



SYDNEY BOYS HIGH
MOORE PARK, SURRY HILLS

2013
HIGHER SCHOOL CERTIFICATE
ASSESSMENT TASK #1

Mathematics Extension 2

General Instructions:

- Reading time – 5 minutes
- Working time – 1.5 hours
- Write using black or blue pen
Black pen is preferred
- Board approved calculators may be used
- A table of standard integrals is provided on the back of the Multiple Choice answer sheet
- All necessary working should be shown in every question if full marks are to be awarded
- Marks may NOT be awarded for messy or badly arranged work
- Answer in simplest exact form unless otherwise stated

Total marks – 70 Marks

Section I Pages 2-3
7 marks

- Attempt Questions 1-7
- Answer on the Multiple Choice answer sheet provided.
- Allow about 10 minutes for this section

Section II Pages 4-6
63 marks

- Attempt Questions 8-10
- Allow about 1 hour 20 minutes for this section
- For Questions 8-10, start a new answer booklet per question

Examiner: Mr D. Hespe

This is an assessment task only and does not necessarily reflect the content or format of the Higher School Certificate.

Section I—7 marks

Select the alternative A, B, C, or D that best answers the question. Fill in the response oval on your multiple choice answer sheet.

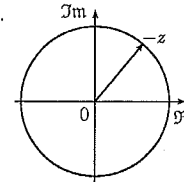
Marks

1. The probability that a randomly chosen angle has a sine which is less than a half is

1

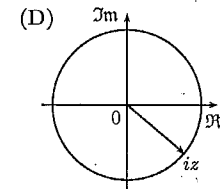
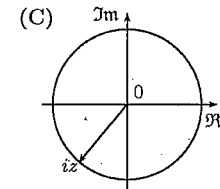
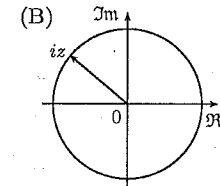
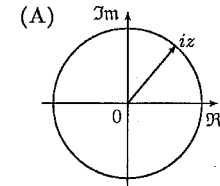
- (A) $\frac{1}{2}$
(B) $\frac{2\pi}{3}$
(C) $\frac{2}{3}$
(D) $\frac{4\pi}{3}$

2.



This graph is a representation of $-z$ shown in an Argand diagram. Which of the following shows iz ?

1



3. If $f(x) = e^{x+2}$ what is the inverse function $f^{-1}(x)$?

1

- (A) $f^{-1}(x) = e^{x-2}$
(B) $f^{-1}(x) = e^{x+2}$
(C) $f^{-1}(x) = \log_e x - 2$
(D) $f^{-1}(x) = \log_e x + 2$

4. What is the domain and range of $y = \cos^{-1}\left(\frac{2x}{5}\right)$? 1

- (A) Domain: $\left[-\frac{5}{2}, \frac{5}{2}\right]$; Range: $[0, \pi]$
 (B) Domain: $[-1, 1]$; Range: $[0, \pi]$
 (C) Domain: $\left[-\frac{5}{2}, \frac{5}{2}\right]$; Range: $[-\pi, \pi]$
 (D) Domain: $[-1, 1]$; Range: $[-\pi, \pi]$

5. Form a polynomial $f(x)$ with real coefficients having the given degree and zeroes.
 Degree: 3, Zeroes: $1 + i$ and -5 . 1

- (A) $f(x) = x^3 + x^2 - 8x + 10$
 (B) $f(x) = x^3 - 5x^2 - 8x - 12$
 (C) $f(x) = x^3 + 3x^2 - 8x + 10$
 (D) $f(x) = x^3 + 3x^2 + 10x - 8$

6. Find the indefinite integral $\int 2t^2(1+t^3)^4 dt$ using the substitution $u = 1+t^3$. 1

- (A) $\frac{1}{5}(1+t^3)^5 + c$
 (B) $\frac{2}{5}(1+t^3)^5 + c$
 (C) $\frac{2}{3}(1+t^3)^5 + c$
 (D) $\frac{2}{15}(1+t^3)^5 + c$

7. If $(x-3)^2 + (y+2)^2 = 0$, then $x+y =$ 1

- (A) 1
 (B) 2
 (C) 3
 (D) 5

Section II — 63 marks

Marks

Question 8 (21 marks) (use a separate answer booklet)

(a) You are dealt a hand of 5 cards from a standard deck of 52 (in 4 suits of 13 cards).

