SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2013
YEAR 12 Mathematics Extension 1
HSC Task #2

Mathematics Extension 1

General Instructions

- Reading Time – 5 Minutes
- Working time – 90 Minutes
- Write using black or blue pen. Pencil may be used for diagrams.
- Board approved calculators maybe used.
- Each Section is to be returned in a separate bundle.
- Marks may **NOT** be awarded for messy or badly arranged work.
- All necessary working should be shown in every question.
- Answers must be given in simplest exact form unless otherwise stated.

Total marks - 65

Multiple Choice Section (5 marks)
- Answer Questions 1-5 on the Multiple Choice answer sheet provided.

Sections A, B and C (60 marks)
- Start a new answer booklet for each section.

Examiner: *D.McQuillan*

STANDARD INTEGRALS

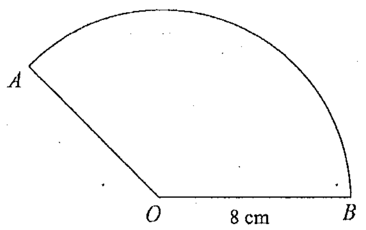
$$\int x^n dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0, \text{if } n < 0$$
$$\int \frac{1}{x} dx = \ln x,\ x > 0$$
$$\int e^{ax} dx = \frac{1}{a}e^{ax},\ a \neq 0$$
$$\int \cos ax\, dx = \frac{1}{a}\sin ax,\ a \neq 0$$
$$\int \sin ax\, dx = -\frac{1}{a}\cos ax,\ a \neq 0$$
$$\int \sec^2 ax\, dx = \frac{1}{a}\tan ax,$$
$$\int \sec ax \tan ax\, dx = \frac{1}{a}\sec ax,\ a \neq 0$$
$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ a \neq 0$$
$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\frac{x}{a},\ a > 0,\ -a < x < a$$
$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right),\ x > a > 0$$
$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x,\ x > 0$

Multiple Choice Section (5 marks)

Use the multiple-choice answer sheet for Questions 1-5

1 AOB is a sector of a circle, centre O and radius 8 cm. The sector has an area of 20π cm^2.



Not to scale

What is the arc length of the sector?

(A) 2π

(B) 5π

(C) 8π

(D) 10π

2 How many ways can 3 boys and 2 girls be arranged about a circular table?

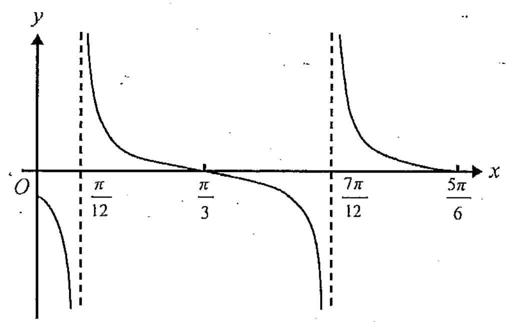
(A) 12

(B) 24

(C) 60

(D) 120

3 Part of the graph of $y = f(x)$ is show below



$f(x)$ could be

(A) $y = -\tan\left(2x - \frac{\pi}{6}\right)$

(B) $y = -\tan\left(2x - \frac{\pi}{3}\right)$

(C) $y = \cot\left(2x - \frac{\pi}{12}\right)$

(D) $y = \cot\left(2x - \frac{\pi}{6}\right)$

4 On a particular day, the temperature y, in degrees Celsius, can be modelled by the function with equation $y = 18 - 5\sin\left(\frac{\pi t}{12}\right)$, where t is the time in hours after midnight. The maximum temperature for this particular day occurs at

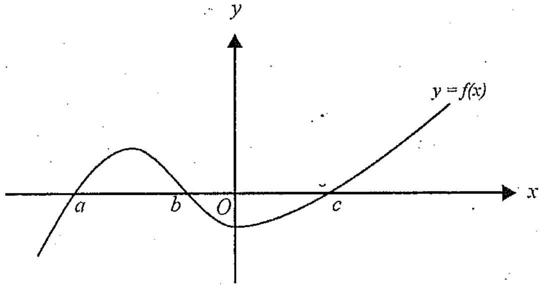
(A) 3.00 am

(B) 6.00 am

(C) 12.00 noon

(D) 6.00 pm

5 Part of the graph of the function $f(x)$ is shown below.



The total area, bounded by the curve of $y = f(x)$ and the x-axis on the interval $a \leq x \leq c$, is given by

