NSW INDEPENDENT SCHOOLS

2011
Higher School Certificate
Trial Examination

Mathematics
Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
- Board approved calculators may be used.
- A table of standard integrals is provided
- All necessary working should be shown in every question
- Write your student number and/or name at the top of every page

Total marks – 84

- Attempt Questions 1 – 7
- All questions are of equal value

This paper MUST NOT be removed from the examination room

STUDENT NUMBER/NAME: ....

HSC STANDARD INTEGRAL SHEET

$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0,\ \text{if } n < 0$

$\int \frac{1}{x}\,dx = \ln x,\ x > 0$

$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\ a \neq 0$

$\int \cos ax\,dx = \frac{1}{a}\sin ax,\ a \neq 0$

$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\ a \neq 0$

$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\ a \neq 0$

$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax,\ a \neq 0$

$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ a \neq 0$

$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\ a > 0,\ -a < x < a$

$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right),\ x > a > 0$

$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$

NOTE : $\ln x = \log_e x,\ x > 0$

Student name / number ....

Marks

Question 1 **Begin a new booklet**

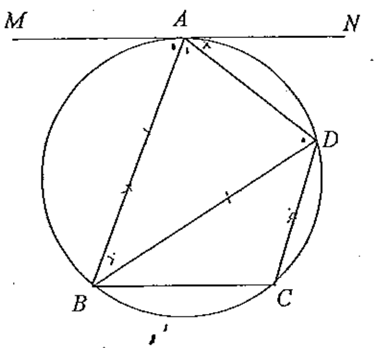
(a) Find $\lim_{x \to 0} \frac{\sin 5x}{2x}$. **2**

(b) Consider the points $A(-3, 2)$ and $B(6, -4)$. Find the coordinates of the point $P(x, y)$ that divides the interval AB internally in the ratio $2:1$. **2**

(c) Solve the inequality $\frac{2}{x+1} < 1$. **2**

(d) Use the substitution $t = \tan\frac{x}{2}$ to show that $\frac{1-\cos x}{\sin x} = \tan\frac{x}{2}$. **2**

(e)



$ABCD$ is a cyclic quadrilateral in which $AB = DB$. The tangent at A to the circle through A, B, C and D is parallel to BC.

(i) Copy the diagram showing this information.
(ii) Show that CD is parallel to BA, giving reasons. **4**

Student name / number ....

Marks

Question 2 **Begin a new booklet**

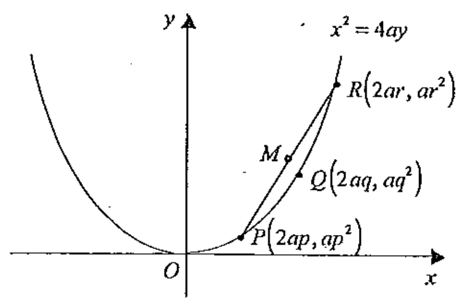
(a) Find the values of k such that $(x-2)$ is a factor of the polynomial $P(x) = x^3 - 2x^2 + kx + k^2$. **2**

(b) Find correct to the nearest degree the acute angle between the lines $y = 3x + 1$ and $x + y - 5 = 0$. **2**

(c) Find the number of ways in which 2 consonants and 3 vowels can be chosen from the letters of the word EQUATION. **2**

(d) Evaluate $\int_0^1 \frac{1}{\sqrt{4-x^2}}\,dx$. **2**

(e)



$Q(2aq, aq^2)$ is a fixed point on the parabola $x^2 = 4ay$ where $a > 0$.
$P(2ap, ap^2)$ and $R(2ar, ar^2)$ are variable points which move on the parabola such that the chord PR is parallel to the tangent to the parabola at Q.

(i) Show that $p + r = 2q$. **2**

(ii) Find in terms of a and q the equation of the locus of the midpoint M of PR. State any restrictions on this locus. **2**

Student name / number ....

Marks

Question 3 **Begin a new booklet**

(a) Find $\dfrac{d}{dx} x\cos^{-1}x$. **2**

(b) Find $\int \sin^2 3x \, dx$. **2**

(c) Consider the function $f(x)=\dfrac{x^2}{x^2+1}$.