(i) What is the chance that you have a flush (i.e. all cards from the same suit)? 1

(ii) What is the chance of four of a kind (i.e. 4 cards of the same value)? 1

(b) (i) What is $\int \sec^2 \psi \, d\psi$? 1

(ii) Evaluate $\int_0^\pi \sin^3 \theta \, d\theta$. 3

(iii) Find $\int \frac{dx}{\sqrt{x^2 - 4x + 20}}$ by using the substitution $y = x - 2$. 3

(c) Mark on an Argand diagram the points representing the numbers $2 + 3i$ and $-3 + 4i$. 2

(d) Find, correct to three significant figures, the modulus and argument of $\frac{1}{12 + 5i}$. 2

(e) Simplify:

(i) $(5 - 3i)(2 + i)$, 1

(ii) $\left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}\right)\left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6}\right)$, 1

(iii) $\frac{1}{5 - 3i} - \frac{1}{5 + 3i}$. 1

(f) The cubic equation $x^3 + ax - b$ has roots α, β, γ . Given that $\gamma = \alpha\beta$, express each of a and b in terms of γ only, and hence show that $(a+b)^2 = b$. 5

Question 9 (21 marks) (use a separate answer booklet)

Marks

(a) The letters of the word EXCELLENT are arranged in a random order. Find the probability that:

(i) the same letter occurs at each end. 2

(ii) X, C and N occur together in any order. 2

(iii) the letters occur in alphabetical order. 1

(b) Evaluate

(i) $\sin\left(\sin^{-1}\frac{1}{2}\right)$ 1

(ii) $\sin\left(\cos^{-1}\frac{1}{2}\right)$ 1

(iii) $\cos^{-1}\left(\cos\frac{4\pi}{3}\right)$ 1

(iv) Sketch the function $f(x) = -2\pi \cos^{-1}\left(\frac{3x}{\pi}\right)$. 3

(c) Use the substitution $x = \tan \theta$ to evaluate $\int_0^1 \frac{x^2 dx}{(1+x^2)^2}$. 3

(d) (i) Use limits to show why $\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$. 1

(ii) Prove that $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$. 3

(iii) Hence find a primitive of $\frac{x^2}{1+x^2}$. 3

Question 10 (21 marks) (use a separate answer booklet)

Marks

(a) A company has to place three orders for supplies among five distributors. Each order is randomly assigned, and a distributor may receive multiple orders. Find the probabilities of the following events.

(i) Each order goes to a different distributor. 1

(ii) All orders go to the same distributor. 1

(iii) Exactly two of the three go to one of the distributors. 1

(b) (i) Use De Moivre's theorem to express $\cos 3\theta$ in terms of $\cos \theta$ and $\sin 3\theta$ in terms of $\sin \theta$. 2

(ii) Use the result to solve the equation 3

$$8x^3 - 6x + 1 = 0.$$

(c) When a rational integral function is divided by $(x - \alpha)^2$ the remainder is $R_2(x - \alpha) + R_1$, i.e. $f(x) = (x - \alpha)^2 Q(x) + R_2(x - \alpha) + R_1$.

(i) Prove that $R_1 = f(\alpha)$ and $R_2 = f'(\alpha)$, where $f'(x)$ is the differential coefficient of $f(x)$ with respect to x . 2

(ii) Show that $x^n - nx + n - 1$ is exactly divisible by $(x - 1)^2$ for any integral value of n greater than 1. 2

(iii) Deduce that $2^{4n} - 15n - 1$ is exactly divisible by 225 for any integral value of n greater than 1. 2

(d) (i) If a, b are the complex numbers represented by points A, B in the Argand diagram, what geometrical properties correspond to the modulus and argument of b/a ? 2

(ii) Show that, if the four points representing the complex numbers z_1, z_2, z_3, z_4 are concyclic, the fraction 5

$$\frac{(z_1 - z_2)(z_3 - z_4)}{(z_3 - z_2)(z_1 - z_4)}$$

must be real.

