MARCELLIN COLLEGE RANDWICK

YEAR 12

EXTENSION 1

HSC ASSESSMENT TASK 1

2016

STUDENT NAME: ______________________ MARK /55

TEACHER: ______________________

TIME ALLOWED: 90 minutes
WEIGHTING: 30%

Directions:

- Use a separate sheet for each question.
- Show all necessary working. Where more than one mark is allocated to a question, full marks may not be awarded for answers only.
- Marks may not be awarded for careless or badly arranged work.
- Calculators may be used

Section I

5 marks
Use the multiple-choice answer sheet for Questions 1-5

---

1 Using a suitable substitution $\int_0^{\frac{\pi}{2}}(\cos^4 x \times \sin x)dx$ can be written as:

(A) $-\int_0^{\frac{\pi}{2}} u^4 du$

(B) $\int_0^{\frac{\pi}{2}} u^4 du$

(C) $-\int_0^{1} u^4 du$

(D) $\int_0^{1} u^4 du$

---

2 What is the value of $\int_1^2 \frac{1}{\sqrt{4-x^2}}dx$?

(A) $\frac{\pi}{6}$

(B) $\frac{\pi}{4}$

(C) $\frac{\pi}{3}$

(D) $\frac{\pi}{2}$

---

3 A polynomial equation has $f(-2)<0$ and $f(-3)>0$. By halving the interval, it is found that $f(-2\frac{1}{2})<0$. A root of the equation lies between:

(A) $x=-3$ and $x=-2\frac{1}{2}$

(B) $x=-2\frac{1}{2}$ and $x=-2$

(C) $x=-2\frac{1}{2}$ and $x=-2\frac{1}{4}$

(D) $x=-2\frac{3}{4}$ and $x=-2\frac{1}{2}$

4 The polynomial $P(x)=x^3+2x+k$ has $(x-2)$ as a factor.
What is the value of k?

(A) -12
(B) -10
(C) 10
(D) 12

---

5 What is the solution to the equation $|x-2|=2x+1$?

(A) $x=-3$
(B) $x=-\frac{1}{3}$
(C) $x=\frac{1}{3}$
(D) $x=3$

End of Section 1

Section 2

50 marks

Attempt Questions 6 – 9

Answer each question on a SEPARATE page.

Question 6 *(15 marks)* **Marks**

(a) Solve the inequality $\frac{2x+3}{x-2}<1$ **2**

(b) (i) Show that $\frac{1+\cos 2x}{\sin 2x}=\cot x$ **2**

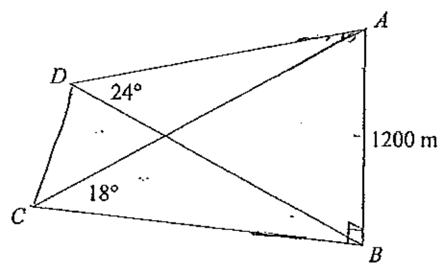
(ii) Hence show the exact value of $\cot 15^\circ$ is $2+\sqrt{3}$. **1**

(c) The point $C(-3,8)$ divides the interval AB externally in the ratio $k:1$. Find the value of k if A is the point $(6,-4)$ and B is the point $(0,4)$. **2**

(d) Use Newton's method to find a second approximation to the positive root of $x-2\sin x=0$. Take $x=1.8$ as the first approximation. **2**

(e) Express $2\sin x-\cos x$ in the form $A\sin(x-\alpha)$ for $0^\circ\le\alpha\le 90^\circ$
Hence solve the equation $2\sin x-\cos x=1$ for $0^\circ\le x\le 360^\circ$ **3**

(f) A mountain AB is 1200 metres above sea level. Daniel is on a boat with a bearing of 140° to the mountain and angle of elevation to the top of the mountain of 24°. Chloe is on a boat with a bearing of 110° to the mountain and angle of elevation to the top of the mountain of 18°.



What is the distance between Daniel and Chloe? Answer to the nearest metre. **3**

Question 7 *(12 marks)* **Marks**

(a) Use the principle of mathematical induction to prove that for all positive integers of n that $5^n \geq 1+4n$. **3**

(b) Prove by mathematical induction that $5^n + 12n - 1$ is divisible by 16 for all positive integers. **3**

(c) Use the substitution $u = \log_e x$ to evaluate $\int \frac{\log_e x}{x}\,dx$. **2**

(d) Use the substitution $x = \frac{1}{4}\tan\theta$ to evaluate $\int \frac{1}{1+16x^2}\,dx$. **2**

(e) Find $\int \frac{1}{\sqrt{25-4x^2}}\,dx$ **2**

Question 8 *(13 marks)* **Marks**

(a) Find the exact value of $\int_{-1}^{1} \sqrt{4-x^2}\,dx$, using the substitution $x = 2\sin\theta$. **3**

(b) Differentiate $\sin^{-1}(\log_e x)$. **1**

(c) Evaluate $\lim_{x\to 0} \frac{3\sin 7x}{5x}$ **2**

(d) Consider the function $y = \frac{1}{2}\cos^{-1}(x-1)$.