(i) Show that the curve $y=f(x)$ has a minimum turning point at $(0,0)$. **2**

(ii) Sketch the curve $y=f(x)$ showing clearly the equation of the horizontal asymptote. **2**

(d) Use the method of Mathematical Induction to show that for all positive integers $n \geq 1$, **4**

$$\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\ldots+\frac{n}{(n+1)!}=1-\frac{1}{(n+1)!}.$$

Question 4 **Begin a new booklet**

(a) Four fair dice are thrown together. Find in simplest exact form

(i) the probability that all four scores are different. **2**

(ii) the probability that there is at most one 6. **2**

(b) Use the substitution $u=x^2+1$ to evaluate $\displaystyle\int_1^7 \frac{x}{\left(1+x^2\right)^2}\,dx$. **4**

(c) At time t years after the start of the year 2000, the number of individuals in a population is given by $N=80+Ae^{0 \cdot 1t}$ for some constant $A>0$.

(i) Show that $\dfrac{dN}{dt}=0 \cdot 1(N-80)$. **1**

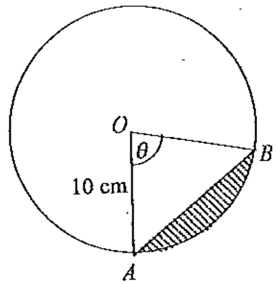
(ii) If there were 100 individuals in the population at the start of the year 2000, find the year in which the population size is expected to reach 200. **3**

Student name / number ....

Marks

Question 5 **Begin a new booklet**

(a)



A circle has centre O and radius 10 cm. OA is a fixed radius of the circle. OB is a variable radius which moves so that $\angle AOB=\theta$ is increasing at a constant rate of $0 \cdot 01$ radians per second. The minor segment of the circle cut off by the chord AB has area S cm^2. **3**

Find the rate at which S is increasing when $\theta=\frac{\pi}{3}$.

(b) A particle is moving in a straight line. At time t seconds it has displacement x metres from a fixed point O in the line, velocity v ms^{-1} given by $v=\dfrac{1}{x+1}$ and acceleration a ms^{-2}. Initially the particle is at O.

(i) Express a as a function of x. **1**

(ii) Express x as a function of t. **3**

(c) A particle is performing Simple Harmonic Motion in a straight line. At time t seconds its displacement from a fixed point O in the line is x metres given by $x=1+\sqrt{2}\cos\left(3t-\frac{\pi}{4}\right)$.

(i) Show by differentiation that $\ddot{x}=-9(x-1)$. **1**

(ii) Find the time taken for the particle to first pass through the point O. **2**

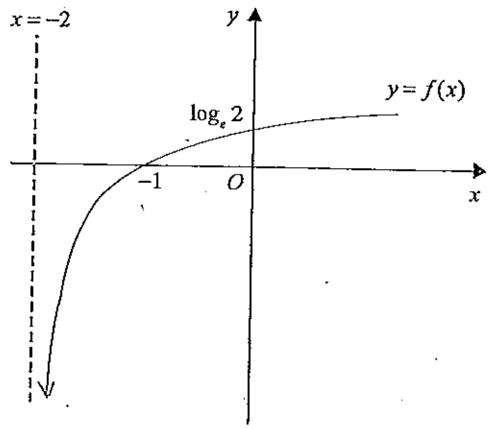
(iii) Find in simplest exact form the average speed of the particle during one complete oscillation of its motion. **2**

Student name / number ....

Marks

Question 6 **Begin a new booklet**

(a)



The diagram shows the graph of the function $f(x)=\log_e(x+2)$.

(i) Copy the diagram and on it draw the graph of the inverse function $f^{-1}(x)$ showing the intercepts on the axes and the equation of the asymptote. **2**

(ii) Show that the x coordinates of the points of intersection of the curves $y=f(x)$ and $y=f^{-1}(x)$ satisfy the equation $e^x-x-2=0$. **2**

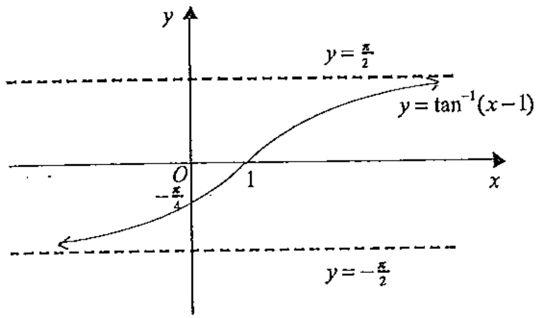
(iii) Show that the equation $e^x-x-2=0$ has a root α such that $1<\alpha<2$. **2**

(iv) Use one application of Newton's method with an initial approximation $\alpha_0=1\cdot 2$ to find the next approximation for the value of α, giving your answer correct to one decimal place. **2**

Student name / number ....

