Using algebra

Sequences and terms

Suppose we are told that the nth term of a particular sequence is $n^2 - 1$.
This means that the 1st term is $1^2 - 1 = 0$,
the 2nd term is $2^2 - 1 = 3$,
the 3rd term is $3^2 - 1 = 8$, and so on.
So the sequence is 0, 3, 8, 15, 24, ...

Convince yourself that for the sequence
1, 3, 7, 15, 31, ... the nth term is $2^n - 1$.

It helps to look for factors, square and cube numbers, numbers going up by the same amount, ... when you are looking for patterns in sequences.

Solving problems

Nick is thinking of a number. If he multiplies it by 8 and subtracts 5 he gets the same as the square of the number plus 10.
What number is Nick thinking of?

Let the number be x. We need to form an equation in x and solve it.

$$8x - 5 = x^2 + 10$$
$$x^2 - 8x + 15 = 0$$
$$(x-3)(x-5) = 0$$
$$x = 3 \text{ or } 5.$$

So Nick must be thinking of the number 3 or 5.

Check by substituting in the original explanation, not your equation in case it is wrong.

Solving linear equations ► page 22
Solving quadratic equations by factors ► page 31
Solving simultaneous equations without a graph ► page 24

1 Write the first five terms of the sequence whose nth term is
(a) $4n - 1$ (b) $n^2 + 3$ (c) $2^{n-1} + 1$ (d) $n(n-1)$

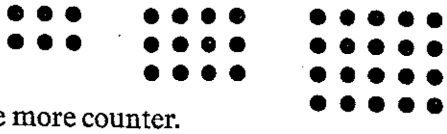
2 Find in terms of n, the nth term of each of these sequences.
(a) 3, 6, 9, 12, 15, ... (b) 3, 9, 27, 81, ...
(c) 3, 6, 12, 24, 48, ... (d) 2, 8, 18, 32, 50, ...

3 Write down an algebraic expression for the results of multiplying the square of x by 5 and adding 3 times x.

4 Ravinder is thinking of a positive number.
If she doubles it and adds 18 it is equal to the square of the difference between the number and three.
What number is Ravinder thinking of?

5 Maria has two brothers, Aaron and Jason.
Maria is x years old.
Aaron is three years older than Maria.
Jason is two years younger than Maria.
(a) Given that the product of the two brothers' ages is 126, show that $x^2 + x - 132 = 0$.
(b) Use the equation to find the ages of the three children. MEG (SMP)

6 Glyn is making rectangular patterns from counters.
The length of each rectangle is one more than the width.
The first three patterns are shown.
Each time, to get the next pattern, he makes the width one more counter.

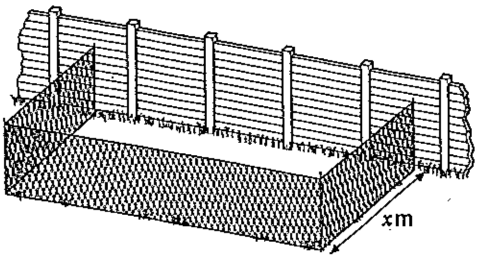


(a) Write an expression for the number of counters in the nth pattern.

Glyn notices that there is also a pattern in how many extra counters he needs each time to make the next pattern.

(b) How many extra counters does he need to make the $(n + 1)$th pattern from the nth pattern? MEG/ULEAC (SMP)

7 James wants to make a screen round part of his garden against a straight fence. He buys a 30 m roll of netting and uses it to make three sides of a rectangle.
(The other side is the fence.)
The length of one side is x m, as shown in the diagram.



(a) Write down an expression for the area of garden inside the screen in square metres.

The area inside the screen is $100\,m^2$.

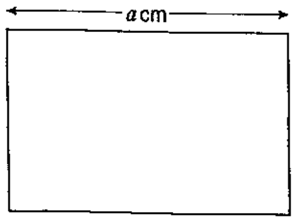
(b) Show that $x^2 - 15x + 50 = 0$ and solve the equation.

(c) Comment on your solutions. MEG/ULEAC (SMP)

8 If n is a positive integer, then $n - 1$, n and $n + 1$ are three consecutive integers.
The sum of the squares of the integers is 365.
Form an equation in n and solve it to find the three integers.

9 The sum of the first n terms of the sequence 4, 10, 16, 22, ... is $n(3n + 1)$.
How many terms add up to 1344?

10 This rectangle has a perimeter of 16 cm and an area of $10\,cm^2$.
The length of one side is a cm.



(a) Show how this information can be used to form the equation $a^2 - 8a + 10 = 0$.

