MARCELLIN COLLEGE RANDWICK

YEAR 12

EXTENSION 1

HSC ASSESSMENT TASK 2

2016

STUDENT NAME: ____________________ **MARK** /32

TEACHER: ____________________

TIME ALLOWED: 45 minutes
WEIGHTING: 20%

Directions:

- Use a separate sheet for each question.
- Show all necessary working. Where more than one mark is allocated to a question, full marks may not be awarded for answers only.
- Marks may not be awarded for careless or badly arranged work.
- Calculators may be used

Attempt Questions 1 – 2

Answer each question on a SEPARATE page.

Question 1 ***(15 marks)*** **Marks**

(a) A population of birds on an island changes according to the equation $\frac{dP}{dt} = k(P - 3500)$. Initially there were 40 individuals and after 5 years the population had increased to 125.

i. Verify that $P = 3500 - Ae^{kt}$ satisfies the above equation, where P is the population, t is the time in years and, A and k are constants. **1**

ii. Find the value of A and find the value of k to 3 significant figures. **3**

iii. Sketch the function $P = 3500 - Ae^{kt}$ for $t \geq 0$ showing any intercepts and asymptotes. **2**

(b) i. Show that $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$. **2**

ii. If $v^2 = 24 - 6x - 3x^2$, find the acceleration of the particle at the particle's greatest displacement from the origin. **3**

(c) A particle is undergoing simple harmonic motion such that its displacement x centimetres from the origin after t seconds is given by:

$$x + 2 = 4\sin\left(2t + \frac{\pi}{3}\right).$$

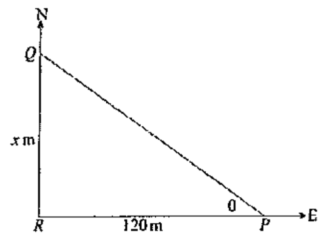
i. Between which two positions is the particle oscillating? **1**

ii. At what time does the particle first move through the origin in the positive direction? **3**

Answer each question on a SEPARATE page.

Question 2 *(17 marks)* **Marks**

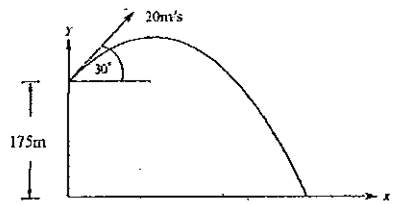
(a)



An observer stands at P, 120 *metres* East of R. A second person is at Q, x metres due North of R and continues to move North. Let angle $RPQ = \theta$. Suppose θ is changing at 0.2 radians/minute.

Find the rate at which x is changing when $x = 90$ metres. **3**

(b) A man who is standing on top of a vertical cliff throws a stone into the air at an angle θ to the horizontal. The top of the cliff is 175 metres above a flat sea.



The initial velocity of the stone is $20m/s$. Acceleration due to gravity is $-10m/s^2$ and the angle of projection of the stone to the horizontal 30°.

i. Show that the parametric equations are given by

$$x = 10\sqrt{3}t \text{ and } y = 10t - 5t^2 + 175$$ **2**

ii. Find the time it takes for the stone to hit the water. **3**

iii. Find the speed at which the stone hits the water. **2**

Continued Question 2 *Marks*

(c) A particle moves according to the equation $v^2 = 16 - 4x^2$ where x its position in metres is and v is its velocity in metres per second.

i. Prove the particle has acceleration in the form $a = -n^2x$ where n is a constant. **1**

ii. Find the amplitude of the motion, explaining how this is found. **2**

iii. Use integration to find an expression for x as a function of time t, beginning with $v = \sqrt{16 - 4x^2}$. **4**