Marks

(b)



The region in the first quadrant bounded by the curve $y=\tan^{-1}(x-1)$ and the y-axis between the lines $y=0$ and $y=\frac{\pi}{4}$ is rotated through one complete revolution about the y-axis.

(i) Show that the volume V of the solid of revolution is given by **1**

$$V=\pi\int_0^{\frac{\pi}{4}}(1+\tan y)^2\,dy.$$

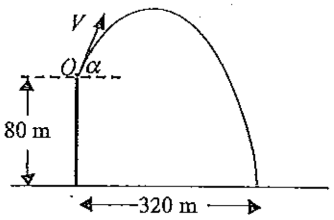
(ii) Hence find the value of V in simplest exact form. **3**

Student name / number ....

Marks

Question 7 **Begin a new booklet**

(a)



A particle is projected with speed V ms^{-1} at an angle α above the horizontal from a point O at the top edge of a vertical cliff which is 80 m above horizontal ground. The particle moves in a vertical plane under gravity where the acceleration due to gravity is 10 ms^{-2}. It reaches its greatest height after 3 seconds and hits the ground at a horizontal distance 320 m from the foot of the cliff.

The horizontal and vertical displacements, x and y metres respectively, of the particle from the point O after t seconds are given by $x = Vt\cos\alpha$ and $y = -5t^2 + Vt\sin\alpha$. (Do NOT prove these results.)

(i) Show that $V\sin\alpha = 30$. **1**

(ii) Show that the particle hits the ground after 8 seconds. **2**

(iii) Show that $V\cos\alpha = 40$. **1**

(iv) Hence find the exact value of V and the value of α correct to the nearest minute. **2**

(v) Find the time after projection when the direction of motion of the particle first makes an angle of 45° below the horizontal. **2**

(b)(i) Write down the binomial expansion of $(1+x)^{2n}$ in ascending powers of x and differentiate both sides with respect to x. **1**

(ii) Hence show that $2\,{}^{2n}C_2 + 4\,{}^{2n}C_4 + 6\,{}^{2n}C_6 + \ldots + 2n\,{}^{2n}C_{2n} = n\,2^{2n-1}$ for $n \geq 1$. **3**

Question 1

a. Outcomes assessed : H5

Marking Guidelines

Criteria	Marks
• rearranges to enable use of a known limiting value	1
• uses this limiting value to complete the evaluation	1

Answer

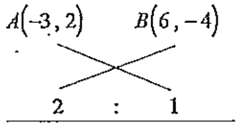
$$\lim_{x\to 0}\frac{\sin 5x}{2x}=\frac{5}{2}.\lim_{x\to 0}\frac{\sin 5x}{5x}=\frac{5}{2}.1=\frac{5}{2}$$

b. Outcomes assessed : H5

Marking Guidelines

Criteria	Marks
• finds x coordinate of P	1
• finds y coordinate of P	1

Answer



$$P\left(\frac{2\times 6+1\times(-3)}{2+1},\ \frac{2\times(-4)+1\times 2}{2+1}\right)\qquad \therefore P(3,-2)$$

c. Outcomes assessed : PE3

Marking Guidelines

Criteria	Marks
• either considers cases $x<-1$, $x>-1$ when multiplying by $(x+1)$, or multiplies by $(x+1)^2$	
• obtains one inequality for x	1
• obtains the second inequality for x, and indicates union rather than intersection	1

Answer

$$\frac{2}{x+1}<1$$

$2(x+1)<(x+1)^2$ and $x\neq -1$

$0<(x+1)^2-2(x+1)$

$0<(x+1)(x+1-2)$

$0<(x+1)(x-1)$

$\therefore 0<x^2-1$ and $x\neq -1$

$x^2>1$

$\therefore x<-1$ or $x>1$

1. d. Outcomes assessed : H5

Marking Guidelines

Criteria	Marks
• substitutes expressions for $\sin x$, $\cos x$ in terms of t	1
• simplifies to obtain result	1