(A) $\int_a^c f(x)\,dx$

(B) $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx$

(C) $-\int_a^0 f(x)\,dx + \int_0^c f(x)\,dx$

(D) $\int_a^b f(x)\,dx + \int_c^b f(x)\,dx$

End of Multiple Choice Section

Section A

Start a new writing booklet for each section

Question 6

(a) Find [3]

(i)

$$\int \frac{x^4}{7} dx$$

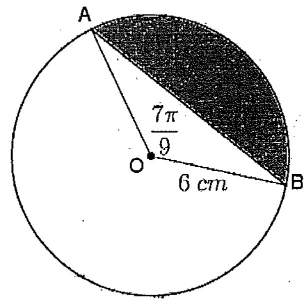
(ii)

$$\int \frac{7}{x^4} dx$$

(b) Evaluate [2]

$$\int_1^2 2t(t^2 - 1)dt$$

(c) [3]



(i) Find the length of the minor arc AB.

(ii) Find the area of the shaded segment.

(d) A restaurant offers these choices: [3]

ENTRÉE	MAIN COURSE	DESSERT
Garlic prawns	Fillet steak	Strawberries
Soup of the day	Chicken	Apple pie and cream
Oysters	Fish	

(i) How many different 3 course dinners can be chosen?

(ii) If I was late and someone ordered for me what is the probability that they would choose what I wanted, assuming they knew I did not like oysters?

Question 7

(a) Sketch the graph of $y = 1 - \cos(\pi x)$ on the domain $-1 \le x \le 2$. [3]

(b) Find the area of the region in the first quadrant bounded by the graphs of $y = \frac{1}{8}x^3$ and $y = 2x$. [4]

(c) [4]

(i) How many arrangements can be made with the letters of the word ARRANGE?

(ii) How many arrangements can be made with the letters of the word ARRANGE if the R's must remain together?

End of Section

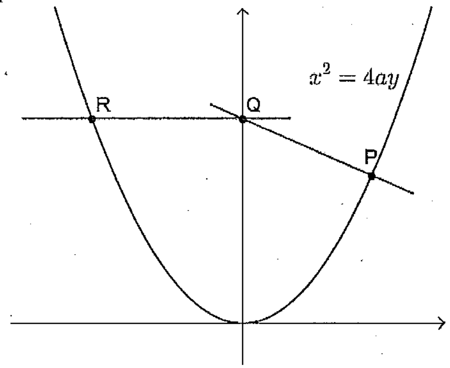
Section B

Start a new writing booklet for each section

Question 8

(a) Use the Trapezoidal Rule, with 3 function values, to approximate the volume generated by rotating $y = \sin x$ about the x-axis between $x = 0$ and $x = \frac{\pi}{2}$. [4]

(b) The diagram below shows the graph of the parabola $x^2 = 4ay$. The normal to the parabola at the variable point $P(2at, at^2), t > 0$, cuts the y-axis at Q. Point R lies on the parabola. [6]



(i) Show that the equation of the normal to the parabola at P is $x + ty = at^3 + 2at$.

(ii) Find the coordinates of R given that QR is parallel to the x-axis and $\angle PQR > 90°$.

(iii)Let M be the midpoint of RQ. Find the Cartesian equation of the locus of M.

Question 9

(a) A committee of 6 is to be chosen from 8 men and 6 women so as to contain at least 3 men and 2 women. [4]

(i) In how many ways can this be done?

(ii) In how many ways can it be done if 2 particular men refuse to serve together?

(b) Use induction to prove that if n is positive and odd, then $4^n + 5^n + 6^n$ is divisible by 15. [5]

End of Section

Section C

Start a new writing booklet for each section.

