Centre Number | 1 | 2 | 5 |

Student Number | | | | | | | | |

CRANBROOK SCHOOL

2016

Term 1 test

Assessment Task 1

Extension 1 Mathematics

Reading time	None
Writing time	90 mins
Total Marks	47
Task weighting	15%

General Instructions

- Write using blue or black pen
- A Board-approved calculator may be used
- Use the Formula Sheet
- Use the Multiple-Choice Answer Sheet provided
- All relevant working should be shown for each question

Additional Materials Needed

- Formula Sheet/Reference Sheet
- Multiple Choice Answer Sheet
- 3 writing booklets

Structure & Suggested Time Spent

Section I

Multiple Choice Questions

- Answer Q1 – 9 on the multiple choice answer sheet
- Allow 20 minutes for this section

Section II

Extended response Questions

- Attempt Q10 – 12 in a separate writing booklet
- Allow about 70 minutes for this section

This paper must not be removed from the examination room

Disclaimer

The content and format of this paper does not necessarily reflect the content and format of the HSC examination paper.

Section I

10 Marks

Allow about 20 minutes for this section

Use the multiple choice answer sheet for Questions 1 to 10.

Question 1

Which of the following is polynomial?

(A) $A(x)=\dfrac{1}{1+3x^2+4x^3-5x^4}$

(B) $B(x)=5$

(C) $C(x)=1-\sqrt{x}$

(D) $D(x)=5x^{\frac{3}{2}}-7x^{\frac{1}{2}}+3$

Question 2

Which of the following is the equation for the normal to the parabola $x^2 = 4ay$ at the point $T(2at,\ at^2)$?

(A) $x + ty - at(2 - t^2) = 0$

(B) $x + ty - at(2 + t^2) = 0$

(C) $tx + y - at(2 - t^2) = 0$

(D) $tx + y - at(2 + t^2) = 0$

Question 3

What is the value of $\int_1^2 \frac{x^2}{5} - \frac{1}{x^2}\, dx$?

(A) $-\frac{1}{30}$

(B) $\frac{1}{30}$

(C) $-\frac{7}{40}$

(D) $\frac{7}{40}$

Question 4

The polynomial $P(x) = 3x^3 + 10x^2 - 5x - 2$ has a root somewhere between 0 and 1. Using the halving the interval method twice, between which two values does the root lie?

(A) 0 and 0.25

(B) 0.25 and 0.5

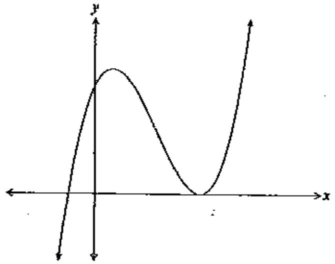
(C) 0.5 and 0.75

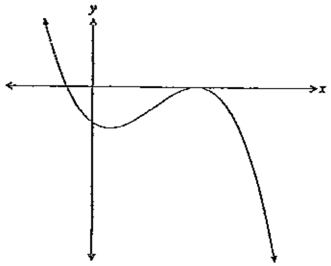
(D) 0.75 and 1

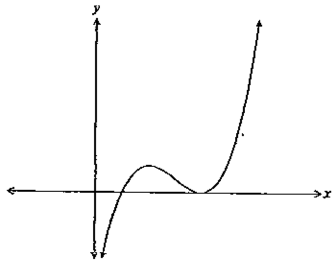
Question 5

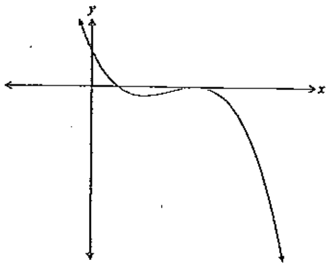
A polynomial $P(x)$ has a double root at 2, a negative constant term and is divisible by $2x-1$.

Which of the following could be the sketch of $y = P(x)$?

(A) 

(B) 

(C) 

(D) 

Question 6

The parametric equations $y = \dfrac{t^2}{t+1}$ and $x = \dfrac{4-t}{3+t}$ define a function.

What are the x and y intercepts of the function?

