OUR LADY OF THE SACRED HEART COLLEGE KENSINGTON

STUDENT – NAME / NUMBER ______________

MATHEMATICS TEACHER ______________

2013

HSC – Half Yearly

Extension 2 Mathematics

Time allowed : 3 hours + 5minutes reading

Assessed Outcomes

- **E3** uses the algebraic and geometric representations of complex numbers
- **E4** uses algebraic manipulation in dealing with polynomials
- **E6** combines algebra and calculus to determine features of a graph
- **E8** applies further techniques of integration including partial fractions

Directions to Candidates

- Section I: 10 marks
- Section 2: 6 Questions – 15 marks each
- Total mark is 100
- Show all working
- READ the questions carefully
- Board approved calculators may be used

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0,\ \text{if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x,\ x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\ a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\ a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\ a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\ a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax,\ a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\ a > 0,\ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln(x+\sqrt{x^2-a^2}),\ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln(x+\sqrt{x^2+a^2})$$

NOTE : $\ln x = \log_e x,\ x > 0$

Section 1:

Worth 10 marks

Each question is worth 1 mark

Fill in the multiple choice sheet for this section.

1. What are the values of real numbers p and q such that $1-i$ is a root of the equation $z^3+pz+q=0$?

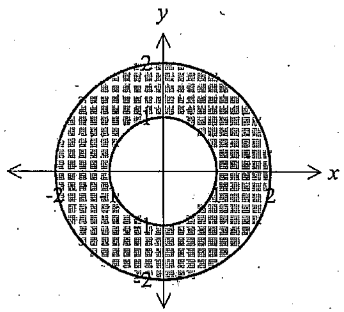
   A. $p=-2$ and $q=-4$

   B. $p=-2$ and $q=4$

   C. $p=2$ and $q=4$

   D. $p=2$ and $q=4$

2. Consider the Argand diagram below.



   Which inequality could define the shaded area?

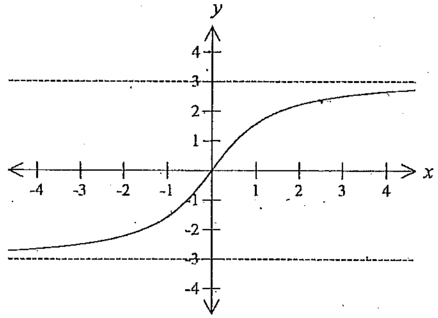
   A. $0\le|z|\le 2$

   B. $1\le|z|\le 2$

   C. $0\le|z-1|\le 2$

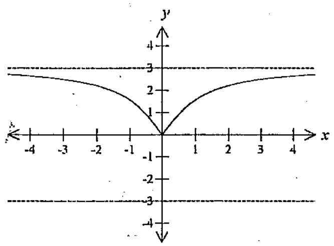
   D. $1\le|z-1|\le 2$

3. The diagram shows the graph of the function $y=f(x)$.

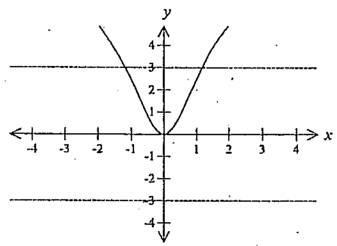


   Which of the following is the graph of $y=\sqrt{f(x)}$?

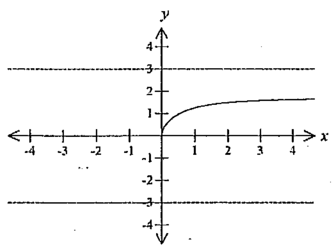
A.



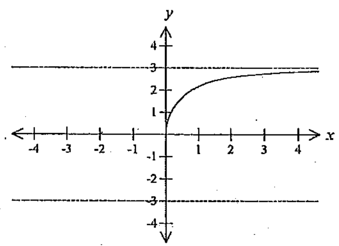
B.



C.



D.



4. The polynomial $P(x) = x^5 - 3x^4 + 4x^3 - 4x^2 + 3x - 1$ has $x = 1$ as a root of multiplicity 3 and $x = i$ as a root. Which of the following expressions is a factorised form of $P(x)$ over the complex numbers?

A. $P(x) = (x+1)^3(x-1)(x+1)$

B. $P(x) = (x-1)^3(x-1)(x+1)$

C. $P(x) = (x+1)^3(x-i)(x+i)$

D. $P(x) = (x-1)^3(x-i)(x+i)$

5. The asymptotes to $y = \dfrac{x^2 - x - 1}{x + 1}$ are:

A. $x = -1$

B. $y = x - 2$ *and* $x = -1$

C. $y = x$ *and* $x = -1$

D. $y = x - 2$ *and* $x = 1$

6. What is the value of $\int_1^3 x(x-2)^5\,dx$? Use the substitution $u = x - 2$.

A. $\frac{1}{7}$

B. $\frac{2}{7}$

C. $\frac{1}{3}$

D. $\frac{2}{3}$

7. Let $z = 3 - i$. What is the value of $\overline{iz}$?

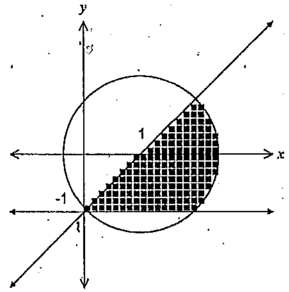
A. $-1-3i$

B. $-1+3i$

C. $1-3i$

D. $1+3i$

8. Consider the Argand diagram below.



Which inequality could define the shaded area?

A. $|z-1| \le \sqrt{2}$ and $0 \le \arg(z-i) \le \frac{\pi}{4}$

B. $|z-1| \le \sqrt{2}$ and $0 \le \arg(z+i) \le \frac{\pi}{4}$

C. $|z-1| \le 1$ and $0 \le \arg(z-i) \le \frac{\pi}{4}$

D. $|z-1| \le 1$ and $0 \le \arg(z+i) \le \frac{\pi}{4}$

9. Which of the following is an expression for $\int \frac{2}{x^2+4x+13}dx$?

A. $\frac{1}{3}\tan^{-1}\frac{(x+2)}{3}+c$

B. $\frac{2}{3}\tan^{-1}\frac{(x+2)}{3}+c$

C. $\frac{1}{9}\tan^{-1}\frac{(x+2)}{9}+c$

D. $\frac{2}{9}\tan^{-1}\frac{(x+2)}{9}+c$

10. Let $I_n = \int_0^{\pi} x^n \sin x dx$, where $0 \le x \le \frac{\pi}{2}$.

Which of the following is the correct expression for I_n?