(i) Find the domain and range of the function. **2**

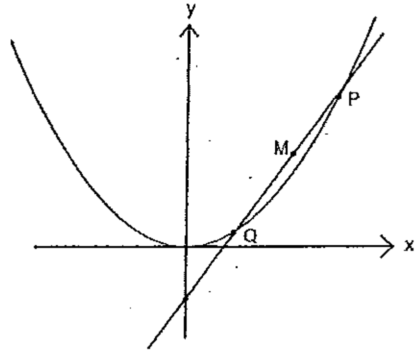
(ii) Hence sketch the graph showing clearly the coordinates of the end points. **1**

(e) (i) Determine the vertical asymptotes for $y = \frac{x+1}{x^2-16}$ **2**

(ii) Hence sketch the curve $y = \frac{x+1}{x^2-16}$ **2**

Question 9 *(10 marks)* **Marks**

(a)



$P\left(2ap,\, ap^2\right)$ and $Q\left(2aq, aq^2\right)$ are two points on the parabola $x^2 = 4ay$.

(i) Find the gradient of the chord PQ. **2**

(ii) Hence show the equation of the chord PQ is $y - \frac{1}{2}(p+q)x + apq = 0$. **2**

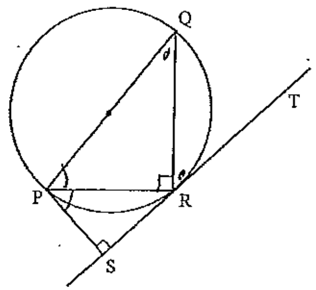
(iii) P and Q move so that the chord PQ passes through the point $(0, -a)$.
Show that $pq = -1$. **1**

(iv) Given that M is the midpoint of PQ with coordinates
$\left[a(p+q), \frac{1}{2}a\left(p^2+q^2\right)\right]$ (DO NOT PROVE THIS).
Find the locus of M. **2**

Question 8 *continued* **Marks**

(b) In the diagram PQ is a diameter of the circle. TS is a tangent to the circle at R and PS is the interval from P perpendicular to TS.

Copy or trace the diagram into your examination booklet.
Prove that PR bisects $\angle QPS$. **3**



NOT TO SCALE

End of Examination

MARCELLIN COLLEGE

2016

Year 12

HSC Assessment Task 1

Half Yearly Examination

Name ______________________

Mathematics Extension 1

Multiple Choice Answer Sheet

Completely fill the oval representing the best response.

1. A ○ B ○ C ○ D ●
2. A ○ B ○ C ● D ○
3. A ● B ○ C ○ D ○
4. A ● B ○ C ○ D ○
5. A ○ B ○ C ● D ○

Start here.

6a) $\frac{2x+3}{x-2} < 1$ $x \neq 2$

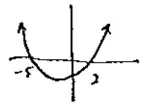
$(2x+3)(x-2) < (x-2)^2$

$(2x+3)(x-2) - (x-2)^2 < 0$

$(x-2)(2x+3-(x-2)) < 0$

$(x-2)(x+5) < 0$

$-5 < x < 2$



6b) $\frac{1+\cos 2x}{\sin 2x} = \cot x$

LHS $= \frac{1+\cos^2 x - \sin^2 x}{2\sin x\cos x}$

$= \frac{2-2\sin^2 x}{2\sin x\cos x} = \frac{1-\sin^2 x}{\sin x\cos x}$

$= \frac{\cos^{\cancel{2}} x}{\sin x\cancel{\cos x}}$

$= \frac{\cos x}{\sin x}$

$= \cot x = $ RHS

$\therefore \frac{1+\cos 2x}{\sin 2x} = \cot x.$

II) $\therefore \cot 15 = \frac{1+\cos 30}{\sin 30} = \frac{1+\frac{\sqrt{3}}{2}}{\frac{1}{2}} = 2+\sqrt{3}$ (as required)

c) $k:-1$

$(6,-4)$ / $B(0,4)$

$x = \frac{mx_2 + nx_1}{m+n}$

$= \frac{0 - 6}{k-1} = -3$ $\quad y = \frac{4k+4}{k-1} = 8$

$-6 = -3k + 3$ $\quad 4(k+1) = 8k - 8$

$3k = 9$ $\quad k+1 = 2k - 2$

$\boxed{\therefore k = 3}$ $\quad k = 3$

d) $x - 2\sin x = 0$

$\frac{d}{dx} = 1 - 2\cos x$

$x_2 = 1.8 - \frac{f(1.8)}{f'(1.8)}$

$\doteq 1.901$

$= 1.90$ (2 dp.)