End of Paper



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Mathematics Extension 2 Solutions

Section I—7 marks

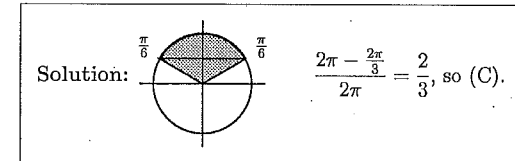
Select the alternative A, B, C, or D that best answers the question.
Fill in the response oval on your multiple choice answer sheet.

Marks

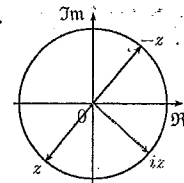
1. The probability that a randomly chosen angle has a sine which is less than a half is

1

- (A) $\frac{1}{2}$
(B) $\frac{2\pi}{3}$
(C) $\frac{2}{3}$
(D) $\frac{4\pi}{3}$

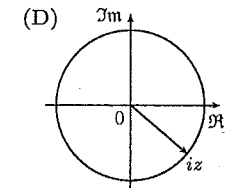
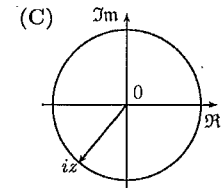
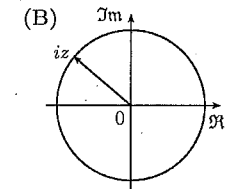
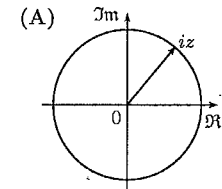


2.



This graph is a representation of $-z$ shown in an Argand diagram. Which of the following shows iz ?

1



Solution: z is π from $-z$ and iz is rotated $\frac{\pi}{2}$ anticlockwise from z , so (D).

3. If $f(x) = e^{x+2}$ what is the inverse function $f^{-1}(x)$?

1

- (A) $f^{-1}(x) = e^{y-2}$
(B) $f^{-1}(x) = e^{y+2}$
(C) $f^{-1}(x) = \log_e x - 2$
(D) $f^{-1}(x) = \log_e x + 2$

Solution: $\log_e x = f^{-1}(x) + 2$,
 $f^{-1}(x) = \log_e x - 2$, so (C).

4. What is the domain and range of $y = \cos^{-1}\left(\frac{2x}{5}\right)$?

(A) Domain: $\left[-\frac{5}{2}, \frac{5}{2}\right]$; Range: $[0, \pi]$

(B) Domain: $[-1, 1]$; Range: $[0, \pi]$

(C) Domain: $\left[-\frac{5}{2}, \frac{5}{2}\right]$; Range: $[-\pi, \pi]$

(D) Domain: $[-1, 1]$; Range: $[-\pi, \pi]$

Solution: $-1 \leq \frac{2x}{5} \leq 1,$
 $-\frac{5}{2} \leq x \leq \frac{5}{2}.$
 $0 \leq y \leq \pi,$ so (A).

1

5. Form a polynomial $f(x)$ with real coefficients having the given degree and zeroes.
 Degree: 3, Zeroes: $1+i$ and -5 .

(A) $f(x) = x^3 + x^2 - 8x + 10$

(B) $f(x) = x^3 - 5x^2 - 8x - 12$

(C) $f(x) = x^3 + 3x^2 - 8x + 10$

(D) $f(x) = x^3 + 3x^2 + 10x - 8$

Solution: $S_1 = (1+i) + (1-i) - 5,$
 $= -3.$
 $S_2 = (1+i)(1-i) + (1+i)(-5) + (1-i)(-5),$
 $= -8.$
 $S_3 = (1+i)(1-i)(-5),$
 $= -10.$

So the solution is $f(x) = x^3 + 3x^2 - 8x + 10,$ i.e. (C).