Answer

$$t=\tan\tfrac{x}{2}\ \Rightarrow\quad \frac{1-\cos x}{\sin x}=\left(1-\frac{1-t^2}{1+t^2}\right)\div\frac{2t}{1+t^2}$$
$$=\frac{(1+t^2)-(1-t^2)}{1+t^2}\times\frac{1+t^2}{2t}$$
$$=\frac{2t^2}{2t}$$
$$=t$$
$$=\tan\tfrac{x}{2}$$

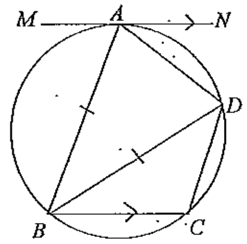
e. Outcomes assessed : PE2, PE3

Marking Guidelines

Criteria	Marks
ii • quotes alternate segment theorem to show $\angle MAB=\angle ADB$	1
• uses equal alt. $\angle$'s between ∥ lines and equal $\angle$'s in the isos. Δ to deduce $\angle CBA=\angle DAB$	1
• uses property of cyclic quad. to deduce $\angle DAB$, $\angle BCD$ supplementary	1
• establishes supplementary cointerior $\angle$'s to deduce $CD \parallel BA$	1

Answer

i.



ii.

$\angle CBA=\angle MAB$ (*Alternate $\angle$'s equal since $MN \parallel BC$*)

$\angle MAB=\angle ADB$ (*$\angle$ between a tangent and a chord drawn to the point of contact is equal to the $\angle$ subtended by that chord in the alternate segment*)

$\angle ADB=\angle DAB$ (*in ΔABD, $\angle$'s opp. equal sides are equal*)

$\therefore \angle CBA+\angle BCD=\angle DAB+\angle BCD$ (*since $\angle CBA=\angle DAB$*)

But $\angle DAB+\angle BCD=180°$ (*opp. $\angle$'s of cyclic quad. ABCD are supplementary*)

$\therefore \angle CBA+\angle BCD=180°$

$\therefore CD \parallel BA$ (*supplementary cointerior $\angle$'s on transversal BC*)

Question 2

a. Outcomes assessed : PE3

Marking Guidelines

Criteria	Marks
• uses the factor theorem to obtain quadratic equation in k	1
• solves this equation	1

Answer

$P(x)=x^3-2x^2+kx+k^2$

$(x-2)$ is a factor of $P(x) \Rightarrow P(2)=0$

$\therefore 8-8+2k+k^2=0$

$k(k+2)=0$

$k=0$ or $k=-2$

b. Outcomes assessed : H5

Marking Guidelines

Criteria	Marks
• uses gradients to find numerical expression for $\tan\theta$	1
• evaluates $\tan\theta$ and hence θ	1

Answer

$y=3x+1$ has gradient 3

$x+y-5=0$ has gradient -1

$\therefore \tan\theta=\left|\frac{3-(-1)}{1+3\times(-1)}\right| = 2$

$\therefore \theta \approx 63°$

c. Outcomes assessed : PE3

Marking Guidelines

Criteria	Marks
• writes numerical expression in terms of binomial coefficients	1
• evaluates	1

Answer

Consonants Q T N Vowels E U A I O $\therefore\ {}^3C_2 \times {}^5C_3 = 3\times 10 = 30$ ways

d. Outcomes assessed : HE4

Marking Guidelines

Criteria	Marks
• finds primitive	1
• evaluates by substituting limits and simplifying	1

Answer

$$\int_0^1 \frac{1}{\sqrt{4-x^2}}\,dx=\left[\sin^{-1}\frac{x}{2}\right]_0^1=\sin^{-1}\tfrac{1}{2}-\sin^{-1}0=\tfrac{\pi}{6}$$

2. e. Outcomes assessed : PE3, PE4

Marking Guidelines

Criteria	Marks
i • shows by differentiation that tangent at Q has gradient q	1
• equates gradient of chord and tangent and simplifies	1
ii • deduces $x=2aq$ at M	1
• notes the restriction on the locus given by $y \ge aq^2$	1

Answer

i. $x=2at \Rightarrow \frac{dx}{dt}=2a$

$y=at^2 \Rightarrow \frac{dy}{dt}=2at$

$\therefore \frac{dy}{dx}=\frac{dy}{dt}\div\frac{dx}{dt}=t$

Hence tangent at $Q(2aq, aq^2)$ and $\parallel$ chord PR each have gradient q.