e) $2\sin x - \cos x$

$A\sin(x-\alpha)$

$= A(\sin x\cos\alpha - \sin\alpha\cos x)$

$A = \sqrt{2^2+1^2}$

$= \sqrt{5}$

$\sqrt{5}\cos\alpha = 2$

$\sqrt{5}\sin\alpha = 1$ $\quad \therefore \alpha = 26^\circ 34'$

$\therefore 2\sin x - \cos x = \sqrt{5}\sin(x - 26^\circ 34')$

P.T.O

e) cont.

$2\sin x - \cos x = 1$

$\sqrt{5}\sin(x - 26^\circ 34') = 1$

$\therefore x - 26^\circ 34' = 26^\circ 34',\ 153^\circ 66'$

$\therefore x = 53^\circ 8'$ & 180°

f) $DB = \frac{\sin 66 \times 1200}{\sin 24}$

$CB = \frac{\sin 72 \times 1200}{\sin 18}$

$DC^2 = CB^2 + DB^2 - 2CB.DB\cos 30$

$DC = \sqrt{3{,}663{,}157.421}$

$\therefore DC = 1{,}913.94\ M$

$\therefore DC = 1{,}914\ M$ (Nearest M)

You may ask for an extra Writing Booklet if you need more space

Start here.

7(a) $5^n \geq 1+4n$

Prove true for $n=1$

$5^1 \geq 1+4(1)$

$5 \geq 1+4$ ✓ $\therefore$ True for $n=1$

$5 \geq 5$

Assume true for $n=k$

$5^k \geq 1+4k$

Prove true for $n=k+1$

$5^{k+1} \geq 1+4(k+1)$

$5.5^k \geq 1+4k+4$ ✓

$5.5^k \geq 5+4k$ ✓ $5^k \geq 1+4k$ (from above)

$5(1+4k) \geq 5+4k$

$\cancel{5}+20k \geq \cancel{5}+4k$ ✓

$16k \geq 0$

$\therefore$ statement is true for all integers $n \geq 1$ (proved by mathematical inductio

b) $5^n+12n-1$ $\vdots 16$

Prove true for $n=1$

$5^1+12(1)-1$ ✓

$=5+12-1$

$=16$ Which is $/16$

$\therefore$ True for $n=1$

Assume true for $n=k$

$5^k+12k-1=16P$ $\therefore 5^k=16P-12k+1$

Prove true for $n=k+1$

$5^{k+1}+12(k+1)-1=16Q$

$5.5^k+12k+11=16Q$ ✓

$5(16P-12k+1)+12k+11=16Q$ (from above)

$80P-60k+5+12k+11=16Q$ ✓

$80P-48k+16=16Q$

$16(5P-3k+1)=16Q$

which is $/16$

$\therefore$ Statement is ~~true~~ true for all integers $n \geq 1$ (proved by mathematical induction)