1

6. Find the indefinite integral $\int 2t^2(1+t^3)^4 dt$ using the substitution $u = 1+t^3$.

(A) $\frac{1}{5}(1+t^3)^5 + c$

(B) $\frac{2}{5}(1+t^3)^5 + c$

(C) $\frac{2}{3}(1+t^3)^5 + c$

(D) $\frac{2}{15}(1+t^3)^5 + c$

Solution: $du = 3t^2 dt,$
 $I = \frac{2}{3} \int 3t^2(1+t^3)^4 dt,$
 $= \frac{2}{3} \int u^4 du,$
 $= \frac{2}{3} \times \frac{u^5}{5} + c,$
 $= \frac{2(1+t^3)^5}{15},$ so (D).

1

7. If $(x-3)^2 + (y+2)^2 = 0,$ then $x+y =$

(A) 1

(B) 2

(C) 3

(D) 5

Solution: $(x-3)^2 \geq 0, (y+2)^2 \geq 0,$
 $(x-3)^2 + (y+2)^2 = 0,$
 $\therefore x-3=0, y+2=0,$
 $x-3+y+2=0,$
 $x+y=1,$ so (A).

1

Section II— 63 marks

Question 8 (21 marks) (use a separate answer booklet)

- (a) You are dealt a hand of 5 cards from a standard deck of 52 (in 4 suits of 13 cards).
- (i) What is the chance that you have a flush (i.e. all cards from the same suit)?

$$\text{Solution: } \frac{4 \times \binom{13}{5}}{\binom{52}{5}} = \frac{33}{16660}$$

- (ii) What is the chance of four of a kind (i.e. 4 cards of the same value)?

$$\text{Solution: } \frac{13 \times (52 - 4)}{\binom{52}{5}} = \frac{1}{4165}$$

- (b) (i) What is $\int \sec^2 \psi \, d\psi$?

$$\text{Solution: } \tan \psi + c$$

- (ii) Evaluate $\int_0^\pi \sin^3 \theta \, d\theta$.

$$\begin{aligned} \text{Solution: } I &= \int_0^\pi \sin^2 \theta \sin \theta \, d\theta, \\ &= \int_0^\pi (1 - \cos^2 \theta) \sin \theta \, d\theta, \\ &= \int_0^\pi \sin \theta \, d\theta - \int_0^\pi \cos^2 \theta \sin \theta \, d\theta, \\ &= [-\cos \theta]_0^\pi - \left[-\frac{\cos^3 \theta}{3} \right]_0^\pi, \\ &= 1 - -1 - \left(\frac{1 - -1}{3} \right), \\ &= \frac{4}{3} \end{aligned}$$

- (iii) Find $\int \frac{dx}{\sqrt{x^2 - 4x + 20}}$ by using the substitution $y = x - 2$.

Marks

1

1

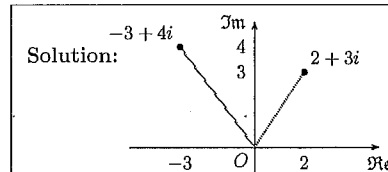
1

3

3

$$\begin{aligned} \text{Solution: } x^2 - 4x + 4 + 16 &= (x - 2)^2 + 4^2, \\ dy &= dx, \\ I &= \int \frac{dy}{\sqrt{y^2 + 4^2}}, \\ &= \ln(y + \sqrt{y^2 + 4^2}) + c, \\ &= \ln(x - 2 + \sqrt{x^2 - 4x + 20}) + c. \end{aligned}$$

- (c) Mark on an Argand diagram the points representing the numbers $2 + 3i$ and $-3 + 4i$.