(A) $x = \dfrac{16}{5}$ and $y = \dfrac{4}{3}$

(B) $x = \dfrac{4}{3}$ and $y = \dfrac{16}{5}$

(C) $x = 4$ and $y = 0$

(D) $x = 0$ and $y = 4$

Question 7

Which of the following is the answer to $\displaystyle\int \frac{1}{\sqrt{4-3x}}\,dx$?

(A) $\dfrac{-2\sqrt{4-3x}}{3} + C$

(B) $-6\sqrt{4-3x} + C$

(C) $\dfrac{-2}{3\sqrt{(4-3x)^3}} + C$

(D) $\dfrac{-6}{\sqrt{(4-3x)^3}} + C$

Question 8

The polynomial $M(x)=rx^3-3rx-p+3$ is divisible by $x+1$ and has a remainder of rp^3+3 when it is divided by $x-p$. What are the possible values of r?

(A) $-\frac{3}{2}$

(B) $-\frac{1}{3}$

(C) $-\frac{1}{3}$ or $-\frac{3}{2}$

(D) No real solutions.

Question 9

The integral $\int_0^8 x\sqrt[3]{8-x}\,dx$ is to be solved using the substitution $u^3=8-x$. Which of the following is the correct substitution step?

(A) $3\int_2^0 u^6-8u^3\,du$

(B) $3\int_0^2 u^6-8u^3\,du$

(C) $\int_2^0 u^4-8u\,du$

(D) $\int_0^2 u^4-8u\,du$

END OF SECTION I

Section II

46 Marks

Allow about 70 minutes for this section

Answer question 11 - 13 in separate booklets.

Question 10 **Begin a new booklet** **12 Marks**

(a) Let $f(x)=(1-x)(x-4)$.

(i) Sketch $y=f(x)$ and the line $x=5$. **1**

(ii) By first setting up the appropriate integral, find the area bound by the curve, the x axis and the lines $x=1$ and $x=5$. **3**

(b) Using Mathematical induction, prove that $13\times 6^n+2$ is divisible by 5 for all positive integer values of n. **3**

(c) The polynomial $G(x)=8x^3+34x^2+27x-9$ has a root at $x=-3$.

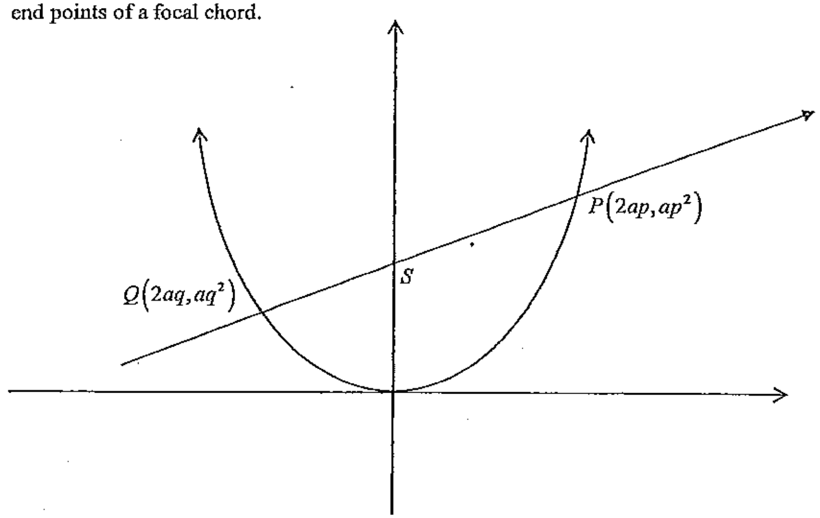
By first factorising $G(x)$, sketch $y=G(x)$ **3**

(d) The function $y=3x-\frac{7}{2x-1}$ has a root near $x=1$. Use one application of Newton's Method to find a better approximation. **2**

Question 11 **Begin a new booklet** **15 Marks**

(a) Prove that $4^n > 2n+1$ by Mathematical Induction for $n \geq 1$. **3**

(b) The points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the parabola $x^2 = 4ay$ and define the end points of a focal chord.



