QUESTION THREE

Which of the following integration statements is correct?

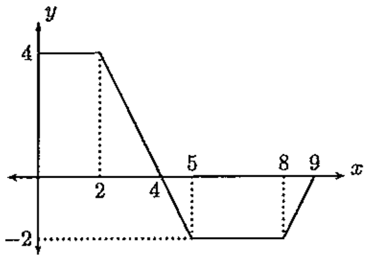
(A) $\int (x^2+1)^2\,dx = \dfrac{(x^2+1)^3}{6x} + C$

(B) $\int \ln x\,dx = \dfrac{1}{x} + C$

(C) $\int \dfrac{3}{x^2}\,dx = -\dfrac{1}{x^3} + C$

(D) $\int \dfrac{2x+6x^2}{x}\,dx = 2x + 3x^2 + C$

QUESTION FOUR




The function $y = f(x)$ is sketched above. The correct value of $\int_0^5 f(x)\,dx$ is:

(A) 4

(B) 11

(C) 13

(D) 20

QUESTION FIVE

Which statement is true of the quadratic $y = 4x^2 + 24x + 36$?

(A) It is positive definite;

(B) It has two unreal zeroes;

(C) It is a perfect square;

(D) The zeroes add to 6.

Exam continues overleaf ...



QUESTION SIX

The correct solution of $\dfrac{x}{x-3} > 0$ is:

(A) $x < 0$ or $x > 3$

(B) $0 < x < 3$

(C) $x > 0$

(D) $x > 0$ or $x > 3$

QUESTION SEVEN

A function $y = f(x)$ is evaluated at points in the interval $0 \le x \le 4$, as in the table below.

| $x$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $f(x)$ | 2 | 4 | 3 | 5 | 4 |

An estimate of $\int_0^4 f(x)\,dx$ using two applications of Simpson's rule is:

(A) 5

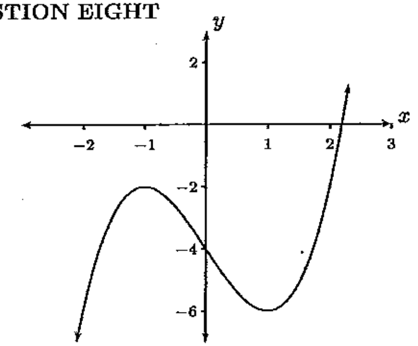
(B) 8

(C) 16

(D) 24

Exam continues next page ...

**QUESTION EIGHT**




The graph of $y = x^3 - 3x - 4$ is sketched above. What is the smallest value of a constant $c$ such that $y = c$ intersects the graph of the cubic at least twice?

(A) $-2$

(B) $-4$

(C) $-6$

(D) $-7$

**QUESTION NINE**

Which of the following functions does not have a horizontal asymptote $y = 1$?

(A) $y = 1 + e^x$

(B) $y = \dfrac{x^2+1}{x^2-1}$

(C) $y = 3 - \dfrac{2x+1}{x+1}$

(D) $y = \dfrac{3x^2+1}{3x+1}$

——— End of Section I ———

**Exam continues overleaf ...**



**SECTION II - Written Response**

**Answers for this section should be recorded in the booklets provided.**

**Show all necessary working.**

**Start a new booklet for each question.**

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**QUESTION TEN** (13 marks) Use a separate writing booklet. **Marks**

(a) Find the exact value of $x^3 + x^2 - 3x + 1$ when $x = -\sqrt{2}$. [2]

(b) Solve $x - \dfrac{4}{x-1} = 1$. [2]

(c) Find the exact solution of $\log_e(2x-4) = 3$. [2]

(d) Differentiate:

(i) $y = \ln(3x+1)$ [1]

(ii) $y = 5xe^x$ [2]

(e) For what values of $x$ is the function $y = x^2 - 6x + 3$ decreasing? [2]

(f) Simplify $\dfrac{a+b}{\frac{1}{a}+\frac{1}{b}}$. [1]

(g) Solve $\tan 2\theta = 0$, for $0^\circ \le \theta \le 360^\circ$. [1]

**Exam continues next page ...**

QUESTION ELEVEN (13 marks) Use a separate writing booklet. **Marks**

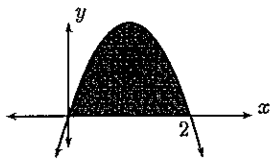
(a) Find:

(i) $\int \frac{2}{x}\,dx$ [1]

(ii) $\int 4\sqrt{x}\,dx$ [1]

(iii) $\int \frac{2x}{x^2+1}\,dx$ [1]

(b)




Find the area bounded by the curve $y = 6x - 3x^2$ and the $x$-axis, as shaded above. [2]

(c) Prove that $y = \frac{3}{x^2}$ is concave up for all $x \neq 0$. [2]

(d) Solve $2\sin^2\theta - \sin\theta - 1 = 0$, for $0° \leq \theta \leq 360°$. [3]

(e) Find all values of $a$ for which $\int_1^a (x+1)\,dx = 6$. [3]

QUESTION TWELVE (13 marks) Use a separate writing booklet. **Marks**

(a) A certain function has derivative $y' = 8x^3 - 6x^2 + 4x - 2$ and passes through the point $(2, 9)$. Find an expression for the function $y$. [3]

(b) Find constants $a$, $b$ and $c$ such that $2x^2 - 3x + 5 \equiv a(x-1)^2 + b(x-1) + c$. [3]

(c) Consider the quadratic $(k+3)x^2 + 2kx + 4$, where $k$ is a constant.

(i) Find a simplified expression for its discriminant. [1]

(ii) For what values of $k$ does the quadratic have no real zeroes? [2]

(iii) Explain why there are no values of $k$ for which the quadratic is negative definite. [1]

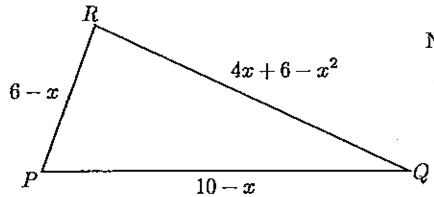
(d) The quadratic equation $2x^2 - 14x + c = 0$, where $c$ is a constant, is known to have roots that differ by 3. By letting the two roots be $\alpha$ and $\alpha + 3$, find the value of $c$. [3]

**Exam continues overleaf ...**



QUESTION THIRTEEN (13 marks) Use a separate writing booklet. **Marks**

(a) The length of the three sides of $\triangle PQR$ pictured below are known in terms of $x$, where $0 \leq x \leq 3$.



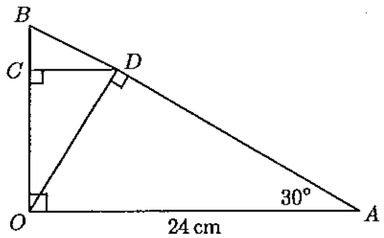

NOT TO SCALE

(i) Write down a simplified expression for the perimeter of the triangle. [1]

(ii) Use calculus to find the **maximum** possible value of the perimeter as $x$ varies. [3]

(iii) What is the **minimum** value of the perimeter? [2]

(b)




In the diagram above, $OA = 24$ cm and $\angle BAO = 30°$.
Also $\angle BOA = \angle ODA = \angle OCD = 90°$.

Find the exact length of $BC$. [3]

(c) (i) Shade the region bounded by $y = x^2 + 1$, the coordinate axes and the line $x = 2$. [1]

(ii) The region in part (i) is rotated about the $x$-axis to form a solid of revolution. Calculate the volume of this solid. [3]

**Exam continues next page ...**

**QUESTION FOURTEEN** (13 marks) Use a separate writing booklet. **Marks**

(a) Consider the function $f(x) = x^2(16 - x^2)^8$.

(i) Find the derivative $f'(x)$. **2**

(ii) Factorise your answer in part (i) and hence find the $x$-coordinates of all stationary points of the function $y = f(x)$. **2**

(b) The country of Pecunia is suffering from an extremely high rate of inflation. A certain family buys a loaf of bread at the start of each week. Over the course of a fifty-two week period, the price of a loaf of bread at their bakery increased by 2% each week. The cost of the loaf at the beginning of the first week was \$2.

Let $T_n$ be the price of the loaf at the beginning of week $n$ and assume it is modelled by a geometric sequence.