c) $u = \ln x$

$\frac{du}{dx} = \frac{1}{x}$

$du = \frac{dx}{x}$

~~Whan~~

$$\int \frac{\ln x}{x}\,dx$$

$$= \int u\,du$$

$$= \frac{u^2}{2} + c$$

$$= \frac{(\ln x)^2}{2} + c$$

d) $$\int \frac{1}{1+16x^2}\,dx$$

Let $x = \frac{1}{4}\tan\theta$ $\quad \theta = \tan^{-1} 4x$

$dx = \frac{1}{4}\sec^2\theta\,d\theta$

$$= \int \frac{1}{1+\tan^2 x}\,\frac{1}{4}\sec^2\theta\,d\theta$$

$$= \frac{1}{4}\int \frac{1}{\sec^2\theta}\,\sec^2\theta\,d\theta$$

$$= \frac{1}{4}\int d\theta$$

$$= \frac{1}{4}\theta + c$$

$$= \frac{1}{4}\tan^{-1} 4x + c$$

e) $$\int \frac{1}{\sqrt{25-4x^2}}\,dx$$

Let $x = \frac{5}{2}\sin\theta$ $\quad \theta = \sin^{-1}\frac{2x}{5}$

$dx = \frac{5}{2}\cos\theta\,d\theta$

$$= \int \frac{1}{\sqrt{25-25\sin^2\theta}}\cdot\frac{5}{2}\cos\theta\,d\theta$$

$$= \int \frac{1}{\sqrt{25\cos^2\theta}}\,\frac{5}{2}\cos\theta\,d\theta$$

$$= \int \frac{1}{5\cos\theta}\,\frac{5}{2}\cos\theta\,d\theta$$

$$= \int \frac{1}{2}\,d\theta$$

$$= \frac{1}{2}\theta + c$$

$$= \frac{1}{2}\sin^{-1}\frac{2x}{5} + c$$

8. a) $\int_{-1}^{1} \sqrt{4-x^2}\,dx$

$x = 2\sin\theta$ When $x=1,\ \theta = \frac{\pi}{6}$

$dx = 2\cos\theta\, d\theta$ $x=-1,\ \theta = -\frac{\pi}{6}$

$$= \int_{-\pi/6}^{\pi/6} \sqrt{4-4\sin^2\theta}\cdot 2\cos\theta\, d\theta$$

$$= \int_{-\pi/6}^{\pi/6} 4\cos^2\theta\, d\theta$$

$$\left(\begin{array}{l} 2\cos^2\theta - 1 = \cos 2\theta \\ 2\cos^2\theta = 1+\cos 2\theta \\ \cos^2\theta = \frac{1}{2}+\frac{1}{2}\cos 2\theta \end{array}\right)$$

$$= 2\int_0^{\pi/6} 4\left(\tfrac{1}{2}+\tfrac{1}{2}\cos 2\theta\right) d\theta$$

$$= 4\int_0^{\pi/6} (1+\cos 2\theta)\, d\theta$$

$$= 4\left[\theta + \tfrac{1}{2}\sin 2\theta\right]_0^{\pi/6}$$

$$= 4\left(\tfrac{\pi}{6} + \tfrac{1}{2}\sin\tfrac{\pi}{3} - (0+0)\right)$$

$$= 4\left(\tfrac{\pi}{6} + \tfrac{\sqrt{3}}{4}\right)$$

$$= \frac{2\pi}{3} + \sqrt{3}$$

b) $\frac{d}{dx}\sin^{-1}(\log_e x) = \frac{1}{\sqrt{1-(\log_e x)^2}} \times \frac{1}{x}$

$$= \frac{1}{x\sqrt{1-(\log_e x)^2}}$$

c) $\lim_{x\to 0} \frac{3\sin 7x}{5x} = \lim_{x\to 0} \frac{3\times 7}{5}\frac{\sin 7x}{7x}$

$$= \frac{21}{5}\lim_{x\to 0}\frac{\sin 7x}{7x}$$

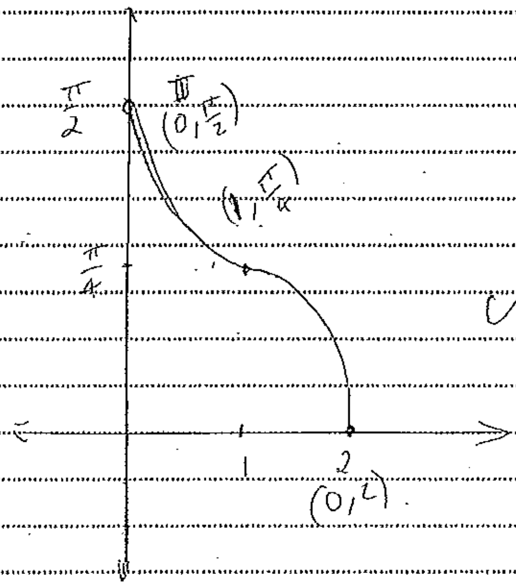
$$= \frac{21}{5}\times 1 = \frac{21}{5}$$

d I) $y = \frac{1}{2}\cos^{-1}(x-1)$

D: $0 \le x \le 2$

R: $0 \le y \le \frac{\pi}{2}$

II)