$\therefore \frac{a(p^2-r^2)}{2a(p-r)}=q$

$\frac{(p-r)(p+r)}{2(p-r)}=q$

$\therefore p+r=2q$

ii. At M, $x=\frac{1}{2}.2a(p+r)$, $y=\frac{1}{2}a(p^2+r^2)$

Hence locus of M has equation $x=2aq$, where $y\ge aq^2$ since M lies vertically above Q.

Question 3

a. Outcomes assessed : HE4

Marking Guidelines

Criteria	Marks
• knows the derivative of $\cos^{-1}x$	1
• applies the product rule	1

Answer

$$\frac{d}{dx}x\cos^{-1}x=1.\cos^{-1}x+x.\frac{-1}{\sqrt{1-x^2}}=\cos^{-1}x-\frac{x}{\sqrt{1-x^2}}$$

b. Outcomes assessed : H5

Marking Guidelines

Criteria	Marks
• uses appropriate double-angle identity	1
• finds primitive	1

Answer

$$\int \sin^2 3x\,dx = \tfrac{1}{2}\int(1-\cos 6x)\,dx$$
$$=\tfrac{1}{2}(x-\tfrac{1}{6}\sin 6x)+c$$
$$=\tfrac{1}{12}(6x-\sin 6x)+c$$

3. c. Outcomes assessed : H6

Marking Guidelines

Criteria	Marks
i • shows $f'(0)=0$	1
• applies test to determine that this stationary point is a minimum turning point	1
ii • sketches curve with correct shape and position	1
• gives equation of horizontal asymptote	1

Answer

i. $f(x)=\dfrac{x^2}{x^2+1}=1-\dfrac{1}{x^2+1}$

$f'(x)=\dfrac{2x}{(x^2+1)^2}$

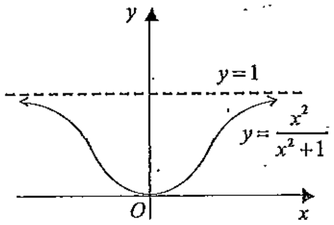
$\therefore f(0)=0$ and $f'(0)=0$

Also $f'(x)<0$ for $x<0$

and $f'(x)>0$ for $x>0$

$\therefore (0,0)$ is a minimum turning point.

ii. $f(x)$ is an even function, and $f(x)\to 1$ as $x\to\infty$.



d. Outcomes assessed : HE2

Marking Guidelines

Criteria	Marks
• defines an appropriate sequence of statements $S(n)$ and shows that the first is true	1
• writes the LHS of $S(k+1)$ in terms of the RHS of $S(k)$, conditional on the truth of $S(k)$	1
• simplifies the resulting expression to produce the RHS of $S(k+1)$	1
• completes the process of Mathematical Induction	1

Answer

For $n=1,2,3,\ldots$, consider the sequence of statements $S(n):\ \dfrac{1}{2!}+\dfrac{2}{3!}+\dfrac{3}{4!}+\ldots+\dfrac{n}{(n+1)!}=1-\dfrac{1}{(n+1)!}$

Consider $S(1)$: $\quad LHS=\dfrac{1}{2!}=\dfrac{1}{2} \qquad RHS=1-\dfrac{1}{2!}=1-\dfrac{1}{2}=\dfrac{1}{2} \qquad \therefore S(1)$ is true.

If $S(k)$ is true: $\quad \dfrac{1}{2!}+\dfrac{2}{3!}+\dfrac{3}{4!}+\ldots+\dfrac{k}{(k+1)!}=1-\dfrac{1}{(k+1)!} \qquad **$

Consider $S(k+1)$:

$$\begin{aligned} LHS&=\left\{\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\ldots+\frac{k}{(k+1)!}\right\}+\frac{k+1}{(k+2)!}\\ &=1-\frac{1}{(k+1)!}+\frac{k+1}{(k+2)!} \qquad \text{if } S(k) \text{ is true, using } **\\ &=1-\frac{(k+2)-(k+1)}{(k+2)!}\\ &=1-\frac{1}{(k+2)!}\\ &=RHS \end{aligned}$$

Hence if $S(k)$ is true, then $S(k+1)$ is true. But $S(1)$ is true, hence $S(2)$ is true and then $S(3)$ is true and so on. $\therefore$ by Mathematical Induction, $S(n)$ is true for all positive integers $n\ge 1$.