(b) Solve the equation to find the length and width of the rectangle.
Give your answers to an appropriate degree of accuracy.

11 12, 13, 14 and 15 are four consecutive numbers.
If you multiply the outer pair you get 180.
If you multiply the inner pair you get 182.

(a) Choose three more sets of four consecutive numbers and carry out similar calculations.
What do you notice?

(b) Use algebra to prove that your rule always works with any four consecutive numbers.

(c) Use algebra to find a similar rule if you start with four consecutive *odd* numbers.

Answers and hints ► page 114

Using algebra (page 34)

1 (a) 3, 7, 11, 15, 19
(b) 4, 7, 12, 19, 28
(c) 2, 3, 5, 9, 17 *Remember that $2^0 = 1$.*
(d) 0, 2, 6, 12, 20

2 (a) $3n$ (b) 3^n (c) $3 \times 2^{n-1}$ (d) $2n^2$

3 $5x^2 + 3x$ or $x(5x + 3)$

4 Let the number be n.
Then $2n + 18 = (n-3)^2$
$2n + 18 = n^2 - 6n + 9$
$n^2 - 8n - 9 = 0$
$(n-9)(n+1) = 0$
$n = 9$
The solution $n = {}^-1$ is impossible because n is positive.

5 (a) Aaron is $x + 3$ years old.
Jason is $x - 2$ years old.
So $(x + 3)(x - 2) = 126$ and
$x^2 + x - 132 = 0$
(b) *Solve the quadratic equation to find x and hence the ages of all three children.*
$(x + 12)(x - 11) = 0$
$x = {}^-12$ or 11
So Maria is 11, Aaron is 14 and Jason is 9 years old.

6 (a) *In questions like this it usually helps to first find the next couple of terms to see how the pattern is built up. In this case we have rectangle numbers; the 4th term is $5 \times 6 = 30$ and the 5th term is $6 \times 7 = 42$.*
The number of counters in the nth pattern is $(n + 1)(n + 2) = n^2 + 3n + 2$.
Either form of the expression would gain full marks but if you wrote $n + 1 \times n + 2$ you would lose a mark.
(b) The sequence is 6, 12, 20, 30, 42, …
The differences are 6, 8, 10, 12, …
The nth difference (which added to the nth term will give the $(n + 1)$th term) is $2(n + 2)$ or $2n + 4$.
So Glyn will need an extra $2n + 4$ counters.

7 (a) $x(30 - 2x)$
(b) $x(30 - 2x) = 100$
$30x - 2x^2 = 100$
$x^2 - 15x + 50 = 0$
$(x - 5)(x - 10) = 0$
$x = 5$ or 10
(c) Both solutions are valid but they give different shapes ($x = 5$ gives a 5 m by 20 m rectangle and $x = 10$ gives a square of side 10 m).

8 $(n-1)^2 + n^2 + (n+1)^2 = 365$
$n^2 - 2n + 1 + n^2 + n^2 + 2n + 1 = 365$
$3n^2 = 363$
$n^2 = 121$
$n = \pm 11$
The three positive numbers are 10, 11 and 12.

9 $n(3n + 1) = 1344$
$3n^2 + n - 1344 = 0$
Using the quadratic formula,
$$n = \frac{{}^-1 \pm \sqrt{(1 + 16128)}}{6}$$
$$= \frac{{}^-1 \pm 127}{6}$$
$= 21$ or ${}^-21\frac{1}{3}$
21 terms add up to 1344.

10 (a) The length of the second side of the rectangle is $(8 - a)$ cm.
So $a(8 - a) = 10$
$8a - a^2 = 10$
$a^2 - 8a + 10 = 0$
(b) $$a = \frac{8 \pm \sqrt{(64 - 40)}}{2}$$
$= 6{\cdot}45$ or $1{\cdot}55$ (to 2 d.p.)

11 (a) The product of the inner pair is always two more than the product of the outer pair.
(b) Let the numbers be $n, n + 1, n + 2$ and $n + 3$.
The difference betwen the two respective products is
$(n + 1)(n + 2) - n(n + 3)$
$= n^2 + 3n + 2 - n^2 - 3n$
$= 2$
So the rule found in (a) always works.
(c) Let the numbers be $n, n + 2, n + 4$ and $n + 6$, where n is odd.
$(n + 2)(n + 4) - n(n + 6)$
$= n^2 + 6n + 8 - n^2 - 6n$
$= 8$
So, for consecutive odd numbers, the difference between the two products is 8.
Note that the difference would be the same for consecutive even numbers.

More help or practice
Finding rules for sequences ► Book Y4 pages 53 to 61
Pattern spotting ► Book YR+ pages 44 to 46