e I) $\frac{x+1}{x^2-16}$

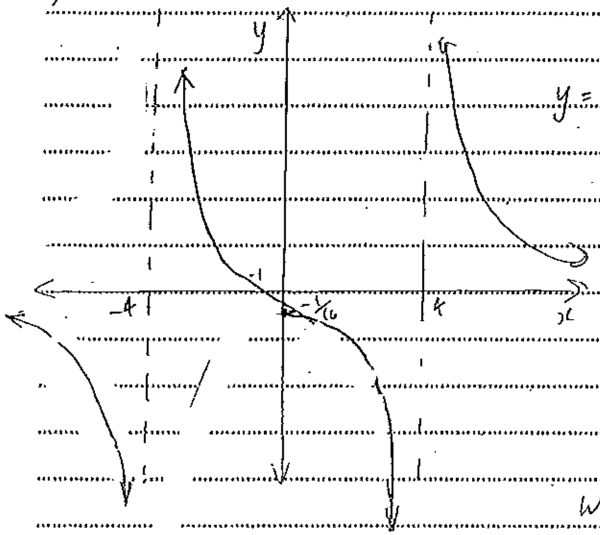
$x \neq \pm 4$

as 0 cannot be on the denominator

$x^2-16 \neq 0 \to x^2 \neq 16$

$\therefore x \neq \pm 4$

II)



Horiz. asympt.

$y = \dfrac{\frac{x}{x^2} + \frac{1}{x^2}}{\frac{x^2}{x^2} - \frac{16}{x^2}} \rightarrow \dfrac{0+0}{1-0}$

$= 0$ as $x \to \infty$

at $x = 0$, $y = -\dfrac{1}{16}$

When $y = 0$,

$\dfrac{x+1}{x^2-16} = 0 \Rightarrow x = -1$

as $x \to \infty$, $y = \dfrac{\frac{1}{x} + \frac{1}{x^2}}{1 - \frac{16}{x^2}} \longrightarrow 0^+$

$x \to -\infty$, $y = \dfrac{\frac{1}{x} + \frac{1}{x^2}}{1 - \frac{16}{x^2}} \longrightarrow 0^-$

$x \to -4^-$, $y = \dfrac{x+1}{(x+4)(x-4)} \longrightarrow -\infty$

$x \to -4^+$, " $\longrightarrow \infty$

$x \to 4^-$, " $\longrightarrow -\infty$

$x \to 4^+$, " $\longrightarrow \infty$

You may ask for an extra Writing Booklet if you need more space

Start here.

9a) I) $\dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{ap^2 - aq^2}{2ap - 2aq} = \dfrac{\cancel{pa}(p-q)(p+q)}{2(p-q)}$

$= \dfrac{p+q}{2}$

II) $(y - ap^2) = \dfrac{p+q}{2}(x - 2ap)$

$2y - \cancel{2ap^2} = (p+q)x - \cancel{2ap^2} - 2apq$

$(p+q)x - 2y - 2apq = 0$ $\div -2$

$y - \frac{1}{2}(p+q)x + apq = 0$ (as required

III) $(0, -a)$ satisfies the equation $y - \frac{1}{2}(p+q)x + apq = 0$

So, $\therefore -a - 0 + apq = 0$

$apq = -a$

$\therefore pq = -1$

IV) $x = a(p+q)$ $\quad y = \frac{1}{2}a(p^2+q^2)$

$p+q = \frac{x}{a}$ ①

① into ②

② $y = \frac{1}{2}a(p+q)^2 - 2pq$

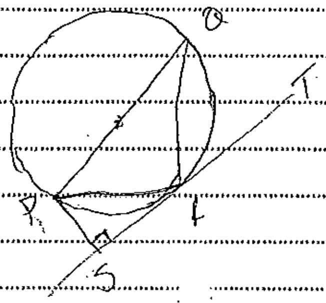
$y = \frac{1}{2}a\left(\frac{x^2}{a^2}\right) + 2$ $\quad pq = -1$ (from above)

$y = \frac{1}{2}\frac{x^2}{a} + 2$

$2(y-2) = \frac{x}{a}$

$\therefore x^2 = 2a(y-2)$

b)



$\angle PRQ = 90^\circ$
($\angle$ at the centre is twice the $\angle$ at the circumference standing on the same arc)

$\angle QPR = \angle QRT$ ($\angle$ at the tangent with a chord is equal to the $\angle$ in the alternate segment)

$\therefore \angle PRS = 180 - 90 - \angle QRT$ (supp. $\angle$'s)
$= 90 - \angle QPR$

$\angle RPS = 180 - 90 - \angle PRS$
$= \cancel{90} - (\cancel{90} - \angle QPR)$
$\therefore \angle RPS = \angle QPR$

$\therefore$ the line PR bisects $\angle QPS$