Question 4

a. Outcomes assessed : HE3

Marking Guidelines

Criteria	Marks
i • counts the ordered selections of 4 numbers chosen from 6 different numbers	1
• divides by the number of possible outcomes and simplifies	1
ii • recognises the binomial distribution and writes an expression for the probability	1
• evaluates this expression	1

Answer

i. $\dfrac{6.5.4.3}{6^4}=\dfrac{5}{18}$

ii. ${}^4C_0\left(\tfrac{1}{6}\right)^0\left(\tfrac{5}{6}\right)^4+{}^4C_1\left(\tfrac{1}{6}\right)^1\left(\tfrac{5}{6}\right)^3=\dfrac{5^4+4.5^3}{6^4}=\dfrac{125}{144}$

b. Outcomes assessed : HE6

Marking Guidelines

Criteria	Marks
• writes du in terms of dx and converts x limits to u limits	1
• writes definite integral in terms of u	1
• finds the primitive	1
• evaluates	1

Answer

$u=x^2+1$

$du=2x\,dx$

$x=1\Rightarrow u=2$

$x=7\Rightarrow u=50$

$$\begin{aligned} \int_1^7\frac{x}{\left(1+x^2\right)^2}\,dx&=\frac{1}{2}\int_2^{50}\frac{1}{u^2}\,du\\ &=\frac{-1}{2}\left[\frac{1}{u}\right]_2^{50}\\ &=\tfrac{-1}{2}\left(\tfrac{1}{50}-\tfrac{1}{2}\right)\\ &=\tfrac{6}{25} \end{aligned}$$

c. Outcomes assessed : HE3

Marking Guidelines

Criteria	Marks
i • shows result by differentiation	1
ii • evaluates A	1
• writes equation for t and obtains t as a logarithm	1
• evaluates t as a decimal and states the required year	1

Answer

i. $N=80+Ae^{0\cdot 1t}$

$\dfrac{dN}{dt}=0\cdot 1Ae^{0\cdot 1t}=0\cdot 1\left(N-80\right)$

ii. $N=100$ when $t=0\ \Rightarrow\ A=20$

$200=80+20e^{0\cdot 1t}$

$e^{0\cdot 1t}=6$

$0\cdot 1t=\ln 6$

$t=10\ln 6$

$\therefore t\approx 17\cdot 92$

Hence population reaches 200 during 2017.

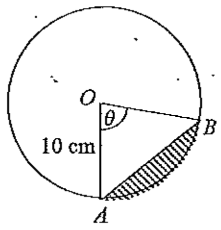
Question 5

a. Outcomes assessed : HE5

Marking Guidelines

Criteria	Marks
• expresses S in terms of θ	1
• finds $\frac{dS}{dt}$ in terms of θ and $\frac{d\theta}{dt}$	1
• calculates the required rate of increase of the area	1

Answer



$S=\frac{1}{2}.10^2\left(\theta-\sin\theta\right)$

$\frac{dS}{dt}=50\left(1-\cos\theta\right)\frac{d\theta}{dt}$

$=50\left(1-\frac{1}{2}\right)\times 0\cdot 01$

$=0\cdot 25$

Area increases at $0\cdot 25\ \text{cm}^2/\text{s}$

b. Outcomes assessed : HE5, HE7

Marking Guidelines

Criteria	Marks
i • uses either $a=v\frac{dv}{dx}$ or $a=\frac{d}{dx}\left(\frac{1}{2}v^2\right)$ to write a in terms of x.	1
ii • integrates to find t as a function of x	1
• rearranges to find $(x+1)^2$ as a function of t	1
• chooses the appropriate square root to find x as a function of t	1