(i) Show that the equation of the tangent at point P is **2**

$$y - px + ap^2 = 0$$

(ii) It is known that the equation of the chord PQ is $y-\frac{(p+q)x}{2}+apq=0$

(You do not need to prove this).

Show that $pq=-1$. **1**

(iii) Hence show that the tangents at P and Q meet on the directrix and are perpendicular. You may assume the equation of the tangent at Q is $y-qx+aq^2=0$ **2**

(c) (i) Show that $\frac{d}{dx}\sqrt{4-\sqrt{x}}=-\frac{1}{4\sqrt{x\left(4-\sqrt{x}\right)}}$ **2**

(ii) Hence evaluate $\int_1^4 \frac{3}{\sqrt{x\left(4-\sqrt{x}\right)}}\,dx$ **2**

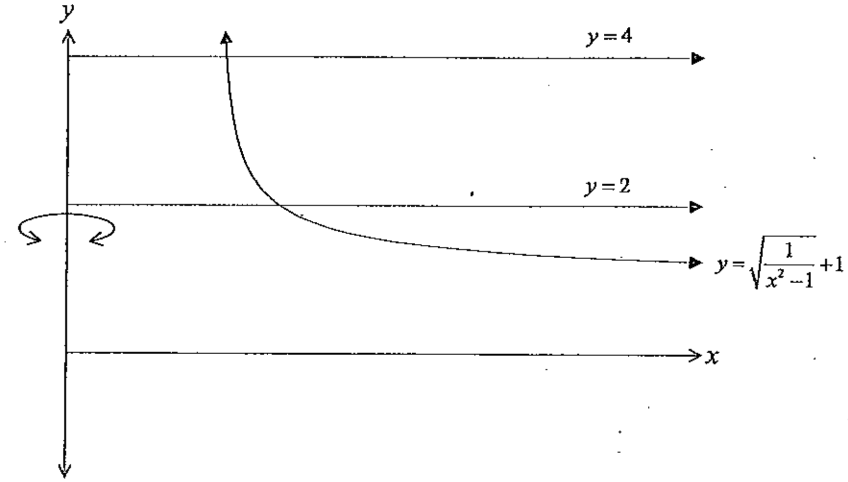
(b) Prove, by Mathematical induction, the following statement. **3**

$$\sum_{r=2}^{n} 3^r = \frac{9\left(3^{n-1}-1\right)}{2} \text{ for } n \geq 2$$

Question 12 **Begin a new booklet** **11 Marks**

(a) Find $\int_1^2 \frac{3x}{\sqrt[3]{3x+1}}\,dx$ using the substitution $u=3x+1$ **3**

(b) The function $y=\sqrt{\frac{1}{x^2-1}}+1$ is sketched below.



Find the volume of the solid generated when the area between the y axis, the lines $y=2$ and $y=4$ and the curve is rotated around the y axis. **4**

(c) Let $E(x) = 6x^3 + 5x^2 + 39x - 5a$ where a is a nonzero constant.

(i) Find the remainder when $E(x)$ is divided by $2x - 1$ in terms of a. 1

(ii) Show that $x - a$ is not a factor of $E(x)$ 3

END OF SECTION II

END OF EXAM

BOSTES | BOARD OF STUDIES TEACHING & EDUCATIONAL STANDARDS NSW

2016 HIGHER SCHOOL CERTIFICATE EXAMINATION

REFERENCE SHEET

– Mathematics –

– Mathematics Extension 1 –

– Mathematics Extension 2 –

Multiple Choice

1) B

2) $y - at^2 = -\frac{1}{t}(x - 2at)$

$ty - at^3 = -x + 2at$

$x + ty = at(2 + t^2)$

B.