END OF EXAMINATION

Q1

1) a) $\frac{DP}{dt} = k(P-3500)$ @ $t=0$, $P=40$
@ $t=5$, $P=125$

I) $P = 3500 - Ae^{kt}$ $\therefore -Ae^{kt} = P - 3500$

$\frac{dP}{dt} = -kAe^{kt}$

$= k(-Ae^{kt})$

$= k(P-3500)$

II) $P = 3500 - Ae^{kt}$

$40 = 3500 - Ae^{0}$

$\therefore Ae^{0} = 3460$

$\therefore A = 3460$

$125 = 3500 - 3460e^{5k}$

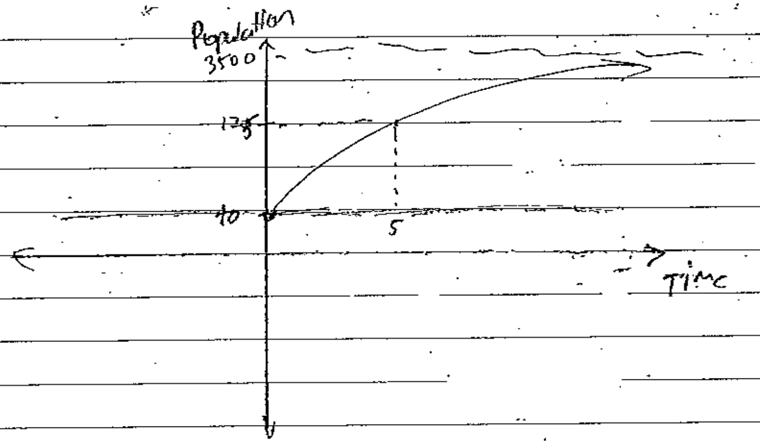
$\therefore e^{5k} = \frac{3375}{3460}$

$\therefore k = \frac{\ln\left|\frac{3375}{3460}\right|}{5}$

$\therefore k = -0.004974652\ldots$

$= -0.00497$ (3 sig. figs)

III)



b I) $\ddot{x} = \frac{d^2x}{dt^2}$

$= \frac{dV}{dt}$

$= \frac{dv}{dx} \times \frac{dx}{dt}$

$= \frac{dv}{dx} \times V$

$= \frac{d}{dv}\left(\frac{1}{2}v^2\right) \times \frac{dv}{dx}$

$= \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

II) $V^2 = 24 - 6x - 3x^2$

$\therefore \frac{1}{2}v^2 = 12 - 3x - \frac{3x^2}{2}$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -3 - 3x$

$= \ddot{x}$

$= -3 - 3a$

$= -3 - 6$

$= -9$

$= -3 + 3(4)$

$= 9$

$\therefore$ Max acceleration $= 9\ m/s^2$

$V^2 = -(3x^2 + 6x - 24)$

$= 3(x^2 + 2x - 8)$

$= 3(x+4)(x-2)$

$\therefore x = 2, -4$

Max acceleration occurs at $x = 2, -4$

c) on New Sheet.

Q 1 part 2.

c) $x = 4\sin(2t + \frac{\pi}{3}) - 2$

I) Centre of motion $= -2$

Amplitude $= 4$

∴ oscillating between -6 & 2

II) $x = 4\sin(2t + \frac{\pi}{3}) - 2$ let $2t + \frac{\pi}{3} = \alpha$

$0 = 4\sin\alpha - 2$

$4\sin\alpha = 2$

$\sin\alpha = \frac{1}{2}$

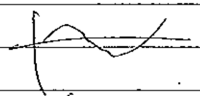
∴ $\alpha = \frac{\pi}{6}, \frac{5\pi}{6}, \dots$

$2t + \frac{\pi}{3} = \frac{\pi}{6}, \frac{5\pi}{6}$

$2t = \cancel{-\frac{\pi}{6}}, \frac{\pi}{2}$

∴ $t = \cancel{-\frac{\pi}{12}}, \frac{\pi}{4}$ $t \geq 0$

∴ first moves through the origin at $t = \frac{\pi}{4}$, however is negative



∴ Moves through the origin positively at $t = -\frac{\pi}{12} + \pi$

~~∴ $t = \frac{23\pi}{12}$ seconds~~

∴ $t = \frac{11\pi}{12}$ seconds.