A. $\pi^n - n(n-1)I_{n-2}$

B. $\pi^n + n(n-1)I_{n-2}$

C. $\pi^n - n(n-2)I_{n-2}$

D. $\pi^n + n(n-2)I_{n-2}$

END OF SECTION 1

Section II:

Worth 90 marks
Six questions 15 marks each
All solutions in booklets
Begin a new booklet for each question

Question 11 (15 marks) **Marks**

(a) Use the substitution $u = x^2$ to calculate $\int_0^{\frac{1}{\sqrt{2}}} \frac{xdx}{\sqrt{1-x^4}}$ **3**

(b) Find $\int_0^{\frac{\pi}{2}} \frac{dx}{2+\cos x}$ **3**

(c) (i) Write $\frac{4}{x^2-1}$ as the sum of two fractions. **2**

(ii) Hence, find $\int \frac{4}{x^2-1}dx$. **2**

(d) Sketch (showing critical points) the graph of $y = x^2 - |x|$. **3**

(e) Show that $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$ **2**

Question 12 on page 9

Question 12. (15 marks) Start a new sheet of writing paper. **Marks**

a. If $z_1 = 3 - 2i$ and $z_2 = 3 - 4i$, find in the form $x + iy$:

i. $z_1 - z_2$ **1**

ii. $(z_2)^2$ **1**

iii. $\frac{z_2}{z_1}$ **1**

iv. $z_1 . \overline{z_2}$ **1**

b. Factorise $x^4 + x^2 - 12$ completely over the field of :

i. Rational numbers. **1**

ii. Real numbers. **1**

iii. Complex Numbers. **1**

c. Find the square roots of $z = 1 + i\sqrt{3}$ in the form $\sqrt{z} = \pm(x + iy)$. **3**

d. For any complex number, z show that $z.\overline{z} = |z|^2$. **2**

e. i. Write $z = \sqrt{3} + i$ in the form $r(\cos\theta + i\sin\theta)$. **1**

ii. Hence, or otherwise, find z^5 in the form $x + iy$. **2**

Question 13 on page 10

Question 13: (15 marks) **Marks**

a. Given that $P(x) = x^4 + x^3 - 3x^2 - 5x - 2$ has a 3 – fold root, find all the roots of $P(x)$ **3**

b. Show that if the polynomial $P(x)$ has a root of multiplicity m , then $P'(x)$ has the root α with multiplicity $(m - 1)$. **2**

c. If α, β, γ are the roots of the cubic equation $x^3 + qx + r = 0$, find the value of $\sum\alpha, \sum\alpha\beta, \sum\alpha\beta\gamma$ in terms of q , r .

Hence prove that $(\beta - \gamma)^2 + (\gamma - \alpha)^2 + (\alpha - \beta)^2 = -6q$ **3**

d. Solve the equation $4x^3 - 24x^2 + 23x + 18 = 0$, given that the roots are in arithmetic progression **3**

e. If $$\frac{2x + 31}{(x-1)^3(x+2)} = \frac{a}{(x-1)} + \frac{b}{(x-1)^2} + \frac{c}{(x-1)^3} + \frac{d}{(x+2)}$$

find the values of a, b, c, d. ~~3~~ 4

Question 14 on page 11

Question 14: (15 marks) **Marks**

a. Sketch the graph of $f(x) = (x-4)(2-x)$ and hence draw separate sketches of the following graphs: 1

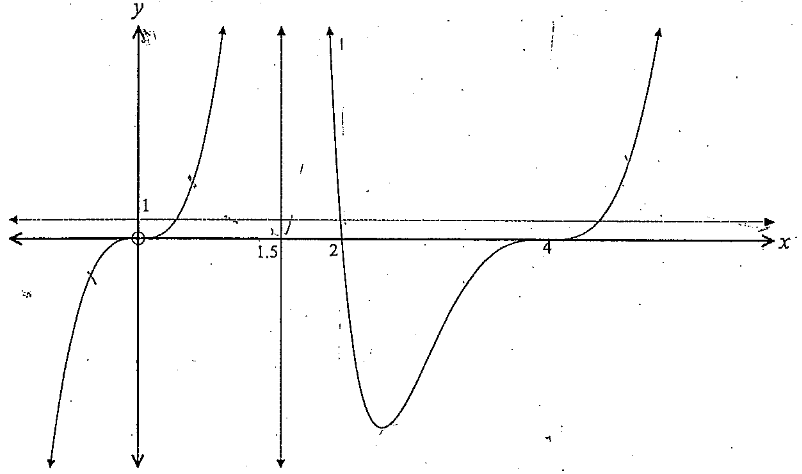
i. $y = \left|\dfrac{1}{f(x)}\right|$ 2

ii. $y = [f(x)]^2$ 1

iii. $y^2 = f(x)$ 2

iv. $y = e^{f(x)}$ 2

b. A sketch of the curve whose equation is $y = f(x)$ is shown below.



Draw neat, half page sketches of:

i. $y = |f(x)|$ 2

ii. $y = (f(x))^2$ 2

iii. $y = \dfrac{1}{f(x)}$ 3

Question 15 on page 12

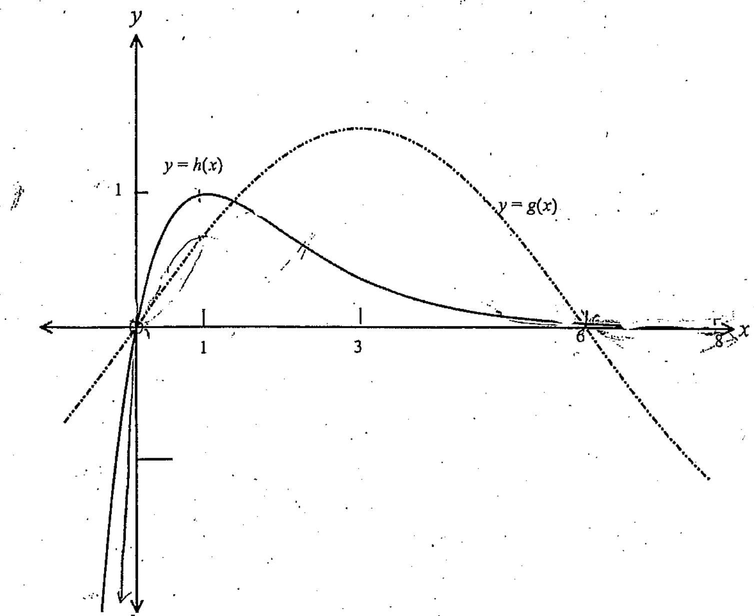
Question 15 (15 marks) Start a new sheet of writing paper. **Marks**