Answer

i. $v=\frac{1}{x+1}$

$a=v\frac{dv}{dx}$

$=\frac{1}{x+1}\cdot\left(\frac{-1}{(x+1)^2}\right)$

$=\frac{-1}{(x+1)^3}$

ii. $\frac{dx}{dt}=\frac{1}{x+1}$

$\frac{dt}{dx}=x+1$

$t=\frac{1}{2}(x+1)^2+c$

$\left.\begin{matrix}t=0\\x=0\end{matrix}\right]\Rightarrow c=-\frac{1}{2}$

$\therefore\ t=\frac{1}{2}(x+1)^2-\frac{1}{2}$

$2t+1=(x+1)^2$

$x+1=\pm\sqrt{2t+1}$

But $t=0\Rightarrow x=0$

$\therefore\ x=-1+\sqrt{2t+1}$

c. Outcomes assessed : HE3

Marking Guidelines

Criteria	Marks
i • shows result by differentiation	1
ii • writes the value of $\cos\left(3t-\frac{\pi}{4}\right)$	1
• finds the smallest positive solution for t	1
iii • expresses the average speed in terms of the amplitude and period of the motion	1
• evaluates this speed in simplest exact form	1

Answer

i. $x=1+\sqrt{2}\cos\left(3t-\frac{\pi}{4}\right)$ $\quad\therefore\ \dot{x}=-3\sqrt{2}\sin\left(3t-\frac{\pi}{4}\right)$

$\ddot{x}=-9\sqrt{2}\cos\left(3t-\frac{\pi}{4}\right)$ $\quad\therefore\ddot{x}=-9(x-1)$

ii. $x=1+\sqrt{2}\cos\left(3t-\frac{\pi}{4}\right)$

When $x=0$,

$\cos\left(3t-\frac{\pi}{4}\right)=-\frac{1}{\sqrt{2}}$

$3t-\frac{\pi}{4}=\frac{3\pi}{4},\ \frac{5\pi}{4},\ \frac{11\pi}{4},\ldots$

$3t=\pi,\ \frac{3\pi}{2},3\pi,\ldots$

Hence first passes through O after $\frac{\pi}{3}$ seconds.

iii. If A is the amplitude and T is the period of the motion, then the average speed during 1 complete oscillation is given by

$\frac{4A}{T}=\frac{4\sqrt{2}}{\left(\frac{2\pi}{3}\right)}=\frac{6\sqrt{2}}{\pi}\ \text{ms}^{-1}$

Question 6

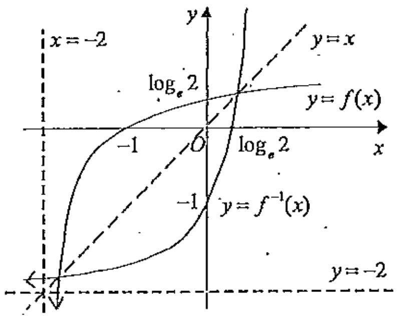
a. Outcomes assessed : PE3, HE4

Marking Guidelines

Criteria	Marks
i • reflects given curve in $y=x$ with intersections on this line and asymptote $y=-2$	1
• shows intercepts on coordinate axes	1
ii • writes logarithmic equation for x obtained from $f(x)=x$	1
• rearranges to find required equation for x	1
iii • shows $g(x)=e^x-x-2$ changes sign between $x=1$ and $x=2$	1
• notes the continuity of $g(x)$ to deduce the required result	1
iv • writes expression for next approximation by applying Newton's method	1
• evaluates this next approximation	1

Answer

i.



ii. Curves intersect on the line $y=x$ where

$\log_e(x+2)=x$

$x+2=e^x$

$e^x-x-2=0$

iii. Let $g(x)=e^x-x-2$

Then $g(x)$ is a continuous function,

$g(1)=e-3<0$ and $g(2)=e^2-4>0$.

$\therefore g(x)=0$ for some $\alpha,\ 1<\alpha<2$

iv. $g'(x)=e^x-1$ $\quad\therefore\ \alpha_1=\alpha_0-\frac{g(\alpha_0)}{g'(\alpha_0)}$

$\therefore\alpha_1=1\cdot 2-\frac{e^{1\cdot 2}-3\cdot 2}{e^{1\cdot 2}-1}\approx 1\cdot 1$

b. Outcomes assessed : H8

Marking Guidelines

Criteria	Marks
i • writes x in terms of y to establish definite integral for V.	1
ii • uses appropriate trig. identities to write integrand in convenient form	1
• finds primitive	1
• evaluates in simplest exact form	1