- (d) Find, correct to three significant figures, the modulus and argument of $\frac{1}{12 + 5i}$.

$$\begin{aligned} \text{Solution: } |12 + 5i| &= \sqrt{144 + 25}, \\ &= 13, \\ \arg(12 + 5i) &= \tan^{-1}\left(\frac{5}{12}\right), \\ &\approx 0.395 \text{ or } 22.6^\circ, \\ \text{So } \left| \frac{1}{12 + 5i} \right| &= \frac{1}{13}, \\ &\approx 0.0769, \\ \arg\left(\frac{1}{12 + 5i}\right) &\approx -0.395. \end{aligned}$$

- (e) Simplify:

- (i) $(5 - 3i)(2 + i)$,

$$\text{Solution: } 10 + 5i - 6i + 3 = 13 - i$$

- (ii) $(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3})(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6})$,

$$\text{Solution: } \cos \frac{\pi}{2} + i \sin \frac{\pi}{2} = i$$

- (iii) $\frac{1}{5 - 3i} - \frac{1}{5 + 3i}$

$$\text{Solution: } \frac{5 + 3i - (5 - 3i)}{25 + 9} = \frac{3i}{17}$$

- (f) The cubic equation $x^3 + ax - b$ has roots α, β, γ . Given that $\gamma = \alpha\beta$, express each of a and b in terms of γ only, and hence show that $(a+b)^2 = b$.

5

Solution:

$$\begin{aligned} 0 &= \alpha + \beta + \gamma, \\ \alpha + \beta &= -\gamma, \\ a &= \alpha\beta + \beta\gamma + \gamma\alpha, \\ &= \gamma + \gamma(\alpha + \beta), \\ &= \gamma - \gamma^2, \\ b &= \alpha\beta\gamma, \\ &= \gamma^2, \\ a + b &= \gamma, \\ (a + b)^2 &= \gamma^2, \\ &= b. \end{aligned}$$

Question 9 (21 marks) (use a separate answer booklet)

Marks

- (a) The letters of the word EXCELLENT are arranged in a random order. Find the probability that:

- (i) the same letter occurs at each end.

2

Solution: $\frac{\frac{7!}{2!} + \frac{7!}{3!}}{3! \times 2!} = \frac{1}{9}$

- (ii) X, C and N occur together in any order.

2

Solution: $\frac{\frac{3! \times 7!}{3! \times 2!}}{3! \times 2!} = \frac{1}{12}$

- (iii) the letters occur in alphabetical order.

1

Solution: $\frac{1}{\frac{9!}{3! \times 2!}} = \frac{1}{30240}$

- (b) Evaluate

(i) $\sin\left(\sin^{-1}\frac{1}{2}\right)$

1

Solution: $\frac{1}{2}$

(ii) $\sin\left(\cos^{-1}\frac{1}{2}\right)$

1

Solution: $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

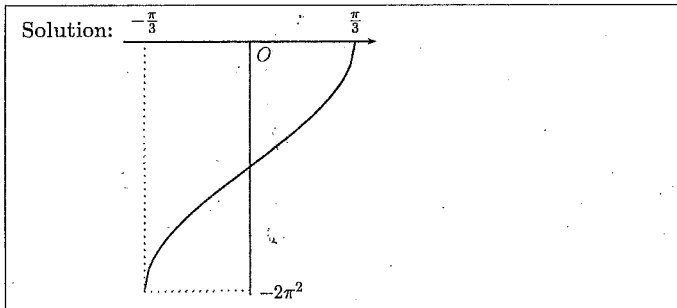
(iii) $\cos^{-1}\left(\cos\frac{4\pi}{3}\right)$

1

Solution: $\cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$

- (iv) Sketch the function $f(x) = -2\pi \cos^{-1}\left(\frac{3x}{\pi}\right)$.

3



(c) Use the substitution $x = \tan \theta$ to evaluate $\int_0^1 \frac{x^2 dx}{(1+x^2)^2}$.