a. Find the fourth roots of $2 + 2\sqrt{3}\, i$. 3

b. The complex number $z = x + iy$ satisfies the relation $(z - \bar{z})^2 + 18(z + \bar{z}) = 36$. Show that the locus of z on the Argand plane is a parabola, and give its focal length and the coordinates of its vertex. 3

c. For the curve with equation $x^2 + 6xy - 4y^2 = 10$, determine the gradient of the tangent at the point (2, 1) on the curve. 3

d. The diagram below shows the graphs of $y = g(x)$ and $y = h(x)$ for the domain, Draw a half page sketch showing the graph of $y = g(x).h(x)$. 4



(e) If $a > b > 0$, show that $1 + \dfrac{b}{a} > 2\sqrt{\dfrac{b}{a}}$. 2

Question 16 on page 13

Question 16: (15 marks) **Marks**

a. i. Show that the recurrence (reduction) formula for

$$I_n = \int tan^n x dx \text{ is } I_n = \frac{1}{n-1}\tan^{n-1} x - I_{n-2}.$$ 3

ii. Hence evaluate $\int_0^{\frac{\pi}{4}} tan^3 x\, dx$ 2

b. i. Show that $\tan(\alpha+\beta+\gamma) = \dfrac{\tan\alpha + \tan\beta + \tan\gamma - \tan\alpha\tan\beta\tan\gamma}{1 - \tan\alpha\tan\beta - \tan\beta\tan\gamma - \tan\gamma\tan\alpha}$. 3

ii. Hence or otherwise, evaluate $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{4} + \tan^{-1}\frac{1}{13}$. 2

c. If α and β are the roots of $x^2 - 2x + 4 = 0$, prove that $\alpha^n - \beta^n = i2^{n+1}\sin\left[\frac{n\pi}{3}\right]$.

Hence find the value of $\alpha^9 - \beta^9$ 5

END OF PAPER

OLSH College Kensington

HSC Half Yearly 2013

Extension 2

Multiple Choice Answer Sheet

MASTER COPY

Name / Number: ____________________

	A	B	C	D
1	A ○	B ●	C ○	D ○ ✓
2	A ○	B ●	C ○	D ○ ✓
3	A ○	B ○	C ●	D ○ ✓
4	A ○	B ○	C ○	D ● ✓
5	A ○	B ●	C ○	D ○ ✓
6	A ○	B ●	C ○	D ○ ✓
7	A ○	B ○	C ● ✓	D ○
8	A ○	B ●	C ○	D ○ ✓
9	A ○	B ●	C ○	D ○ ✓
10	A ○	B ○	C ○	D ● ✓

8/10

11a) $\frac{1}{2}\int_0^{\frac{1}{\sqrt{2}}} \frac{2x\,dx}{\sqrt{1-x^4}}$ let $u=x^2$

$du = 2x\,dx$

$x=\frac{1}{\sqrt{2}}$: $u=\left(\frac{1}{\sqrt{2}}\right)^2$

$= \frac{1}{2}$

$x=0$: $u=0$

$\int_0^{\frac{1}{2}} \frac{du}{\sqrt{1-u^2}}$

$= \left[\sin^{-1} u\right]_0^{\frac{1}{2}}$

$= \sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}(0)$ ✓ ✓

$= \frac{\pi}{6}$ 2

b) $\int_0^{\frac{\pi}{2}} \frac{dx}{2+\cos x}$ let $t = \tan\frac{x}{2}$

$dx = \frac{2}{1+t^2}\,dt$

$\cos x = \frac{1-t^2}{1+t^2}$

$\int_0^1 \frac{\frac{2}{1+t^2}\,dt}{\frac{2(1+t^2)+1-t^2}{\cancel{1+t^2}}}$ $x = \frac{\pi}{2}$: $t = \tan\frac{\pi}{4}$

$= 1$

$x = 0$: $t = 0$

$= \int_0^1 \frac{2}{2+2t^2+1-t^2}\,dt$

$= \int_0^1 \frac{2}{t^2+3}\,dt$ ~~let $u = t^2+3$~~ ~~du~~

$= 2\int_0^1 \frac{1}{t^2+3}\,dt$

3

$= 2\left[\frac{1}{\sqrt{3}}\tan^{-1}\frac{t}{\sqrt{3}}\right]_0^1$

$= 2\left(\frac{1}{\sqrt{3}}\tan^{-1}\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{3}}\tan^{-1}\frac{0}{\sqrt{3}}\right)$

$= 2\left(\frac{1}{\sqrt{3}} \times \frac{\pi}{6}\right) = \frac{2\pi}{6\sqrt{3}} = \frac{\pi}{3\sqrt{3}}$

(11) (c) (i) $\frac{4}{x^2-1} = \frac{4}{(x-1)(x+1)}$

$\frac{4}{(x+1)(x-1)} = \frac{A}{(x+1)} + \frac{B}{(x-1)}$

$4 = A(x-1) + B(x+1)$

let $x = 1$:

$4 = 2B$

$B = 2$ ✓

let $x = -1$

$4 = A(-1-1)$

$A = -2$ ✓

$\frac{4}{x^2+1} = \frac{-2}{(x+1)} + \frac{2}{x-1}$

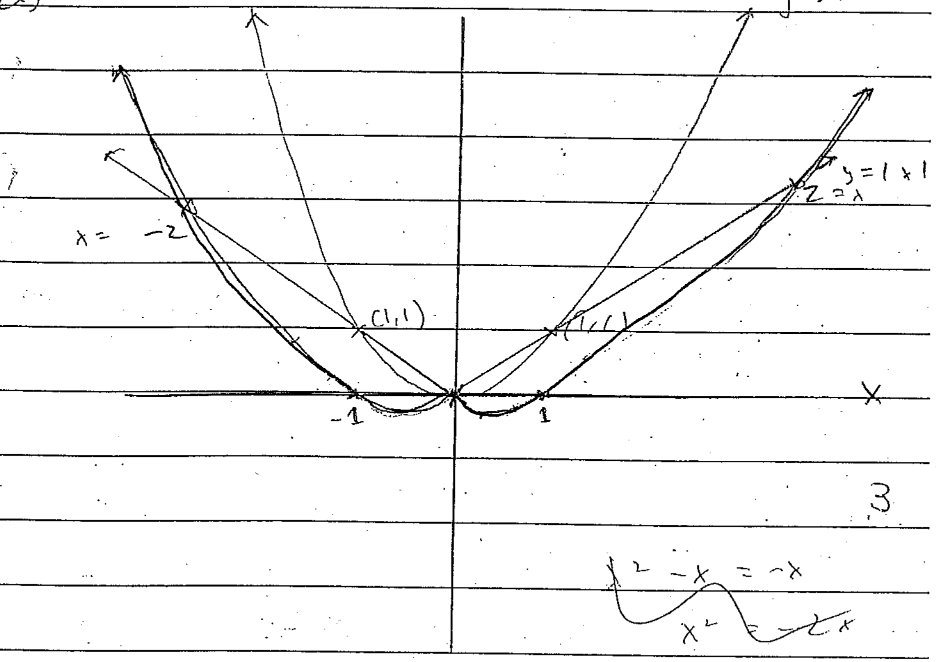
$= \frac{2}{(x-1)} - \frac{2}{(x+1)}$ 2

ii) $\displaystyle\int \frac{2}{(x-1)} - \frac{2}{(x+1)} \, dx$

$= 2\ln|x+1| - 2\ln|x+1| + c$

$= 2\ln\left|\frac{x+1}{x-1}\right| + c$ 2

d)



3

$x^2 - x = -x$

$x^2 = \cancel{-2x}$

e) $\displaystyle\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

LHS:

$\displaystyle\int_0^a f(x)\,dx$ let $\cancel{x} = a - u$ $u = a - x$.

$dx = -du$ $\frac{du}{dx} = -1$

$x = a$: $\cancel{x = a - a - u}$ $du = -1\,dx$

$-\displaystyle\int_a^0 f(a-u)\,du$ $a = a - u$ $-du = dx$

$u = 0$

$x = 0$: $0 = a - u$

$= \displaystyle\int_0^a f(a-u)\,du$ $u = a$

as u is ~~any number~~ a dummy variable for x / interchangeable w/ x

$\therefore = \displaystyle\int_0^a f(a-x)\,dx$ 1

$=$ RHS.

12(a) $z_1 = 3-2i \qquad z_2 = 3-4i$

i) $z_1 - z_2$

$= (3-2i)-(3-4i)$

$= \cancel{3} - 2i - \cancel{3} + 4i$

$= 2i$ 1

ii) $(z_2)^2$

$= (3-4i)(3-4i)$

$= 9 - 12i - 12i + 16i^2$

$= 9 - 24i - 16$

$= -7 - 24i$ 1

iii) $\dfrac{z_2}{z_1}$ $\qquad \dfrac{3-4i}{3+2i}\,\dfrac{(3+2i}{(3+2i}$

$= \dfrac{3-4i}{3-2i}\,\dfrac{(3+2i)}{(3+2i)}$ $\qquad \dfrac{9+6i-12i-8i^2}{9+4}$

$= \dfrac{9+6i-12i-8i^2}{9-4i^2}$ 1

$= \dfrac{17-6i}{13}$

(iv) $z_1 \cdot \overline{z_2}$

$= (3-2i)(3+4i)$

$= 9 + 12i - 6i - 8i^2$

$= 9 + 6i + 8$

$= 17 + 6i$ 1

b) $x^4 + x^2 - 12$

i) $(x^2+4)(x^2-3)$ 1

ii) $(x^2+4)(x-\sqrt{3})(x+\sqrt{3})$ 1

iii) $(x^2-4i^2)(x-\sqrt{3})(x+\sqrt{3})$

$= (x-2i)(x+2i)(x-\sqrt{3})(x+\sqrt{3})$ 1

c) $1+i\sqrt{3} = a^2-b^2+2abi$

$a^2-b^2=1 \qquad 2ab=\sqrt{3}$

$b = \dfrac{\sqrt{3}}{2a}$

$a^2 - \left(\dfrac{\sqrt{3}}{2a}\right)^2 = 1$

$a^2 - \dfrac{3}{4a^2} = 1$

$4a^4 - 4a^2 - \cancel{3}^{12} = 0$

$\dfrac{(4a^2-6)(4a^2+2)}{4}$

$(2a^2-3)(2a^2+1) = 0$

(c) Cont'd

$2a^2 = 3$

$a^2 = \frac{3}{2}$ 3

$a = \pm\sqrt{\frac{3}{2}}$ ✓

$2\left(\frac{\sqrt{3}}{\sqrt{2}}\right)b = \sqrt{3}$

$\pm\left(\sqrt{\frac{3}{2}} + \frac{\sqrt{2}}{2}i\right)$

$b = \sqrt{3} \div \left(\frac{2\sqrt{3}}{\sqrt{2}}\right)$

$= \cancel{\sqrt{3}} \times \frac{\sqrt{2}}{2\cancel{\sqrt{3}}} \quad = \pm\frac{\sqrt{2}}{2}$ ✓

d) $z \cdot \bar{z} = |z|^2$

LHS:

$(x+iy)(x-iy)$

$= x^2 - xiy + xiy + y^2$

$= x^2 + y^2$

RHS:

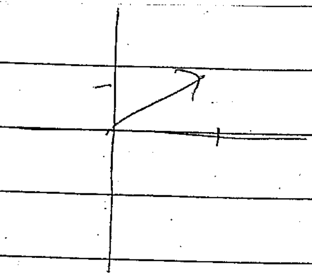
$|z|^2$

$= \left(\sqrt{x^2+y^2}\right)^2$

$= x^2 + y^2$

$\therefore$ LHS = RHS. 2

e) i) $z = \sqrt{3} + i$



$|z| = \sqrt{\sqrt{3}^2 + 1^2}$

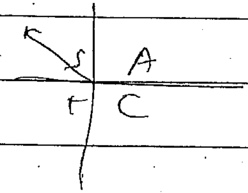
$= 2$

$\arg z = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

$= \frac{\pi}{6}$ 1

$z = 2\,\text{cis}\,\frac{\pi}{6}$

ii) $z^5 = \left(2\,\text{cis}\,\frac{\pi}{6}\right)^5$

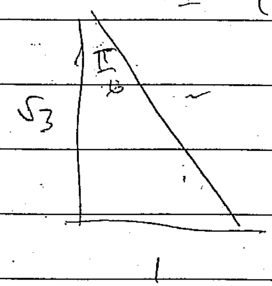


$= 32\,\text{cis}\,\frac{5\pi}{6}$

$= 32\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$

$= 32\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$

$= 16(-\sqrt{3} + i)$ 2



13a) $p(x) = x^4 + x^3 - 3x^2 - 5x - 2$

$p'(x) = 4x^3 + 3x^2 - 6x - 5$

$p''(x) = 12x^2 + 6x - 6$

$= 2x^2 + x - 1$

$= (2x + 2)(2x - 1)$

$2x + 2 = 0 \qquad 2x = 1$

$2x = -2 \qquad x = \frac{1}{2}$

$x = -1$

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & 2 & & 1 & \\ 1 & & 3 & & 3 & & 1 \end{array}$$

test $x = -1$ ✓

$p(-1) = 0$

$\therefore (x+1)^3$ is a root.

$p(x) = (x+1)^3 Q(x)$

$= (x^3 + 3x^2 + 3x + 1)(x - 2)$ 3

By Inspection

roots are $-1, -1, -1, 2$

b) $p(x) = (x - \alpha)^m Q(x)$

$p'(x) = m(x-\alpha)^{m-1} Q(x) + (x-\alpha)^m Q'(x)$

$= (x-\alpha)^{m-1}\left[m Q(x) + (x-\alpha) Q'(x)\right]$

$\therefore p'(x)$ has a root $(x-\alpha)$ multiplicity $m-1$. 2.

c) $p(x) = x^3 + qx + r$

~~$\Sigma\alpha = -b$~~

$= x^3 + 0x^2 + qx + r$

$\Sigma\alpha = \frac{-b}{a}$

$= 0$

$\Sigma\alpha\beta = \frac{c}{a}$

$= q$

$\Sigma\alpha\beta\gamma = \frac{-d}{a}$

$= -r$

$(\beta-\gamma)^2 + (\gamma-\alpha)^2 + (\alpha-\beta)^2 = -6q$

LHS:

$(\beta-\gamma)(\beta-\gamma) + (\gamma-\alpha)(\gamma-\alpha) + (\alpha-\beta)(\alpha-\beta)$

$= \beta^2 - 2\beta\gamma + \gamma^2 + \gamma^2 - 2\alpha\gamma + \alpha^2 + \alpha^2 - 2\alpha\beta + \beta^2$

$= 2(\alpha^2 + \beta^2 + \gamma^2) - 2(\alpha\beta + \beta\gamma + \alpha\gamma)$ ✓

~~$= 2(\alpha+\beta+\gamma)^2 - 4(\alpha\beta+\beta\gamma+\alpha\gamma) -$~~

~~$= 2(0)^2 - 4(q)$~~

$= 2(\alpha+\beta+\gamma)^2 - 4(\alpha\beta+\beta\gamma+\alpha\gamma) - 2(\alpha\beta+\beta\gamma+\alpha\gamma)$

$= -6(q)$

$= -6q$

$= \text{RHS}$ 3

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Student Number: ....

d) $p(x) = 4x^3 - 24x^2 + 23x + 18$

$a, a+d, a-d$

$\Sigma\alpha = \frac{-b}{a}$

$a + a + d + a - d = \frac{24}{4}$

$3a = \frac{24}{4}$

$\boxed{a = 2}$ ✓

roots are
$2, 4.5, -0.5.$

$\Sigma\alpha\beta\gamma = \frac{-d}{a}$

$a(a+d)(a-d) = \frac{-18}{4}$

$a(a^2-d^2) = \frac{-18}{4}$

~~$a(a+d)(a-d) = \frac{-18}{4}$~~

$2(4-d^2) = \frac{-18}{4}$

~~$2(4-d^2) = \frac{-18}{4}$~~

~~$4 - d^2 = \frac{-18}{4}$~~

$4 - d^2 = \frac{-18}{8}$

~~$4 + \frac{18}{4} = d^2$~~

3

$d^2 = \frac{25}{4}$

~~$d = \pm$~~

$d = \pm 2.5$ ✓

let $x = -1$:

$-2 + 31 = A(-1-1)^2(-1+2) + B(-1-1)(-1+2) + 11(-1+2) + (-1)(-1-1)^3$

$= A(4)(1) + B(-2)(1) + 11(1) + (-1)(-2)$

$29 = 4A - 2B + 11 + 8$

$10 = 4A - 2B$

$5 = 2A - B$

$5 = 2(4+B) - B$

$= 8 + 2B - B$

$\boxed{-3 = B}$ ✓

~~$A+4$~~

$A = 4 + B$

$= 4 - 3$

$\boxed{A = 1}$ ✓

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e) $\dfrac{2x+31}{(x-1)^3(x+2)} = \dfrac{A}{(x-1)} + \dfrac{B}{(x-1)^2} + \dfrac{c}{(x-1)^3} + \dfrac{d}{(x+2)}$

$2x+31 = A(x-1)^2(x+2) + B(x-1)(x+2) + C(x+2) + d(x-1)^3$

let $x = 1$:

$2+31 = c(1+2)$

$33 = 3c$

$\boxed{c = 11}$ ✓

let $x = -2$:

$-4+31 = d(-2-1)^3$

$27 = -27d$

$\boxed{d = -1}$ ✓

4

let $x = 0$:

$31 = A(1)(2) + B(-1)(2) + 11(2) \cancel{+11}$

$= 2A - 2B + 22 + 1$

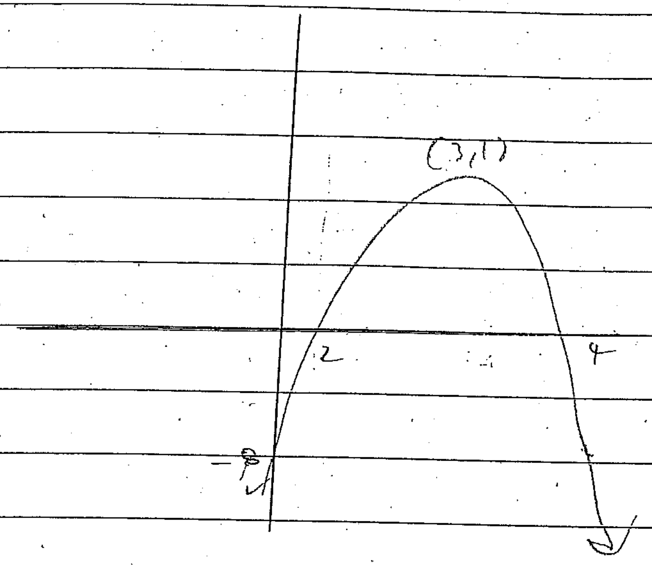
$8 = 2A - 2B$

$4 = A - B$

$\boxed{A = 4 + B}$

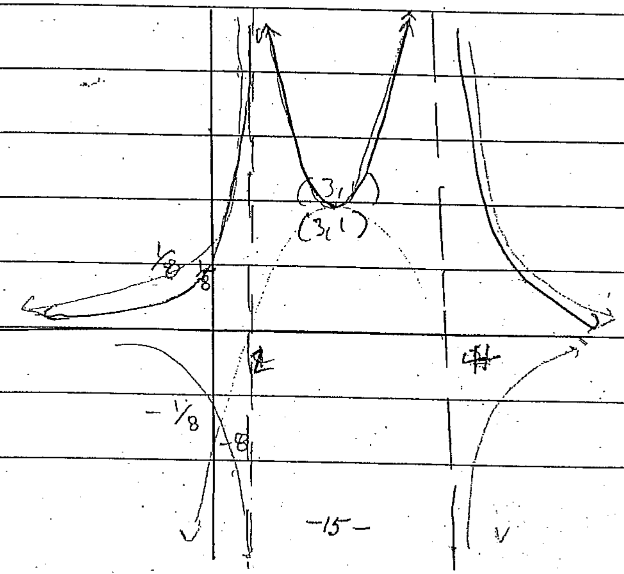
14a) $f(x) = (x-4)(2-x)$

$=$



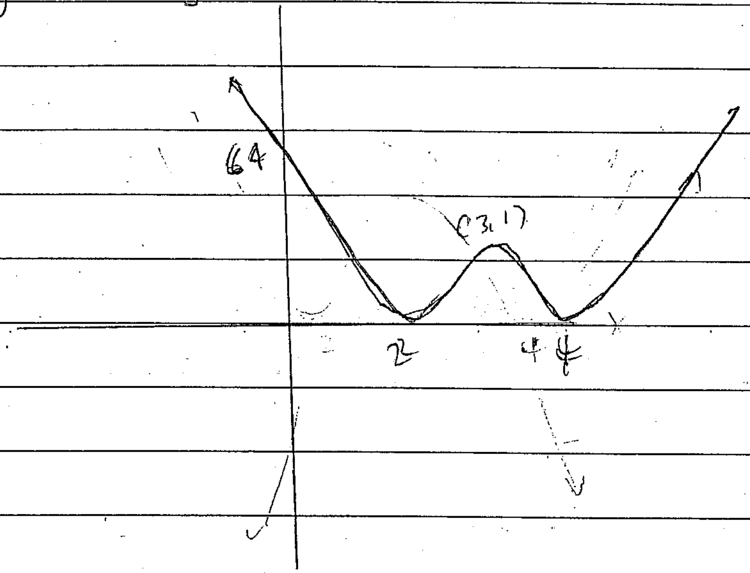
1

i) $\left|\dfrac{1}{f(x)}\right| = y$



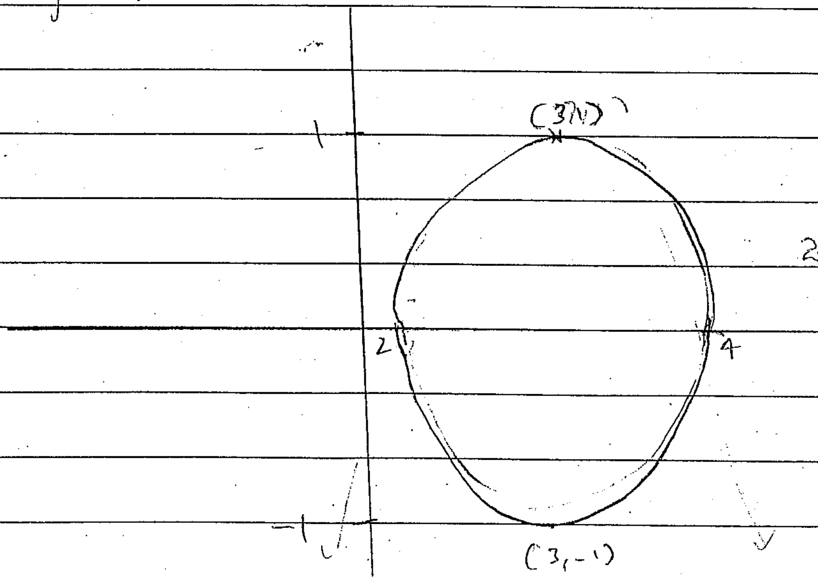
2

ii) $y = [f(x)]^2$ $\quad [(x-4)(2-x)]$



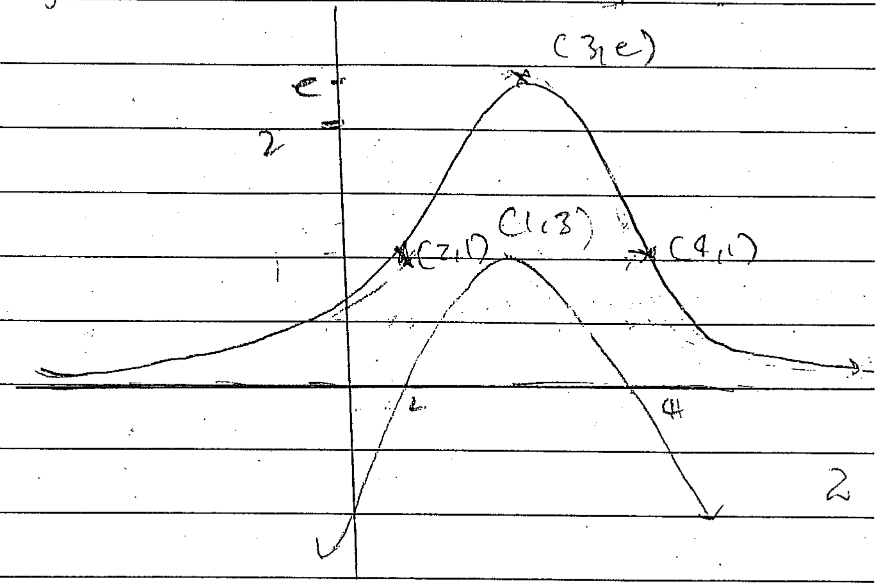
1

iii) $y^2 = f(x)$



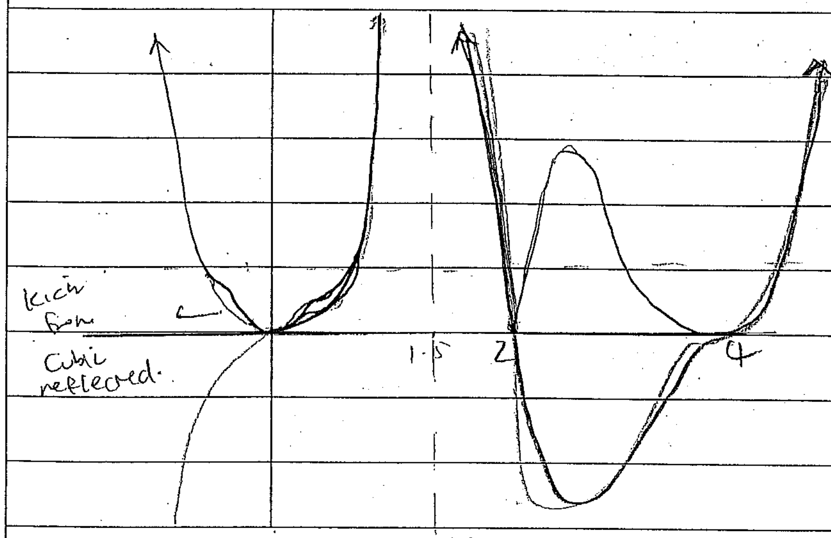
2