Q2

2a) $\frac{d\theta}{dt} = 0.2$ rads/min. $\frac{dx}{dt} = ?$

$\tan\theta = \frac{x}{120}$ $\frac{dx}{dt} = \frac{d\theta}{dt} \times \frac{dx}{d\theta}$

∴ $x = 120\tan\theta$

$\frac{dx}{d\theta} = 120\sec^2\theta$ $\frac{dx}{dt} = 0.2 \times 120\sec^2\theta$

$= 24\sec^2\theta$

when $x = 90$ $= 24\sec^2(\tan^{-1}\frac{90}{120})$

$90 = 120\tan\theta$ $= 24\left(\frac{1}{0.8^2}\right)$

∴ $\tan^{-1}\frac{90}{120} = \theta$

$= \boxed{\frac{24}{0.64} = 37.5 \text{ rads/min}}$

b) $\ddot{x} = 0$

$\dot{x} = 0t + c$

@ $t = 0$, $\dot{x} = V\cos\alpha$

$= 20\cos30$ ∴ $c = 20\cos30$

∴ $\dot{x} = 20\cos30$

∴ $x = 20t\cos30 + c$

@ $t = 0$, $x = 0$ ∴ $c = 0$

∴ $x = 20t\cos30$

$= t\left(\frac{\sqrt{3}}{2}\right) \times 20$

$= 10\sqrt{3}t$ (as required)

$\ddot{y} = -10$

$\dot{y} = -10t + c$

@ $t = 0$, $\dot{y} = 20\sin30$

∴ $c = 10$

∴ $\dot{y} = -10t + 10$

$y = -5t^2 + 10t + c$

@ $t = 0$, $y = 175$

∴ $c = 175$

∴ $y = -5t^2 + 10t + 175$

$= 10t - 5t^2 + 175$

(as required)

P.T.O

II) Stone hits water when $y = 0$

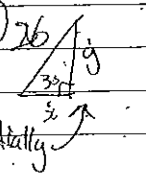
$10t - 5t^2 + 175 = 0$

$t^2 - 2t - 35 = 0$

$(t-7)(t+5) = 0 \quad (t \geq 0)$

$t = 7s$

III) 2b

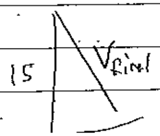


CFPA. projectile

$\therefore \dot x = 10\sqrt{3}$ (no change in $\dot x$ as the ~~pb~~ projectile flies)

initial $\dot y = 10$ m/s after ~~an~~ 2.5 s, $\dot y = -15$ m/s

& $\ddot y = -10$ m/s^2



Final: $15^2 + (10\sqrt{3})^2 = V^2$ (pythag.)

$\therefore V_{final} = 22.9$ m/s (1 dec. pl.)

Q2 part 2.

2e) $v^2 = 16 - 4x^2$

I) $\frac{1}{2}v^2 = 8 - 2x^2$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \ddot x = -4x$

$\therefore$ In the form $-n^2x$, where $n = 2$

II) $n = 2 \quad \therefore$ period $= \pi$

since $\ddot x$ is in the form n^2x, the motion must be in SHM.

Max Velocity occurs at $x = 0$

$v^2 = 16 - 0 \quad \therefore v = \pm 4$

Max displacement occurs at ~~a~~ $V = 0$

$0 = 16 - 4x^2$

$16 = 4x^2$

$4 = x^2 \quad \therefore x = \pm 2$

$\therefore$ amplitude of motion $= 2$ (as ~~max~~ max displacement occurs when $V = 0$)

c) III) $v = \sqrt{16-4x^2}$

$$\frac{dx}{dt} = \sqrt{16-4x^2}$$

$$\frac{dt}{dx} = \sqrt{16-4x^2}$$

$$\int \frac{dt}{dx}\,dx = \int \sqrt{16-4x^2}\,dx$$

Let $x = 2\sin\theta$ $\quad \theta = \sin^{-1}\frac{x}{2}$

$dx = 2\cos\theta\,d\theta$

$$t = \int \sqrt{16(1-\sin^2\theta)} \cdot 2\cos\theta\,d\theta$$

$$= \int 4\cos\theta \cdot 2\cos\theta\,d\theta$$

$2\cos^2\theta - 1 = \cos 2\theta$

$\cos^2\theta = \cos 2\theta + 1$

$$= 4\int (1+\cos 2\theta)\,d\theta$$

$$= 4\left[\theta + \frac{1}{2}\sin 2\theta\right] + c$$

$$= 4\sin^{-1}\frac{x}{2} + 4\sin\theta\cos\theta + c$$

$$= 4\sin\frac{x}{2} + 4\sin\theta\sqrt{1-\sin^2\theta} + c$$

$$= 4\sin\frac{x}{2} + 2x\sqrt{1-\frac{x^2}{4}} + c$$