3

Solution:

$$\begin{aligned}
 dx &= \sec^2 \theta d\theta, \\
 x = 0 &\implies \theta = 0, \\
 x = 1 &\implies \theta = \frac{\pi}{4}, \\
 I &= \int_0^{\frac{\pi}{4}} \frac{\tan^2 \theta \sec^2 \theta d\theta}{(1 + \tan^2 \theta)^2}, \\
 &= \int_0^{\frac{\pi}{4}} \frac{\sin^2 \theta}{\cos^2 \theta} \times \frac{d\theta}{\cos^2 \theta} \times \frac{1}{(\sec^2 \theta)^2}, \\
 &= \int_0^{\frac{\pi}{4}} \sin^2 \theta d\theta, \\
 &= \int_0^{\frac{\pi}{4}} \frac{1 - \cos 2\theta}{2} d\theta, \\
 &= \frac{1}{2} \left[\theta - \frac{\sin 2\theta}{2} \right]_0^{\frac{\pi}{4}}, \\
 &= \frac{1}{2} \left(\frac{\pi}{4} - \frac{1}{2} - (0 - 0) \right), \\
 &= \frac{\pi - 2}{8}.
 \end{aligned}$$

(d) (i) Use limits to show why $\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$.

1

Solution: We consider a small change in x , δx and a corresponding small change in y , δy .

$$\begin{aligned}
 \text{Now } \frac{dx}{dy} &= \lim_{\delta y \rightarrow 0} \frac{\delta x}{\delta y}, \\
 &= \lim_{\delta y \rightarrow 0} \frac{1}{\frac{\delta y}{\delta x}}.
 \end{aligned}$$

If y is a continuous function of x then $\delta y \rightarrow 0$ as $\delta x \rightarrow 0$;

whence

$$\begin{aligned}
 \frac{dx}{dy} &= \lim_{\delta x \rightarrow 0} \frac{1}{\frac{\delta y}{\delta x}}, \\
 &= \frac{1}{\frac{dy}{dx}}.
 \end{aligned}$$

(ii) Prove that $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$.

3

Solution: Let $y = \tan^{-1} x$, $x \in \mathbb{R}$, $-\frac{\pi}{2} < y < \frac{\pi}{2}$,

$$x = \tan y,$$

$$\frac{dx}{dy} = \sec^2 y,$$

$$= 1 + \tan^2 y,$$

$$= 1 + x^2.$$

$$\therefore \frac{dy}{dx} = \frac{1}{1+x^2}, \quad x \in \mathbb{R}, \quad \frac{dy}{dx} > 0.$$

(iii) Hence find a primitive of $\frac{x^2}{1+x^2}$.

3

Solution:

$$\begin{aligned}
 \int \frac{x^2 dx}{1+x^2} &= \int \left(1 - \frac{1}{1+x^2} \right) dx, \\
 &= x - \tan^{-1} x + c.
 \end{aligned}$$

Question 10 (21 marks) (use a separate answer booklet)

(a) A company has to place three orders for supplies among five distributors. Each order is randomly assigned, and a distributor may receive multiple orders. Find the probabilities of the following events.

(i) Each order goes to a different distributor.

Solution: Total of all arrangements = $5 \times 5 \times 5$,
 $= 125$.
 Ways of distributing = $5 \times 4 \times 3$,
 $= 60$.
 $\therefore P(\text{each different}) = \frac{60}{125}$,
 $= \frac{12}{25}$.

1

(ii) All orders go to the same distributor.

Solution: There are only 5 ways this can occur,
 so $P(\text{all to same}) = \frac{5}{125}$,
 $= \frac{1}{25}$.

1

(iii) Exactly two of the three go to one of the distributors.

Solution: Method 1—
 This is the only other possibility
 so $P(2 \text{ of } 3 \text{ to } 1) = 1 - (\frac{12}{25} + \frac{1}{25})$,
 $= \frac{12}{25}$.
Method 2—
 $\binom{3}{2} \times 5 \times 4 = 60$.
 So $P(2 \text{ of } 3 \text{ to } 1) = \frac{60}{125}$,
 $= \frac{12}{25}$.

1

(b) (i) Use De Moivre's theorem to express $\cos 3\theta$ in terms of $\cos \theta$ and $\sin 3\theta$ in terms of $\sin \theta$.