Answer

i. $y=\tan^{-1}(x-1)$

$\tan y = x-1$

$x = 1+\tan y$

$\therefore V=\pi\int_0^{\frac{\pi}{4}}(1+\tan y)^2\,dy$

ii. $V=\pi\int_0^{\frac{\pi}{4}}\left(1+2\tan y+\tan^2 y\right)dy$

$=\pi\int_0^{\frac{\pi}{4}}\left(2\frac{\sin y}{\cos y}+\sec^2 y\right)dy$

$=\pi\left[-2\ln(\cos y)+\tan y\right]_0^{\frac{\pi}{4}}$

$=\pi\left\{-2\left(\ln\frac{1}{\sqrt{2}}-\ln 1\right)+(1-0)\right\}$

$=\pi\left(\ln 2+1\right)$

Question 7

a. Outcomes assessed : HE3

Marking Guidelines

Criteria	Marks
i • writes expression for $\dot{y}$ and substitutes $\dot{y}=0,\ t=3$	1
ii • substitutes $y=-80$ in expression for y to obtain quadratic equation in t	1
• solves this quadratic equation	1
iii • substitutes $x=320,\ t=8$ in expression for x	1
iv • finds V	1
• calculates α	1
v • writes an equation in t using expressions for $\dot{x},\ \dot{y}$	1
• solves to find t	1

Answer

i. $\dot{y}=-10t+V\sin\alpha$ and $\dot{y}=0$ when $t=3$.

$0=-30+V\sin\alpha \qquad \therefore V\sin\alpha=30$

ii. $y=-80\Rightarrow -80=-5t^2+30t$

$\therefore t^2-6t-16=0$ and $t\ge 0$

$(t-8)(t+2)=0 \qquad \therefore t=8$

iii. $x=320$ when $t=8 \Rightarrow 320=8V\cos\alpha$

$\therefore V\cos\alpha=40$

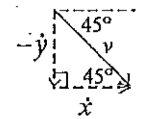
iv. $V^2(\sin^2\alpha+\cos^2\alpha)=30^2+40^2$

$V^2=50^2 \qquad \therefore V=50$

$\frac{V\sin\alpha}{V\cos\alpha}=\frac{30}{40} \Rightarrow \tan\alpha=\frac{3}{4}$

$\therefore \alpha\approx 36°52'$

v.



$\dot{y}=-\dot{x} \Rightarrow -10t+30=-40 \qquad \therefore 10t=70$.

Hence after 7 seconds.

b. Outcomes assessed : HE3

Marking Guidelines

Criteria	Marks
i • writes binomial expansion and differentiates both sides wrt x	1
ii • substitutes $x=1$	1
• substitutes $x=-1$	1
• subtracts to obtain required identity	1

Answer

i. $(1+x)^{2n}\equiv {}^{2n}C_0+{}^{2n}C_1\,x+{}^{2n}C_2\,x^2+{}^{2n}C_3\,x^3+\ldots+{}^{2n}C_r\,x^r+\ldots+{}^{2n}C_{2n}\,x^{2n}$

$2n(1+x)^{2n-1}\equiv 1.{}^{2n}C_1+2.{}^{2n}C_2\,x+3.{}^{2n}C_3\,x^2+\ldots+r.{}^{2n}C_r\,x^{r-1}+\ldots+2n.{}^{2n}C_{2n}\,x^{2n-1}$

ii. Substituting $x=1$, $\quad 2n.2^{2n-1}=1.{}^{2n}C_1+2.{}^{2n}C_2+3.{}^{2n}C_3+\ldots+r.{}^{2n}C_r+\ldots+2n.{}^{2n}C_{2n}$

Substituting $x=-1$, $\quad 0=1.{}^{2n}C_1-2.{}^{2n}C_2+3.{}^{2n}C_3+\ldots-(-1)^r\,r.{}^{2n}C_r+\ldots-2n.{}^{2n}C_{2n}$

By subtraction, $\quad 2n.2^{2n-1}=2\left\{2.{}^{2n}C_2+4.{}^{2n}C_4+6.{}^{2n}C_6+\ldots+2n.{}^{2n}C_{2n}\right\}$

$\therefore 2.{}^{2n}C_2+4.{}^{2n}C_4+6.{}^{2n}C_6+\ldots+2n.{}^{2n}C_{2n}=n.2^{2n-1}$ for $n\ge 1$