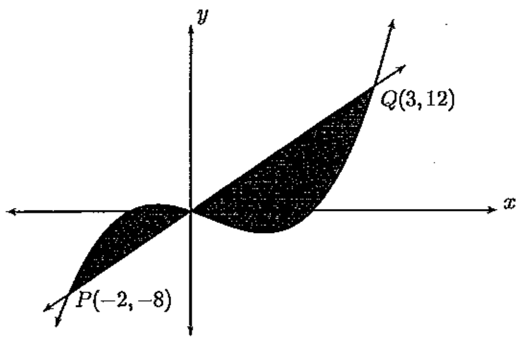
(i) Write down an expression for $T_n$ using the information given. **1**

(ii) What is the cost of their last loaf, bought in the fifty-second week? **1**

(iii) In what week does their loaf first cost them over \$4? **2**

(iv) How much does the family spend on bread over the fifty-two week period? **1**

(c)




The graphs of $y = x^3 - x^2 - 2x$ and $y = 4x$ are sketched above. The graphs intersect at $P(-2,-8)$, $Q(3,12)$ and the origin $O(0,0)$. (You need NOT show this).

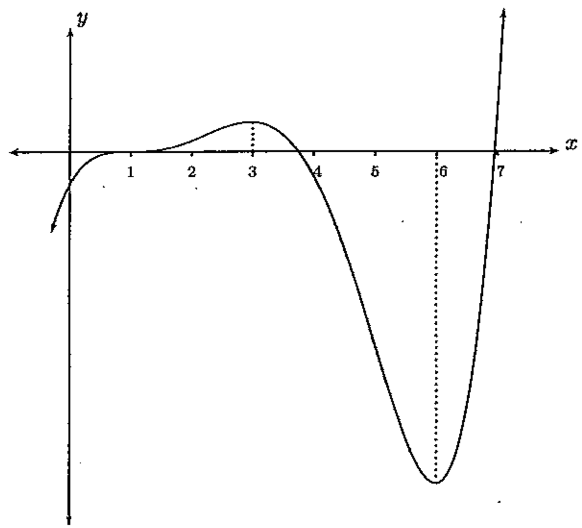
Find the area of the region bounded by the cubic and the line, which is shaded in the diagram. **4**

**Exam continues overleaf ...**



**QUESTION FIFTEEN** (13 marks) Use a separate writing booklet. **Marks**

(a)




The graph above is stationary when $x = 1$, $x = 3$ and $x = 6$ and has points of inflexion when $x = 1$, $x = 2$ and $x = 5$. Sketch a possible graph of its derivative. Be sure to label the $x$-axis in your solution with the integer values from $x = 1$ to $x = 7$. **3**

(b) Consider the curve $y = \dfrac{x+1}{x^2}$.

(i) Write down any intercepts with the coordinate axes. **1**

(ii) Write down the equation of the vertical asymptote. **1**

(iii) With working to justify your answer, find the equation of the horizontal asymptote. **1**

(iv) Find the derivative $y'$. **1**

You may assume that the second derivative is $y'' = \dfrac{2x+6}{x^4}$.

(v) Find any stationary points and determine their nature. **2**

(vi) Find any points of inflexion. **2**

(vii) Sketch the curve, showing clearly the information found above. **2**

**Exam continues next page ...**

**QUESTION SIXTEEN** (13 marks) Use a separate writing booklet. Marks

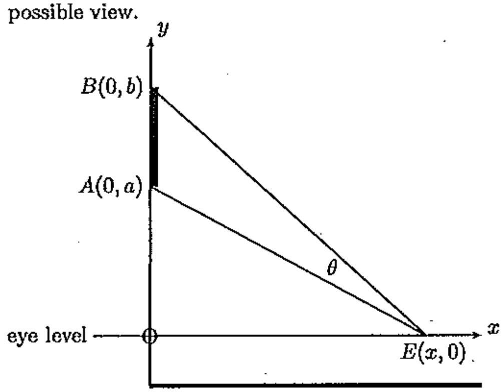
(a) Use the discriminant to find any tangents to the curve $y = x^3 + 2x^2 - 3x$ that pass through the origin. Hint: Let $y = mx$. **3**

(b) (i) Differentiate $x^3 \ln x$. **1**

(ii) Hence find $\int x^2 \ln x \, dx$. **1**

(iii) Given the definition $\int_0^1 x^2 \ln x \, dx = \lim_{\epsilon \to 0} \int_\epsilon^1 x^2 \ln x \, dx$, evaluate $\int_0^1 x^2 \ln x \, dx$. **1**

(c) Staff at an art gallery wish to ensure that patrons viewing the paintings have the best possible view.