Solution: $\cos 3\theta + i \sin 3\theta = (\cos \theta + i \sin \theta)^3$,
 $= \cos^3 \theta + 3i \cos^2 \theta \sin \theta - 3 \cos \theta \sin^2 \theta - i \sin^3 \theta$.
 $\cos 3\theta = \cos^3 \theta - 3 \cos \theta \sin^2 \theta$ (real part),
 $= \cos^3 \theta - 3 \cos \theta (1 - \cos^2 \theta)$,
 $= 4 \cos^3 \theta - 3 \cos \theta$.
 $\sin 3\theta = 3 \cos^2 \theta \sin \theta - \sin^3 \theta$ (imaginary part),
 $= 3(1 - \sin^2 \theta) \sin \theta - \sin^3 \theta$,
 $= 3 \sin \theta - 4 \sin^3 \theta$.

2

Marks

(ii) Use the result to solve the equation

$$8x^3 - 6x + 1 = 0.$$

3

Solution: If we put $x = \cos \theta$, then
 $8 \cos^3 \theta - 6 \cos \theta + 1 = 0$,
 $4 \cos^3 \theta - 3 \cos \theta = -\frac{1}{2}$,
i.e. $\cos 3\theta = -\frac{1}{2}$,
 $3\theta = \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}, \frac{10\pi}{3} \dots$
 $\theta = \frac{2\pi}{9}, \frac{4\pi}{9}, \frac{8\pi}{9}, \frac{10\pi}{9} \dots$
 $x = \cos \frac{2\pi}{9}, \cos \frac{4\pi}{9}, \cos \frac{8\pi}{9}, \cos \frac{10\pi}{9} \dots$
 As it is a cubic, only 3 values of x are possible.
 Now, $\cos \frac{8\pi}{9} = -\cos \frac{\pi}{9}$, $\cos \frac{10\pi}{9} = -\cos \frac{\pi}{9}$, etc.
 Hence $x = -\cos \frac{\pi}{9}, \cos \frac{2\pi}{9}, \cos \frac{4\pi}{9}$ only.

(c) When a rational integral function is divided by $(x - \alpha)^2$ the remainder is $R_2(x - \alpha) + R_1$, *i.e.* $f(x) = (x - \alpha)^2 Q(x) + R_2(x - \alpha) + R_1$.

(i) Prove that $R_1 = f(\alpha)$ and $R_2 = f'(\alpha)$, where $f'(x)$ is the differential coefficient of $f(x)$ with respect to x .

2

Solution: $f(x) = (x - \alpha)^2 Q(x) + R_2(x - \alpha) + R_1 \dots \dots \dots$ [1]
 $f(\alpha) = (\alpha - \alpha)^2 Q(\alpha) + R_2(\alpha - \alpha) + R_1$,
 $= 0 + 0 + R_1$.
i.e. $R_1 = f(\alpha)$.
 From [1], differentiating w.r.t. x ,
 $f'(x) = 2(x - \alpha)Q(x) + (x - \alpha)^2 Q'(x) + R_2 + (x - \alpha)R_2'$,
 $\therefore f'(\alpha) = 2(\alpha - \alpha)Q(\alpha) + (\alpha - \alpha)^2 Q'(\alpha) + R_2 + (\alpha - \alpha)R_2'$,
 $= 0 + 0 + R_2 + 0$,
i.e. $R_2 = f'(\alpha)$.

(ii) Show that $x^n - nx + n - 1$ is exactly divisible by $(x - 1)^2$ for any integral value of n greater than 1.

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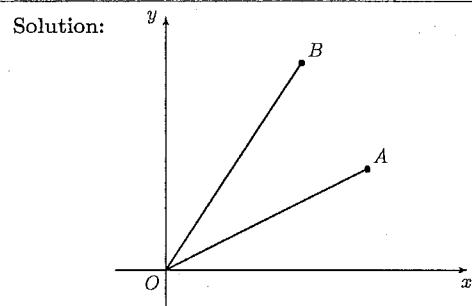
Solution: Let $f(x) = x^n - nx + n - 1$,
 $f(1) = 1^n - n + n - 1$,
 $= 0 = R_1$.
 $f'(x) = nx^{n-1} - n$,
 $f'(1) = n \cdot 1^{n-1} - n$,
 $= n - n = 0 = R_2$.
 $\therefore f(x) = (x - 1)^2 Q(x) + 0 + 0$,
i.e. $x^n - nx + n - 1$ is divisible by $(x - 1)^2$.
 n must be an integer since $x^n - nx + n - 1$ must be a rational function before the above procedure can be used.