3) $\int_1^2 \frac{x^2}{5} - \frac{1}{x^2}\, dx$

$\left[\frac{x^3}{15} + \frac{1}{x}\right]_1^2$

$\frac{8}{15} + \frac{1}{2} - \frac{1}{15} - 1$

$\frac{7}{15} - \frac{1}{2}$

$\frac{7}{15} - \frac{15}{30}$

$= \frac{14}{30} - \frac{15}{30}$

$= \frac{-1}{30}$ A

4) $P(x) = 3x^3 + 10x^2 - 5x - 2$

$P(0) = -2$

$P(1) = 6$

$P(0.5) = \frac{3}{8} + \frac{10}{4} - 2.5 - 2$

$= -ve$

$P(0.75) = 1.14$

> 0

C

5) C

6) $x = 0$

$t = 4$

$y = \frac{16}{5}$

B

7) $\int (4-3x)^{-1/2}$

$\frac{2(4-3x)^{1/2}}{-3} + c$

A

8) $M(x) = rx^3 - 3rx - p + 3$

$M(-1) = -r + 3r - p + 3$

$0 = 2r - p + 3$

$p = 2r + 3$

$M(p) = rp^3 - 3rp - p + 3$

~~2r - p + 3 =~~

$\cancel{rp^3} + \cancel{3} = \cancel{rp^3} - 3rp - p + \cancel{3}$

$3rp + p = 0$

$p(1 + 3r) = 0$

$\cancel{r = -\frac{1}{3}}$

or

$(2r+3)(1+3r) = 0$

$\boxed{r = -\frac{3}{2}, -\frac{1}{3}}$

C

9.) $\int_0^8 x\sqrt[3]{8 - x}\, dx$ let $u^3 = 8 - x$

$x = 0$, $u = 2$

$x = 8$, $u = 0$

$x = 8 - u^3$

$\frac{dx}{du} = -3u^2$

$\int_2^0 (8 - u^3)\sqrt[3]{(u^3)} \times -3u^2\, du$

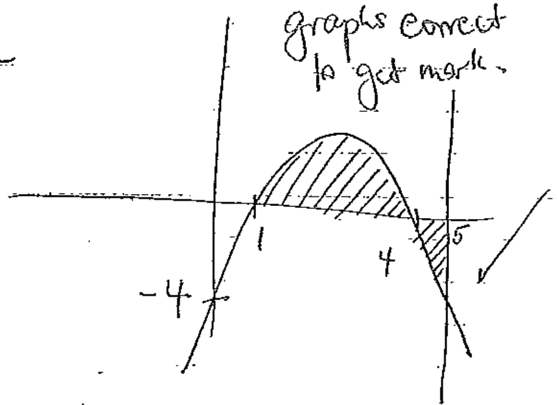
$= \int_2^0 (8u - u^4) \times -3u^2\, du$

$\boxed{= 3\int_2^0 u^6 - 8u^3\ du}$ A

Question 10

$A = (1-x)(x-4)$
$= x - x^2 - 4 + 4x$
$= -x^2 + 5x - 4.$

Needed both graphs correct to get mark.



$A = \int_1^4 -x^2 + 5x - 4 \, dx + \left| \int_4^5 -x^2 + 5x - 4 \, dx \right|$

$= \left[\frac{-x^3}{3} + \frac{5x^2}{2} - 4x \right]_1^4 + \left| \left[\frac{-x^3}{3} + \frac{5x^2}{2} - 4x \right]_4^5 \right|$

$= \left(-\frac{4^3}{3} + \frac{5 \times 4^2}{2} - 4 \times 4 \right) - \left(-\frac{1}{3} + \frac{5}{2} - 4 \right) + \left| \left(-\frac{5^3}{3} + \frac{5 \times 5^2}{2} - 4 \times 5 \right) - \left(\frac{4^3}{3} + \frac{5 \times 4^2}{2} - 4 \times 4 \right) \right|$

$= \frac{9}{2} + \left| -\frac{11}{6} \right|$

$= \frac{9}{2} + \frac{11}{6}$

$= \frac{19}{3} u^2$

$13 \times 6^n + 2$ is divisibly 5.

Step 1
Prove true for $n = 1$

$13 \times 6^1 + 2 = 80$
which is divisible by 5.
$\therefore$ true for $n = 1$

Step 2
Assume true for $n = k$.

$\therefore 13 \times 6^k + 2 = 5M$ where M is any integer.
$6^k = \frac{5M - 2}{13}$

Step 3
Prove true for $n = k+1$

$13 \times 6^{k+1} + 2 = 13 \times 6 . 6^k + 2$
$= 13 \times 6 \left(\frac{5M - 2}{13} \right) + 2$
$= 30M - 12 + 2$
$= 30M - 10$
$= 5(6M - 2)$

$\therefore$ this is true for $n = k+1$

Step 4.