Suppose a painting is hung on the wall, with the top and bottom of the painting at heights $b$ and $a$ metres respectively above eye level. Suppose the painting subtends an angle $\theta$, where $0° \le \theta \le 90°$, at the eye of the viewer standing at $E$, $x$ metres from the wall.

(i) Use the cosine rule in $\triangle ABE$ to show that **2**

$$\cos\theta = \frac{x^2 + ab}{\sqrt{a^2 + x^2}\sqrt{b^2 + x^2}}.$$

(ii) Use the identity $\tan^2\theta = \dfrac{1 - \cos^2\theta}{\cos^2\theta}$ to show that **2**

$$\tan\theta = \frac{(b-a)x}{x^2 + ab}.$$

(iii) Use calculus to find the value of $x$ that maximises $\tan\theta$, and hence the position that maximises the observer's angle of view $\theta$. **3**

End of Section II

END OF EXAMINATION



The following list of standard integrals may be used:

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \ne -1; \quad x \ne 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \ne 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \ne 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \ne 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \ne 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \ne 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \ne 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \; x > 0$

## SECTION I - Multiple Choice

### QUESTION ONE
The correct is answer is [A]

### QUESTION TWO
$y'' > 0$ where is it concave up, thus the correct answer is [B]

### QUESTION THREE
The correct answer is [D]

### QUESTION FOUR
Note the limits – we are not finding the integral over the whole domain.

$$\begin{aligned}\int_0^5 f(x)\,dx &= 4\times 2 + \tfrac{1}{2}\times 4\times 8 - \tfrac{1}{2}\times 2\times 1\\ &= 11\end{aligned}$$

The correct answer is [B]

### QUESTION FIVE
The discriminant is $\Delta = 24^2 - 4\times 4 * 36 = 0$ so it is NOT positive definite.

Since $\Delta = 0$, it doesn't have 2 distinct zeroes.

$4x^2 + 24x + 36 = (2x+6)^2$, so it is a perfect square.

The zeroes add to $-24/4 = -6$. The correct answer is [C]

### QUESTION SIX
Multiply by $(x-3)^2$. Hence $x(x-3) > 0$. Thus $x < 0$ or $x > 3$. The correct answer is [A]

### QUESTION SEVEN

$$\begin{aligned}\int_0^4 f(x)\,dx &= \int_0^2 f(x)\,dx + \int_2^4 f(x)\,dx\\ &\doteqdot \tfrac{1}{6}(2-0)(2+4\times 4+3) + \tfrac{1}{6}(4-2)(3+4\times 5+4)\\ &= \tfrac{1}{6}(2)(2+4\times 4+2\times 3+4\times 5+4)\\ &= 16\end{aligned}$$

The correct answer is [C]

### QUESTION EIGHT
The line $y = -6$ cuts the cubic twice, but a lower horizontal line only cuts once. Hence [C].

### QUESTION NINE
The correct answer is [D].



## SECTION II - Written Response

### QUESTION TEN

(a) $$\begin{aligned}x^3+x^2-3x+1 &= (-\sqrt{2})^3 + (-\sqrt{2})^2 - 3\times(-\sqrt{2}) + 1\\ &= -2\sqrt{2} + 2 + 3\sqrt{2} + 1\\ &= \sqrt{2} + 3\end{aligned}$$

(b) $$\begin{aligned}x - \frac{4}{x-1} &= 1\\ \frac{x(x-1)}{x-1} - \frac{4}{x-1} &= 1\\ \frac{x^2-x-4}{x-1} &= 1\\ x^2 - x - 4 &= x - 1\\ x^2 - 2x - 3 &= 0\\ (x-3)(x+1) &= 0\end{aligned}$$

Hence $x = -1$ or $x = 3$.

(c) $x = \dfrac{1}{2}(4 + e^3)$.