- (iii) Deduce that $2^{4n} - 15n - 1$ is exactly divisible by 225 for any integral value of n greater than 1.

Solution: $x^n - nx + n - 1 = x^n - (x-1)n - 1$,
 Taking the $x = 16$ we have
 $x^n - (x-1)n - 1 = 16^n - 15n - 1$,
 $= 2^{4n} - 15n - 1$.

However $x^n - (x-1)n - 1$ is divisible by $(x-1)^2, n > 1$.
 $\therefore 2^{4n} - 15n - 1$ is divisible by $(15)^2$ for any integral value of $n > 1$.
 i.e. $2^{4n} - 15n - 1$ is divisible by 225 for any integral value of n greater than 1.

- (d) (i) If a, b are the complex numbers represented by points A, B in the Argand diagram, what geometrical properties correspond to the modulus and argument of b/a ?



The lengths of the vectors \vec{OA}, \vec{OB} are equal respectively to $|a|$ and $|b|$ and the angles $\angle xOA, \angle xOB$ are equal to $\arg a$ and $\arg b$.

Since $\left| \frac{b}{a} \right| = \frac{|b|}{|a|} = \frac{OB}{OA}$,

the modulus of b/a corresponds to the ratio OB/OA .

Again, since $\arg \left(\frac{b}{a} \right) = \arg b - \arg a = \angle xOB - \angle xOA$,
 $= \angle AOB$.

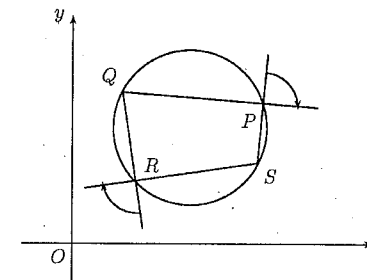
- (ii) Show that, if the four points representing the complex numbers z_1, z_2, z_3, z_4 are concyclic, the fraction

$$\frac{(z_1 - z_2)(z_3 - z_4)}{(z_3 - z_2)(z_1 - z_4)}$$

must be real.

Solution: Let the representative points of the four complex numbers z_1, z_2, z_3, z_4 be P, Q, R, S .

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Clearly, the sum of the marked angles is π (opposite \angle s of a cyclic quad. and vert. opp. \angle s). Since vectors \vec{SP}, \vec{QP} represent respectively the complex numbers $z_4 - z_1$ and $z_2 - z_1$, the angle of turn from \vec{SP} to \vec{QP} is

$$\arg \left(\frac{z_2 - z_1}{z_4 - z_1} \right).$$

Similarly the angle of turn from \vec{QR} to \vec{SR} is $\arg \left(\frac{z_4 - z_3}{z_2 - z_3} \right)$. Hence

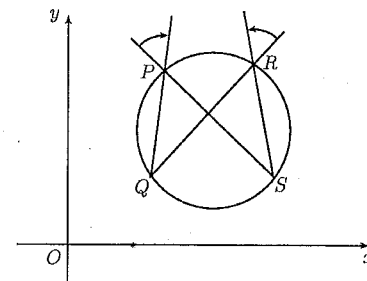
$$\arg \left(\frac{z_2 - z_1}{z_4 - z_1} \right) + \arg \left(\frac{z_4 - z_3}{z_2 - z_3} \right) = \pi.$$

But the sum of the arguments of two complex numbers is equal to the argument of their product, so that

$$\arg \left(\frac{z_2 - z_1}{z_4 - z_1} \times \frac{z_4 - z_3}{z_2 - z_3} \right) = \pi.$$

If the argument of a complex number is π , the number is a negative real number as required.

NOTE: even if we label the points around the circle in some other, non-cyclic order:



Then the sum of the arguments is zero and we have a positive real number.

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