Question 10

(a) Show that Newton's method fails when applied to the equation $\sqrt[3]{x} = 0$ with any initial approximation $x_1 \neq 0$. [3]

(b) A grain silo consists of a cylindrical main section, with height 8 metres, and a hemispherical roof. [6]

(i) In order to achieve a total volume of 60 m^3 (including the part inside the roof section) show that one of the solutions of $\pi r^3 + 12\pi r^2 - 90 = 0$ is the necessary radius, r, of the silo.

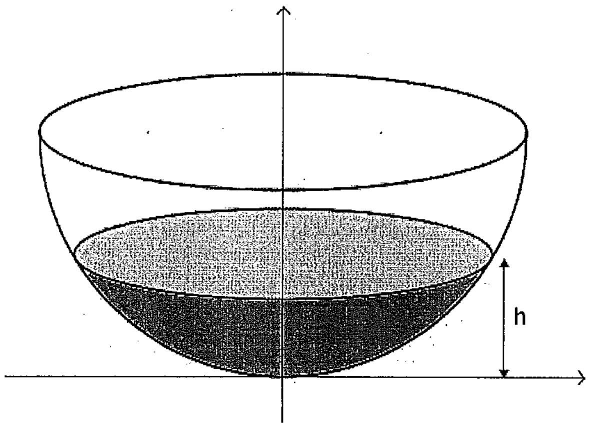
(ii) Use Newton's method with a first approximation of radius $r_1 = 1$ metre to find the third approximation, r_3, to 3 decimal places.

Question 11

(a) [6]

(i) Show that the equation of the lower semi-circle with centre $(0, 20)$ and radius 20 is $y = 20 - \sqrt{400 - x^2}$.

(ii) Find the volume of the solid generated by rotating the semi-circle around the y-axis between the points $y = 0$ and $y = h$.



(iii) If 171π cm^3 of water is poured into a semi-circular bowl with radius 20 cm. What will be the height of the water in the bowl?

(b) Suppose that three points on the parabola $y = x^2$ have the property that their normal lines intersect at a common point. Show that the sum of their x-coordinates is 0. [4]

End of Exam

Student Number: **SOLUTIONS:**

Mathematics Extension 1
Task 2 2013

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample: 2 + 4 = (A) 2 (B) 6 (C) 8 (D) 9

A ○ B ● C ○ D ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ● B ☒ C ○ D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate the correct answer by writing the word *correct* and drawing an arrow as follows.

A ☒ B ☒ (correct) C ○ D ○

Section I: **Multiple choice answer sheet.**

Completely colour the cell representing your answer. Use black pen.

1. A ● C D
2. A ● C D
3. A B C ●
4. A B C ●
5. A B C ●

Question 6.

(a)(i) $\frac{x^5}{35}$ [1] (ii) $-\frac{7}{3x^3}$ [2]

(b) $\int_1^2 (2t^3 - 2t)\,dt = \left[\frac{t^4}{2} - t^2\right]_1^2$

$= 8 - 4 - (\frac{1}{2} - 1) = 4\frac{1}{2}$ [2]

(c)(i) $l = r\theta = 6 \times \frac{7\pi}{9} = \frac{14\pi}{3}$ cm [1]

(ii) $A = \frac{1}{2}r^2(\theta - \sin\theta) = \frac{1}{2} \times 36\left(\frac{7\pi}{9} - \sin\frac{7\pi}{9}\right)$

$= \left(14\pi - 18\sin\frac{7\pi}{9}\right)$ cm^2 [2]

(d)(i) $3 \times 3 \times 2 = 18$ choices [1]

(ii) $2 \times 3 \times 2 = 12$, probability $= \frac{1}{12}$ [2]

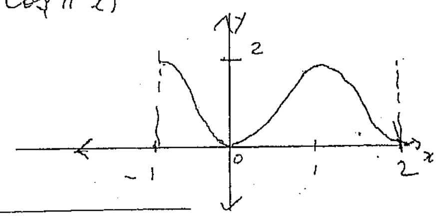
Question 7 (a) $y = 1 - \cos(\pi x)$

$x = -1$ $y = 2$

$x = 0$ $y = 0$

$x = 1$ $y = 2$

$x = 2$ $y = 0$



[3]