(d)

(i) $y' = \dfrac{3}{3x+1}$

(ii) $$\begin{aligned}y' &= 5e^x + 5xe^x\\ &= 5(1+x)e^x\end{aligned}$$

(e) It is decreasing when $y' < 0$. Thus:

$$\begin{aligned}2x - 6 &< 0\\ x &< 3\end{aligned}$$

(f) $$\begin{aligned}\frac{a+b}{\frac{1}{a}+\frac{1}{b}} &= \frac{a+b}{\frac{b+a}{ab}}\\ &= \frac{(a+b)ab}{(b+a)}\\ &= ab\end{aligned}$$

(g) $2\theta = 0^\circ, 180^\circ, 360^\circ, 540^\circ, 720^\circ$ for $0^\circ \le 2\theta \le 720^\circ$

$\theta = 0^\circ, 90^\circ, 180^\circ, 270^\circ, 360^\circ$

### QUESTION ELEVEN

(a) (i) $\displaystyle\int \frac{2}{x}\,dx = 2\log_e x + C$

(ii) $$\begin{aligned}\int 4\sqrt{x}\,dx &= \int 4x^{\frac{1}{2}}\,dx\\ &= \frac{8}{3}x^{\frac{3}{2}} + C\\ &= \frac{8}{3}\sqrt{x^3} + C\end{aligned}$$

(iii) $\displaystyle\int \frac{2x}{x^2+1}\,dx = \log_e(x^2+1) + C$

(b) Area $= \int_0^2 (6x - 3x^2)\,dx$

$= \left[3x^2 - x^3\right]_0^2$

$= (12 - 8) - (0 - 0)$

$= 4\,\text{u}^2$

(c) $y = 3x^{-2}$

$y' = -6x^{-3}$

$y'' = 18x^{-4}$

$y'' = \dfrac{18}{x^4}$

$> 0$ (where $x \neq 0$)

(d) $2\sin^2\theta - \sin\theta - 1 = 0$

$(2\sin\theta + 1)(\sin\theta - 1) = 0$

Hence $\sin\theta = -\frac{1}{2}$ or $\sin\theta = 1$

$\theta = 210^\circ, 330^\circ$ $\qquad \theta = 90^\circ$

(e) $\int_1^a (x+1)\,dx = 6$

$\left[\frac{1}{2}x^2 + x\right]_1^a = 6$

$(\frac{1}{2}a^2 + a) - (\frac{1}{2} + 1) = 6$

$a^2 + 2a - 3 = 12$

$a^2 + 2a - 15 = 0$

$(a+5)(a-3) = 0$

Hence $a = -5$ or $a = 3$.

## QUESTION TWELVE

(a) $y' = 8x^3 - 6x^2 + 4x - 2$

$y = 2x^4 - 2x^3 + 2x^2 - 2x + C,$

for some constant $C$. Since it passes through $(2, 9)$;

$9 = 2(2)^4 - 2(2)^3 + 2(2)^2 - 2(2) + C$

$9 = 32 - 16 + 8 - 4 + C$

$C = -11$

The function is $y = 4x^4 - 2x^3 + 2x^2 - 2x - 11$.

(b) Equating coefficients of $x^2$ gives $a = 2$.

Substituting $x = 1$ gives $c = 4$.

Substituting $x = 0$ gives $a - b + c = 5$, so $b = 1$.

Thus $2x^2 - 3x + 5 = 2(x-1)^2 + 1(x-1) + 4$.

(c) (i) $\Delta = (2k)^2 - 4 \times (k+3) \times 4$

$\Delta = 4(k^2 - 4k - 12)$

$\Delta = 4(k-6)(k+2)$

(ii) The quadratic has no real roots if $\Delta < 0$ i.e. if $-2 < k < 6$.

(iii) To be negative definite we need $\Delta < 0$ and $(k+3) < 0$, so that the quadratic is concave down. But $\Delta < 0$ is never true for $k < -2$, and it particular it is not true for $k < -3$.

(d) Sum of roots: $\alpha + (\alpha + 3) = 7$

So $\alpha = 2$ and the roots are 2 and 5.

Product of roots: $10 = \dfrac{c}{2}$

$c = 20$



## QUESTION THIRTEEN

(a) (i) $P = (6 - x) + (10 - x) + (4x + 6 - x^2)$

$= 2x - x^2 + 22$

(ii) $P' = 2 - 2x$

When $x = 1$, $P' = 0$ and $P'' = -2$.

Hence the stationary point at $x = 1$ when $P = 23$ is a maximum (since $P'' < 0$).

(iii) There are no other stationary points, but the minimum will occur at one end of the domain $0 \le x \le 3$. At $x = 0$, $P = 22$. At $x = 3$, $P = 19$. Hence the minimum perimeter is $P = 19$ units, when $x = 3$.