$\therefore$ This is true ~~for n~~ by M.I.

$$\begin{array}{r} 8x^2+10x-3 \\ x+3 \overline{\smash{)}\,8x^3+34x^2+27x-9} \\ 8x^3+24x^2 \\ \hline 10x^2+27x \\ 10x^2+30x \\ \hline -3x-9 \\ (-3x-9) \\ \hline 0 \end{array}$$

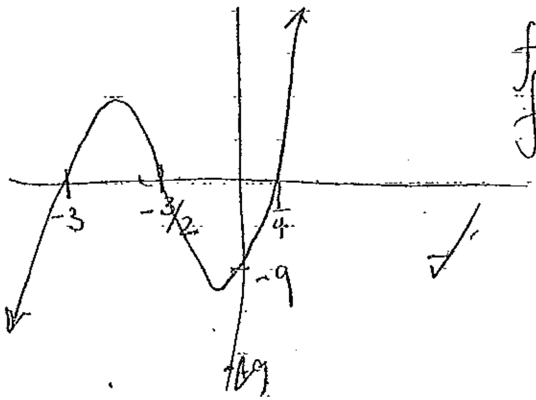
$8x^2+10x-3=0.$

4 -1

2 3

$(4x-1)(2x+3)=0.$

$\therefore\ g(x)=(x+3)(4x-1)(2x+3).$



$f(x)=8x^3+34x^2+27x-9$

$f'(x)=24x^2+68x+27$

Shape, intercepts marked

(too many people put turning point on y int)

$\therefore\ x_1=1-\frac{-4}{17}$

$=1.24$ (2dp)

$f(1)=3-7$

$=-4$

$f'(x)=3-7(2x-1)^{-1}$

$f'(x)=3-7\times-1\times2(2x-1)$

$=3+14(2x-1)^{-1}$

$f'(1)=3+14$

$=17.$

Q 11

(a) Prove $4^n > 2n+1$, $n \geq 1$

1/ Prove true for $n = 1$

$4^1 > 2(1)+1$
$4 > 3$
true
$\therefore$ true for $n = 1$

2/ Assume true for $n = k$

ie assume $4^k > 2k+1$

3/ Prove true for $n = k+1$
ie Prove
$4^{k+1} > 2(k+1)+1$

$4^{k+1} > 2k+3$

Method 1 Prove RHS − LHS > 0

$RHS - LHS = 4(4^k) - (2k+3)$

$= 4(2k+1) - (2k+3)$ (using assumption)

$= 8k + 4 - 2k - 3$
$= 6k+1$
> 0 since $k \geq 1$

$\therefore$ Proven true by Mathematical Induction

PTO FOR an alternative Proof

OR
$4^k > 2k+1$

Prove
$4^{k+1} > 2(k+1)+1$

ie $4(4^k) > 2k+3$ (1)

Starting with assumption

$4^k > 2k+1$

$\times 4$
$4^k \times 4^1 > (2k+1) \times 4$

$4^{k+1} > 8k+4$ (2)

comparing RHSs OF (1) + (2)

$8k+4 > 2k+3$
since $2k > -1$, $k \geq 1$

$\therefore 4^{k+1} > 8k+4 > 2k+3$

$\therefore 4^{k+1} > 2k+3$

$\therefore$ Proven true by MI

(b)

(i) $x^2 = 4ay$

$y = \frac{x^2}{4a}$

$\frac{dy}{dx} = \frac{2x}{4a}$

$= \frac{x}{2a}$

$\therefore$ at $P(2ap, ap^2)$ $\quad M_T = \frac{2ap}{2a}$

$= p$

Must show in detail

Eq'n of tangent is

$y - y_1 = m(x - x_1) \quad \left\{ \begin{array}{l} x_1 = 2ap \\ y_1 = ap^2 \\ m = p \end{array} \right.$