(b)(i) $2x = \frac{1}{8}x^3$

$16 = x^2$, $x = \pm 4$

1st quad $\int_0^4 2x - \frac{1}{8}x^3\,dx$

$= \left[x^2 - \frac{1}{32}x^4\right]_0^4 = 16 - 8 - [0 - 0]$

$= 8u^2$ [4]

c(i) $\frac{7!}{2! \times 2!} = 1260$ arrangements [2]

(ii) $\frac{6! \times 2!}{2! \times 2!} = 360$ arrangements [2]

SECTION C

Q10. a) Aim to show that Newton's method fails for $\sqrt[3]{x}=0$ when 1st approx is x_1.

Let $f(x) = x^{\frac{1}{3}}$.

$\therefore f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$.

Now $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$= x_n - \frac{(x_n)^{\frac{1}{3}}}{\frac{1}{3}(x_n)^{-\frac{2}{3}}}$

$= x_n - 3.(x_n)^1$

$= -2x_n$.

$\therefore x_{n+1} = -2x_n$

now if 1st approx. is x_1

then $x_2 = -2x_1$

$x_3 = -2\times-2x_1$

$= 4x_1$

$x_4 = -2x_3$

$= -8x_1$ etc.

$\therefore \{x_1, -2x_1, +4x_1, -8x_1, \dots\}$ which represents the set of consecutive approximations

Clearly the method fails!

8(a) $y = \sin x$

$V = \pi \int y^2 . dx$

$= \pi \int_0^{\frac{\pi}{2}} \sin^2 x . dx$

$y = \sin^2 x$

x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$
y	0	$\frac{1}{2}$	1

$= \pi \left[\frac{\pi}{8}\{0 + 2\times\frac{1}{2} + 1\}\right]$ (TRAPEZOIDAL)

$= \frac{\pi^2}{4}\ u^3$

5 (i) $x^2 = 4ay$

$\frac{dy}{dx} = \frac{x}{2a}$

$\therefore$ Tangent gradient $= t$ at $x = 2at$

$\therefore$ Normal gradient $= -\frac{1}{t}$

$\therefore$ Normal Equation at $(2at, at^2)$

$y - at^2 = -\frac{1}{t}(x - 2at)$

$x + ty = at^3 + 2at$

(ii)

For Q sub $x = 0$ in Eqn. of normal

$0 + t.y = at^3 + 2at$

$y = at^2 + 2a$

$\therefore$ Q is $(0, at^2 + 2a)$

Sub $y = at^2 + 2a$ in $x^2 = 4ay$ for R

$\therefore x^2 = 4a(at^2 + 2a)$

$x = -2a\sqrt{t^2+2}$

R is $(-2a\sqrt{t^2+2}, at^2+2a)$

(iii) M, M.P of QR is $(-a\sqrt{t^2+2}, at^2+2a)$

Eliminate t^2 in $x^2 = a^2(t^2+2)$ and $y = at^2+2a$

$\therefore x^2 = ay$ is locus

since $t > 0$ we must have $x < 0$

Locus of M $\therefore x = -\sqrt{ay}$

4(a)(i) Committee must have 3 men or 4 men exactly.

No. with exactly

3 men $= {}^8C_3 \times {}^6C_3$

$= 56 \times 20$

$= 1120$

4 men $= {}^8C_4 \times {}^6C_2$

$= 70 \times 15$

$= 1050$

$\therefore$ Total $= 2170$

(ii) Find the number of committees possible with the 2 men in it

3 men $= {}^6C_1 \times {}^6C_3 = 120$

4 men $= {}^6C_2 \times {}^6C_2 = 225$

$\therefore$ Total with the 2 men in $= 345$.