(b) In $\triangle ODA$, $\dfrac{DA}{OA} = \cos 30^\circ$

$DA = 12\sqrt{3}$

In $\triangle BOA$, $\dfrac{OA}{BA} = \cos 30^\circ$

$BA = 24 \div \frac{\sqrt{3}}{2}$

$BA = 16\sqrt{3}$

$BD = BA - DA$

$= 4\sqrt{3}$

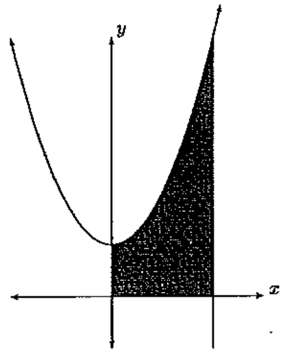
In $\triangle BCD$, $\dfrac{BC}{BD} = \cos 60^\circ$ (Angle sum of triangle)

$BC = 4\sqrt{3} \times \frac{1}{2}$

$= 2\sqrt{3}$

There are many ways of arriving at this result, either by trigonometry or similarity.

(c)




Volume $= \int_0^2 \pi y^2\,dx$

$= \int_0^2 \pi(x^2+1)^2\,dx$

$= \pi \times \int_0^2 (x^4 + 2x^2 + 1)\,dx$

$= \pi \times \left[\frac{1}{5}x^5 + \frac{2}{3}x^3 + x\right]_0^2$

$= \pi \times \left(\dfrac{32}{5} + \dfrac{16}{3} + 2\right)$

$= \dfrac{206}{15}\pi$

QUESTION FOURTEEN

(a) (i) $f'(x) = 2x(16-x^2)^8 + x^2 \times -2x \times 8(16-x^2)^7$

(ii) $f'(x) = 2x(16-x^2)^7(16-x^2-8x^2)$

$= 2x(4-x)(4+x)(16-9x^2)$

$= 2x(4-x)(4+x)(4-3x)(4+3x)$

hence $f'(x) = 0$ when $x = 0$, $x = 4$, $x = -4$, $x = \frac{4}{3}$ or $x = -\frac{4}{3}$.

(b) (i) $T_n = ar^{n-1}$

$= 2 \times 1.02^{n-1}$

(ii) $T_{52} = 2 \times 1.02^{51}$

$= 5.49$

The final loaf costs \$5.49.

(iii) $4.00 < 2 \times 1.02^{n-1}$

$1.02^{n-1} > 2$

$n > 1 + \log_{1.02} 2$

$n > 1 + \log 2 \div \log 1.02$

$n > 36.003$

The price of a loaf has doubled by the start of week 37.

(iv) $S_n = a\dfrac{r^n - 1}{r-1}$

$S_{52} = 2\dfrac{1.02^{52} - 1}{0.02}$

$S_{52} = 180.03$

The total cost over the 52 week period is \$180.03.

(c) For $x < 0$, the cubic lies above the line and this area is:

$$\int_{-2}^{0} \left((x^3 - x^2 - 2x) - (4x)\right) dx = \int_{-2}^{0} \left(x^3 - x^2 - 6x\right) dx$$

$$= \left[\tfrac{1}{4}x^4 - \tfrac{1}{3}x^3 - 3x^2\right]_{-2}^{0}$$

$$= \frac{16}{3}$$

For $x > 0$, the line lies above the cubic and this area is:

$$\int_{0}^{3} \left(4x - (x^3 - x^2 - 2x)\right) dx = \int_{0}^{3} \left(6x - x^3 + x^2\right) dx$$

$$= \left[3x^2 - \tfrac{1}{4}x^4 + \tfrac{1}{3}x^3\right]_0^3$$

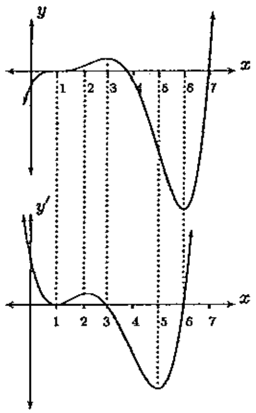
$$= \frac{63}{4}$$

Hence the total area $= \dfrac{16}{3} + \dfrac{63}{4} = \dfrac{253}{12}$.



QUESTION FIFTEEN

(a)




(b) (i) There is one $x$-intercept: $x = -1$

(ii) The vertical asymptote is the vertical line $x = 0$.