$y - ap^2 = p(x - 2ap)$

$y - ap^2 = px - 2ap^2$

$y - px + ap^2 = 0$

(ii) Chord PQ has eq'n

$y - \frac{(p+q)x}{2} + apq = 0$

Focal chord $\therefore$ passes thru' focus ie $(0, a)$

$a - 0 + apq = 0$

$\frac{apq}{a} = -\frac{a}{a}$

$\therefore pq = -1$ as required

(b) (iii) tangent at P is $y - px + ap^2 = 0$ ①

similarly the tangent at Q is $y - qx + aq^2 = 0$ ②

Solve ① + ② simultaneously

② − ①

$-qx + px + aq^2 - ap^2 = 0$

$(p - q)x = a(p^2 - q^2)$

$\cancel{(p-q)}x = a(p+q)\cancel{(p-q)}$

$\therefore x = a(p+q)$ ③

Subst ③ into ①

$y = px - ap^2$

$= p(a(p+q)) - ap^2$

$= \cancel{ap^2} + apq - \cancel{ap^2}$

$= apq$

note: need find y only

$\therefore$ Tangents meet at $(a(p+q), apq)$.

Since $pq = -1$ $\quad y = -a$ ie they meet on the directrix

Also since $pq = -1$ the gradients of the tangents must be negative reciprocals of each other ie tangents are perpendicular

(c) i)

$$\frac{d}{dx}\sqrt{4-\sqrt{x}}$$

$$= \frac{d}{dx}\left(4 - x^{\frac{1}{2}}\right)^{\frac{1}{2}}$$

$$= \frac{1}{2}\left(4 - x^{\frac{1}{2}}\right)^{-\frac{1}{2}} \times -\frac{1}{2}x^{-\frac{1}{2}}$$

$$= -\frac{1}{4}\left(\frac{1}{\sqrt{4-\sqrt{x}} \times \sqrt{x}}\right)$$

$$= \frac{-1}{4\sqrt{x(4-\sqrt{x})}}$$

(ii) from (i) create required integral

(WRITE Result from part (i) along 1 line)

$$\frac{d}{dx}\left(\sqrt{4-\sqrt{x}}\right) = \frac{-1}{4\sqrt{x(4-\sqrt{x}}}$$

$\times -4$: $$-4\frac{d}{dx}\sqrt{4-\sqrt{x}} = \frac{1}{\sqrt{x(4-\sqrt{x})}}$$

$\times 3$: $$-12\frac{d}{dx}\sqrt{4-\sqrt{x}} = \frac{3}{\sqrt{x(4-\sqrt{x})}}$$

now integrate BOTH sides w.r.t. x

$$\int\left[-12\frac{d}{dx}\sqrt{4-\sqrt{x}}\right]dx = \int\frac{3}{\sqrt{x(4-\sqrt{x})}}\,dx$$

(c)(ii)(ctd)

ie $$\int\frac{3}{\sqrt{x(4-\sqrt{x}}}\,dx = -12\int \cancel{\frac{d}{dx}\left(\sqrt{4-\sqrt{x}}\right)dx}$$

$$\therefore \int_1^4 \frac{3}{\sqrt{x(4-\sqrt{x}}} = -12\left[\sqrt{4-\sqrt{x}}\right]_1^4$$

$$= -12\left[(\sqrt{2}) - (\sqrt{3})\right]$$

$$= -12(\sqrt{2}-\sqrt{3})$$

(d) $$\sum_{r=2}^{n} 3^r = \frac{9(3^{n-1}-1)}{2} \quad \text{for } n \geq 2$$

RTP

ie $$3^2 + 3^3 + 3^4 + \dots + 3^n = \frac{9}{2}\left(3^{n-1}-1\right)$$

1, Prove true for $n=2$

$$LHS = 3^2 = 9 \qquad RHS = \frac{9}{2}\left(3^{2-1}-1\right) = \frac{9}{2}(2) = 9$$