$\therefore$ Total without 2 men in $= 2170 - 345$

$= 1825$

(b) Since n is positive and odd we test for $n = 1$

$4^1 + 5^1 + 6^1 = 15$ $\therefore$ true for $n = 1$

Assume true for $n = k$

ie $4^k + 5^k + 6^k = 15P$ (where P = integer)

We need to show true for $n = k+2$

ie $4^{k+2} + 5^{k+2} + 6^{k+2}$ is a multiple of 15

$4^{k+2} + 5^{k+2} + 6^{k+2}$

$= 16\times4^k + 25\times5^k + 36\times6^k$

$= 16(4^k + 5^k + 6^k) + 9\times5^k + 20\times6^k$

{FROM ASSUMPTION} $= 16\times15P + 5\times3\times3\times5^{k-1} + 5\times4\times6\times6^{k-1}$

$= 15(16P + 3\times5^{k-1} + 8\times6^{k-1})$

The contents of the parentheses is an integer for all integral $k > 0$)

$\therefore$ True for $n = k+2$

True for all n, since true for n=

b (i) $V = \frac{1}{2} \times \frac{4}{3}\pi r^3 + \pi r^2 \times 8.$

$\therefore 60 = \frac{2}{3}\pi r^3 + 8\pi r^2.$

$180 = 2\pi r^3 + 24\pi r^2.$

$90 = \pi r^3 + 12\pi r^2$

$\therefore \boxed{\pi r^3 + 12\pi r^2 - 90 = 0}$

(ii). $r_2 = 1 - \frac{f(r_1)}{f'(r_1)}$ where $f(r) = \pi r^3 + 12\pi r^2 - 90.$

$f'(r) = 3\pi r^2 + 24\pi.$

$= 1 - \frac{13\pi - 90}{27\pi}.$

$= 1.57955 - - .$

$\therefore r_3 = 1.57955 - \frac{f(1.57955)}{f'(1.57955)}$ [USE A CALCULATOR]

$\boxed{r_3 = 1.464}$ correct to 3 DEC. PLACES

Q11 (a) (i) Consider the circle centre (0,20) radius 20.

$x^2 + (y-20)^2 = 20^2.$ —— (A)

$(y-20)^2 = 400 - x^2$

$y - 20 = \pm\sqrt{400 - x^2}$

$y = 20 \pm \sqrt{400 - x^2}$

$\therefore$ lower semi-circle is $\boxed{y = 20 - \sqrt{400 - x^2}}$

(ii) $V = \pi \int_0^h x^2\, dy$

Now from (A)

$x^2 = 20^2 - (y-20)^2$

$= 400 - (y^2 - 40y + 400)$

$= 40y - y^2$

$= \pi \int_0^h (40y - y^2)\, dy.$

$= \pi \left[20y^2 - \frac{y^3}{3} \right]_0^h$

$= \pi \left(20h^2 - \frac{h^3}{3} \right)$

ie. $\boxed{20\pi h^2 - \frac{\pi h^3}{3}}$

(iii) let $20\pi h^2 - \frac{\pi h^3}{3} = 171\pi$

$60h^2 - h^3 = 171 \times 3$

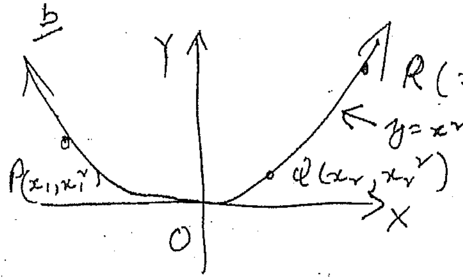
$h^3 - 60h^2 + 513 = 0$

now $h = 3$ is a root of $f(h) = h^3 - 60h^2 + 513$ because

$f(3) = 27 - 540 + 513 = 0.$

$\therefore \boxed{h = 3\text{cm}}$

b



Slope of normal at $P(x_1, x_1^2)$ is $\frac{-1}{2x_1}$.