(iii) $y = \dfrac{x+1}{x^2}$

$y = \dfrac{\frac{1}{x} + \frac{1}{x^2}}{1}$

$y \to 0 \quad$ as $x \to \pm\infty$

Hence $y = 0$ is a horizontal asymptote.

(iv) $y = \dfrac{x+1}{x^2}$

$y' = \dfrac{1(x^2) - (x+1)2x}{x^4}$

$y' = \dfrac{-x^2 - 2x}{x^4}$

$y' = \dfrac{-(x+2)}{x^3}$

(v) Hence $y' = 0$ when $x = -2$.

When $x = -2$, $y = -\frac{1}{4}$

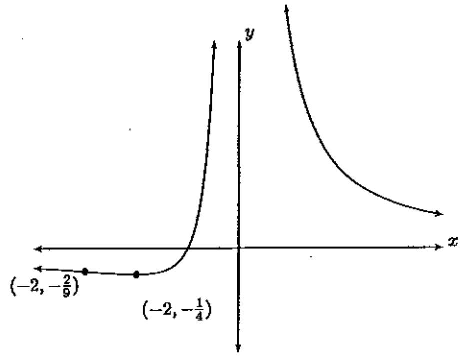
When $x = -2$, $y'' = \dfrac{2(-2)+6}{(-2)^4} > 0$, hence $(-2, -\frac{1}{4})$ it is a local minimum.

(vi) There is a possible point of inflexion when $y'' = 0$, that is when $x = -3$. Testing for a change in concavity, we have:

| $x$ | $-4$ | $-3$ | $-1$ |
|---|---|---|---|
| $y''$ | $-\frac{2}{256}$ ∩ | 0 · | 4 ∪ |

Since there is a change in concavity, $(-3, -\frac{2}{9})$ IS a point of inflexion.

(vii)




QUESTION SIXTEEN

(a) Intersecting the cubic $y = x^3 + 2x^2 - 3x$ and tangent $y = mx$ gives

$$x^3 + 2x^2 - 3x = mx$$
$$x^3 + 2x^2 - (m+3)x = 0$$
$$x(x^2 + 2x - m - 3) = 0$$

The line $y = mx$ will be a tangent if the discriminant of the quadratic $x^2 + 2x - m - 3$ is 0. The discriminant is:

$$\Delta = 4 + 4 \times 1 \times (m+3)$$
$$= 4 \times (m+4)$$

Hence we have a repeated root and thus a tangent when $m = -4$. There will also be a tangent when $m = -3$, because $x(x^2 + 2x) = 0$ has a repeated root $x = 0$.
Thus the two tangents are $y = -3x$ and $y = -4x$.

(b) (i) $\dfrac{d}{dx}(x^3 \ln x) = 3x^2 \ln x + x^3 \times \dfrac{1}{x}$

$\dfrac{d}{dx}(x^3 \ln x) = 3x^2 \ln x + x^2$

(ii) $\dfrac{d}{dx}(x^3 \ln x) = 3x^2 \ln x + x^2$

Hence,

$$x^3 \ln x = \int 3x^2 \ln x \, dx + \tfrac{1}{3}x^3$$
$$\int 3x^2 \ln x \, dx = x^3 \ln x - \tfrac{1}{3}x^3$$
$$\int x^2 \ln x \, dx = \tfrac{1}{3}x^3 \ln x - \tfrac{1}{9}x^3 + C$$

(iii)
$$\int_0^1 x^2 \ln x \, dx = \lim_{\epsilon \to 0} \int_\epsilon^1 x^2 \ln x \, dx$$
$$= \left(-\tfrac{1}{3}(1)^3 \ln 1 - \tfrac{1}{9}(1)^3\right) - \lim_{\epsilon \to 0}\left(-\tfrac{1}{3}(\epsilon)^3 \ln \epsilon - \tfrac{1}{9}(\epsilon)^3\right)$$
$$= \left(0 - \tfrac{1}{9}\right) - (0 - 0)$$
$$= -\tfrac{1}{9}$$

To justify this limit we have used the fact that $\epsilon^3$ dominates $\ln \epsilon$ as $\epsilon \to 0$.