$= 9$

$= LHS$

$\therefore$ true for $n=2$

2/ Assume true for $n = k$

$$\boxed{3^2 + 3^3 + 3^4 + \dots + 3^k} = 9\left(\frac{3^{k-1} - 1}{2}\right)$$

3/ Prove true for $n = k+1$

ie Prove $\boxed{3^2 + 3^3 + 3^4 + \dots + 3^k} + 3^{k+1} = \frac{9(3^k - 1)}{2}$

$$LHS = \boxed{3^2 + 3^3 + 3^4 + \dots + 3^k} + 3^{k+1}$$

$$= 9\left(\frac{3^{k-1} - 1}{2}\right) + 3^{k+1} \quad \text{(using assumption)}$$

(expanding out then factorising may be easiest way ☺)

$$= \frac{9 \times 3^{k-1}}{2} - \frac{9}{2} + 3^{k+1}$$

$$= \frac{\cancel{9}^{3}}{2} \times \frac{3^k}{\cancel{3}} - \frac{9}{2} + 3(3^k)$$

$$= \frac{3}{2}(3^k) - \frac{9}{2} + 3(3^k)$$

$$= \frac{9}{2}(3^k) - \frac{9}{2}$$

$$= \frac{9}{2}(3^k - 1)$$

$$= RHS$$

note: $3^{k-1} = \frac{3^k}{3}$

note: $\frac{3}{2} + 3 = 4\frac{1}{2} = \frac{9}{2}$

$\therefore$ Proven true by Mathematical Induction

Q12) A) $\displaystyle\int_1^2 \frac{3x}{\sqrt[3]{3x+1}}\,dx$

let $u = 3x+1$

$\frac{du}{dx} = 3$

$\frac{du}{3} = dx$

$\frac{u-1}{3} = x$

$x = 1$, $u = 4$

$x = 2$, $u = 7$

$$= \int_4^7 \frac{(u-1)}{3u^{1/3}}\,du$$

$$= \frac{1}{3}\int_4^7 u^{2/3} - u^{-1/3}\,du$$

$$= \frac{1}{3}\left[\frac{3u^{5/3}}{5} - \frac{3u^{2/3}}{2}\right]_4^7$$

$$= \frac{1}{3}\left[\frac{3}{5} \times 7^{5/3} - \frac{3(7)^{2/3}}{2} - \left(\frac{3(4)^{5/3}}{5} - \frac{3(4)^{2/3}}{2}\right)\right]$$

$$= \underline{2.54}$$

① Correct substitution

② Correct steps to ~~calcul~~ integrate

③ All calculations correct.

$$y = \sqrt{\frac{1}{x^2-1}} + 1 \implies (y-1)^2 = \frac{1}{x^2-1}$$

$$\frac{1}{(y-1)^2} = x^2 - 1$$

$$x^2 = 1 + \frac{1}{(y-1)^2} dy$$

$$= \pi \int_2^4 x^2 \, dy \quad ✓$$

$$= \pi \int_2^4 1 + \frac{1}{(y-1)^2} dy \quad ✓$$

$$= \pi \int_2^4 1 + (y-1)^{-2} dy$$

$$\pi \left[y + \frac{(y-1)^{-1}}{-1} \right]_2^4 ✓ \implies \pi \left[y - \frac{1}{(y-1)} \right]_2^4$$

$$\pi \left[4 + \frac{-1}{3} - \left(2 - \frac{1}{1}\right) \right]$$

$$\pi \left[2\tfrac{2}{3} \right]$$

$$= \frac{8\pi}{3} u^3 \quad ✓$$

(1) Correct formula

(2) Correct substitution in of x^2 into formula

(3) Correct integration

(4) Correct evaluation
- Evaluation must be of correct integral or similar.

12c) $E(x) = 6x^3 + 5x^2 + 39x - 5a$

$$E\left(\tfrac{1}{2}\right) = \frac{3}{4} + \frac{5}{4} + \frac{39}{2} - 5a$$

$$= \frac{86}{4} - 5a$$

$$= \frac{43}{2} - 5a \quad ✓$$

$$E(a) = 6a^3 + 5a^2 + 39a - 5a \quad ✓$$

$$= 6a^3 + 5a^2 + 34a$$

$$= a(6a^2 + 5a + 34)$$

$a \neq 0$ ✓

$\Delta = 25 - 4 \times 6 \times 34$

$= -ve$

$\therefore$ No solution. ✓