iv) $y = e^{f(x)}$



2

b) $y = |f(x)|$



2

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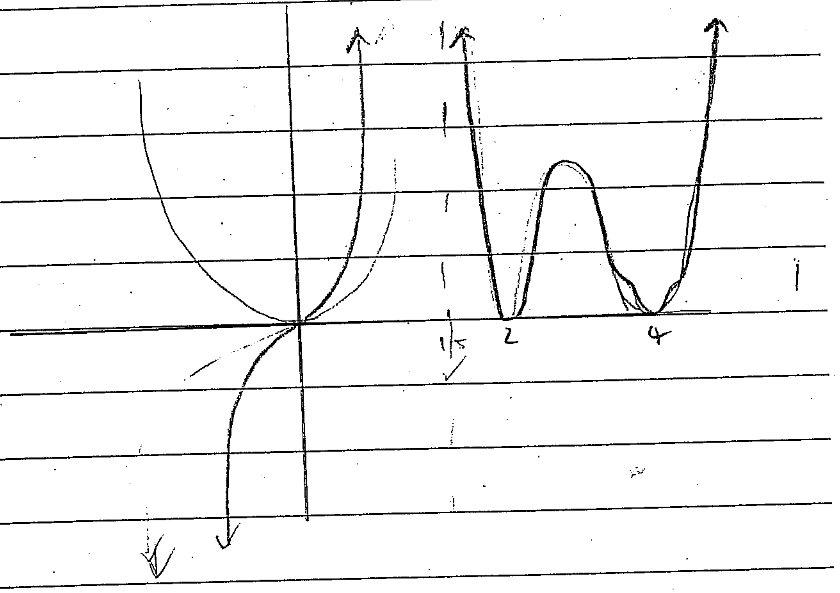
Single roots turn into double roots.

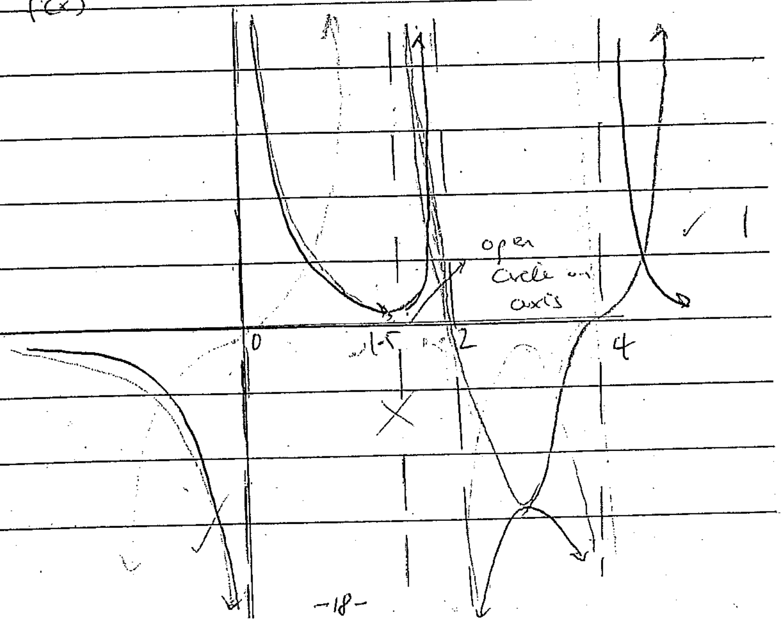
steeper than original

ii) $y = (f(x))^2$



1

iii) $y = \frac{1}{f(x)}$



✓ 1

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16. a) $I_n = \int \tan^n x \, dx = \frac{1}{n-1} \tan^{n-1} x - I_{n-2}$

$\cancel{= \int \tan^{n-1} x \tan x \, dx}$

$\cancel{= \tan^{n-1} x \quad -(n-1)}$

LHS:

$\int \tan^{n-2} x \tan^2 x$

$\int \tan^{n-2} x (\sec^2 x - 1)$ ✓

$\int \tan^{n-2} x \sec^2 x - \int \tan^{n-2} x$

$\int u^{n-2} du - I_{n-2}$ $\qquad u = \tan x$

$\frac{u^{n-1}}{n-1} - I_{n-2}$ $\qquad du = \sec^2 x \, dx$

$\frac{\tan^{n-1} x}{n-1} - I_{n-2}$

3

$= \frac{1}{n-1} \tan^{n-1} x - I_{n-2}$

$= RHS$

ii) $\int_0^{\frac{\pi}{4}} \tan^3 x \, dx = I_3$

$I_3 = \left[\frac{1}{3-1} \tan^{n-1} x\right]_0^{\frac{\pi}{4}} - I_{3-2}$

$= \left[\frac{1}{2} \tan^2 x\right]_0^{\frac{\pi}{4}} - I_1$

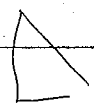
$= \left[\frac{1}{2}\right] - I_1$

$I_1 = \int \tan x \, dx$

$= \int \frac{\sin x}{\cos x} dx$ let $u = \cos x$

$du = -\sin x$

$= -\ln|\cos x|$



$I_3 = \frac{1}{2} - \left(-\ln|\cos x|\right)_0^{\frac{\pi}{4}}$

$= \frac{1}{2} - \left[-\ln\left|\frac{1}{\sqrt{2}}\right|\right]$

$= \frac{1}{2} + \ln|\sqrt{2}|$

2 answers are not crossed out

1

b i) $\tan(\alpha+\beta+\gamma) = \frac{\tan\alpha + \tan\beta + \tan\gamma - \tan\alpha\tan\beta\tan\gamma}{1 - \tan\alpha\tan\beta - \tan\beta\tan\gamma - \tan\gamma\tan\alpha}$

LHS:

$\tan((\alpha+\beta)+\gamma)$

$= \frac{\tan(\alpha+\beta) + \tan\gamma}{1 - \tan(\alpha+\beta)\tan\gamma}$

$\tan(\alpha+\beta)$

$= \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$

$= \dfrac{\frac{\tan\alpha + \tan\beta + \tan\gamma(1 - \tan\alpha\tan\beta)}{1 - \tan\alpha\tan\beta}}{1 - \frac{(\tan\alpha + \tan\beta)\tan\gamma}{(1 + \tan\alpha\tan\beta)}}$

$= \frac{\tan\alpha + \tan\beta + \tan\gamma - \tan\alpha\tan\beta\tan\gamma}{1 + \tan\alpha\tan\beta - \tan\alpha\tan\gamma - \tan\beta\tan\gamma}$

3

= RHS.

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ii) $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{4} + \tan^{-1}\frac{1}{13}$

let $\alpha = \tan^{-1}\frac{1}{12}$, $\beta = \tan^{-1}\frac{1}{4}$, $\gamma = \tan^{-1}(\frac{1}{13})$

$\tan(\alpha+\beta+\gamma) = \cancel{\tan(\tan^{-1}\frac{1}{12}) + \tan(\tan^{-1}\frac{1}{4}) + \tan(\tan^{-1}}$

a) $\tan(\tan^{-1}x) = x$, tan and $\tan^{-1}$ can be removed $\therefore$

$$= \frac{\frac{1}{2} + \frac{1}{4} + \frac{1}{13} - (\frac{1}{2})(\frac{1}{4})(\frac{1}{13})}{1 - (\frac{1}{2})(\frac{1}{4}) - (\frac{1}{4})(\frac{1}{13}) - (\frac{1}{13})(\frac{1}{2})}$$

calculator work!

$= \frac{87}{104} \div \frac{85}{104}$

$= \frac{87}{85}$.

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$x^2 - 2x + 4 = 0$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \frac{+2 \pm \sqrt{4 - 4(1)(4)}}{2}$

$= \frac{2 \pm \sqrt{-12}}{2}$ ~~$\alpha = 1 + i\sqrt{3}$~~

$= \frac{2 \pm \sqrt{4}\sqrt{3i^2}}{2}$ ~~β~~

$\alpha = 1 + i\sqrt{3}$

$= \frac{2 \pm 2i\sqrt{3}}{2}$ $\beta = 1 - i\sqrt{3}$

$x = 1 \pm i\sqrt{3}$ ✓

$(1+\sqrt{3}i)(1+\sqrt{3}i) = 1 + 2\sqrt{3}i - 3$

$(1+i\sqrt{3})^2 - 2(1+i\sqrt{3}) + 4 = 0$ → PTO

$-2 + 2\sqrt{3}i - 2 - 2\sqrt{3}i + 4 = 0$ proof

$0 = 0$ Cont.

$x^2 - 2x + 4 = 0$

~~$(x-1)^2 - 1 + 4 = +1$~~ ~~$(x-1)^2 - 1 + 4 = 0$~~ -1

~~$(x-1)^2 = -2$~~ ~~$(x-1)^2 + 3 = -1$~~

$= 2i$ $(x-1)^2 = -4$

$(x-1)^2 = 4i^2$

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$\alpha = 2\operatorname{cis}\frac{\pi}{3}$ $\beta = 2\operatorname{cis}-\frac{\pi}{3}$

$\alpha+\beta = 2\left(\operatorname{cis}\frac{\pi}{3} + \operatorname{cis}-\frac{\pi}{3}\right)$

$= 2i\sin\frac{\pi}{3}$ ✓

$\alpha^n = 2^n \operatorname{cis}\frac{n\pi}{3}$ $\beta^n = 2^n\operatorname{cis}-\frac{n\pi}{3}$

LHS:

$\alpha^n + \beta^n = 2^n \operatorname{cis}\frac{n\pi}{3} + 2^n \operatorname{cis}-\frac{n\pi}{3}$

$= 2^n\left[\operatorname{cis}\frac{n\pi}{3} + \operatorname{cis}-\frac{n\pi}{3}\right)$

$= 2^n\left[2i\sin\frac{n\pi}{3}\right)$

$= 2^n \cdot 2i\sin\frac{n\pi}{3}$

$= 2^{n+1} i\sin\frac{n\pi}{3}$ 5

$= i2^{n+1}\sin\frac{n\pi}{3}$ ✓

$= RHS$

* part (ii) is answered on first page :)

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~~c) $p(x) = x^2-2x+4$~~

~~α, β are roots~~

~~prove: $\cancel{\alpha^n \; \beta^n \; i2}$~~

~~$\alpha^n - \beta^n = i2^{n+1}\sin\left[\frac{n\pi}{3}\right]$~~

~~$x^2-2x+4=0$~~

~~$(x-1)^2 - 1 + 4 = 0$~~ → PTO.

~~$(x-1)^2 = -\cancel{2}$~~

ii) $\alpha^9 - \beta^9$

$= i2^{9+1}\sin\left[\frac{9\pi}{3}\right]$

$= i2^{10}\sin(3\pi)$

$= i2^{10}(0)$

$= 0$ ✓