(c) (i) As in the diagram on the paper, set up a coordinate system with origin $O$ at the eye of the observer. Then

$BA = b - a$

$AE = \sqrt{a^2 + x^2}$ By Pythagoras in $\triangle AOE$

$BE = \sqrt{b^2 + x^2}$ By Pythagoras in $\triangle BOE$



Hence by the cosine rule in $\triangle ABE$,

$$\cos\theta = \frac{AE^2 + BE^2 - BA^2}{2AE \times BE}$$
$$= \frac{(a^2 + x^2) + (b^2 + x^2) - (b-a)^2}{2\sqrt{a^2+x^2}\sqrt{b^2+x^2}}$$
$$= \frac{a^2 + x^2 + b^2 + x^2 - b^2 - a^2 + 2ab}{2\sqrt{a^2+x^2}\sqrt{b^2+x^2}}$$
$$= \frac{2x^2 + 2ab}{2\sqrt{a^2+x^2}\sqrt{b^2+x^2}}$$
$$= \frac{x^2 + ab}{\sqrt{a^2+x^2}\sqrt{b^2+x^2}}$$

(ii)
$$\tan^2\theta = \frac{1 - \cos^2\theta}{\cos^2\theta}$$
$$= \left(1 - \frac{(x^2+ab)^2}{(a^2+x^2)(b^2+x^2)}\right) \times \frac{(a^2+x^2)(b^2+x^2)}{(x^2+ab)^2}$$
$$= \left((a^2+x^2)(b^2+x^2) - (x^2+ab)^2\right) \times \frac{1}{(x^2+ab)^2}$$
$$= \left(a^2b^2 + a^2x^2 + b^2x^2 + x^4 - (x^4 + a^2b^2 + 2x^2ab)\right) \times \frac{1}{(x^2+ab)^2}$$
$$= \left(a^2x^2 + b^2x^2 - 2x^2ab\right) \times \frac{1}{(x^2+ab)^2}$$
$$= \frac{x^2(a^2 + b^2 - 2ab)}{(x^2+ab)^2}$$
$$= \frac{x^2(a-b)^2}{(x^2+ab)^2}$$

Now since $b > a$ and $\tan\theta > 0$, we have:

$$\tan\theta = \frac{x(b-a)}{x^2+ab}$$

(iii) Since $\tan\theta$ is an increasing function on $0° \le \theta < 90°$, to maximise $\theta$ it is enough to maximise $\tan\theta$.

By the quotient rule on this expression,

$$\frac{d}{dx}\tan\theta = \frac{d}{dx}\left(\frac{x(b-a)}{x^2+ab}\right)$$
$$= \frac{(b-a)(x^2+ab) - x(b-a)2x}{(x^2+ab)^2}$$
$$= \frac{(b-a)(x^2+ab-2x^2)}{(x^2+ab)^2}$$
$$= \frac{(b-a)(ab-x^2)}{(x^2+ab)^2}$$

This function has a stationary point when $x^2 = ab$, i.e. when $x = \sqrt{ab} > 0$.
We need to show that this point is a maximum, either using a table of signs of the derivative, or by the second derivative test.

The second derivative is:

$$\frac{d^2}{dx^2}\tan\theta = \frac{(b-a)(-2x)(x^2+ab)^2 - (b-a)(ab-x^2)2(x^2+ab)(2x)}{(x^2+ab)^4}$$
$$= \frac{-2(b-a)x(x^2+ab)(x^2+ab-2(ab-x^2))}{(x^2+ab)^4}$$
$$= \frac{-2(b-a)x(x^2+ab)(3x^2-ab)}{(x^2+ab)^4}$$
$$= \frac{-2(b-a)\sqrt{ab}(2ab)(2ab)}{(2ab)^4} \quad \text{when } x^2 = ab$$
$$< 0$$

Hence we have a local maximum when $x = \sqrt{ab}$.

If we choose instead to bracket the zero of the derivative in a table of signs, obvious values to test are $x = 0$ and $x = 2\sqrt{ab}$.

| $x$ | $0$ | $\sqrt{ab}$ | $2\sqrt{ab}$ |
|---|---|---|---|
| $\dfrac{d}{dx}\tan\theta$ | $\dfrac{(b-a)(ab)}{(ab)^2}$ | $0$ | $\dfrac{(b-a)(-3ab)}{(3ab)^2}$ |
| sign | $+$ | $0$ | $-$ |

Hence again we see that $x = \sqrt{ab}$ is a local maximum.

BDD