$\therefore$ eqn. of normal at P.

$$\frac{y - x_1^2}{x - x_1} = \frac{-1}{2x_1}$$

$$y - x_1^2 = \frac{-(x - x_1)}{2x_1}$$

$$y - x_1^2 = \frac{-x}{2x_1} + \frac{1}{2}.$$

at Q

$$y - x_2^2 = -\frac{x}{2x_2} + \frac{1}{2}$$

✓ at R.

$$y - x_3^2 = \frac{-x}{2x_3} + \frac{1}{2}$$

Consider pt of intersection of normals at PQ

$$y - x_1^2 = \frac{-x}{2x_1} + \frac{1}{2} \quad \text{— (1)}$$

$$y - x_2^2 = \frac{-x}{2x_2} + \frac{1}{2} \quad \text{— (2)}$$

$$-x_1^2 + x_2^2 = -\frac{x}{2}\left(\frac{1}{x_1} - \frac{1}{x_2}\right)$$

$$x_2^2 - x_1^2 = -\frac{x}{2}\left(\frac{x_2 - x_1}{x_1 x_2}\right)$$

$$x = -2x_1 x_2 (x_1 + x_2)$$

Similarly for normals at P and R.

$$x = -2x_1 x_3 (x_1 + x_3)$$

now these two points are co-incident

$$\therefore -2x_1 x_2 (x_1 + x_2) = -2x_1 x_3 (x_1 + x_3)$$

$$\therefore x_2 (x_1 + x_2) = x_3 (x_1 + x_3)$$

$$x_1 x_2 + x_2^2 = x_1 x_3 + x_3^2.$$

$$x_2^2 - x_3^2 = x_1 (x_3 - x_2)$$

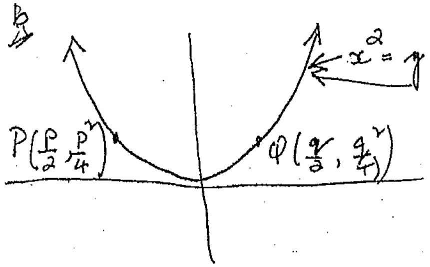
$$(x_2 - x_3)(x_2 + x_3) = x_1 (x_3 - x_2)$$

$$(x_2 + x_3) = -x_1$$

$$\therefore \boxed{x_1 + x_2 + x_3 = 0.} \quad \text{QED}$$

OR AN ALTERNATIVE ON NEXT PAGE(S)

b



$R\left(\frac{r}{2}, \frac{r^2}{4}\right)$

note $a = \frac{1}{4}$

Consider P

Slope of tangent $= p$

Slope of normal $= -\frac{1}{p}$

$\therefore$ eqn. of normal at P

$$\frac{y - \frac{p^2}{4}}{x - \frac{p}{2}} = -\frac{1}{p}$$

$$py - \frac{p^3}{4} = -x + \frac{p}{2}$$

$$4py - p^3 = -4x + 2p$$

$$\boxed{4x + 4py = p^3 + 2p} \quad \text{normal at P.}$$

$$\therefore \boxed{4x + 4qy = q^3 + 2q} \quad \text{normal at Q}$$

$$\boxed{4x + 4ry = r^3 + 2r} \quad \text{normal at R.}$$

Find the intersection of normals at P & Q.

$$4x + 4py = p^3 + 2p \quad (1)$$

$$4x + 4\ y = q^3 + 2q \quad (2)$$

$$y(4p - 4q) = p^3 - q^3 + 2(p - q)$$

$$4y = (p^2 + 2pq + q^2 + 2)$$

Sub in (1)

$$4x + p(p^2 + pq + q^2 + 2) = p^3 + 2p.$$

$$4x + p^2q + pq^2 = 0$$

$$4x = -pq(p+q)$$

$$x = -\frac{pq(p+q)}{4}$$

Similarly for normals at P & R

$$x = -\frac{pr(p+r)}{4}$$

and normals of Q & R

$$x = -\frac{qr(q+r)}{4}$$

now these x-values are all equal

$$\therefore -\frac{pq(p+q)}{4} = -\frac{pr(p+r)}{4}$$

$$q(p+q) = r(p+r)$$

$$pq + q^2 = rp + r^2$$

$$pq - pr + q^2 - r^2 = 0$$

$$p(q-r) + (q-r)(q+r) = 0$$

$$(q-r)(p+q+r) = 0$$

Clearly $r \neq q$

$$\therefore p + q + r = 0$$

$$\therefore \boxed{\frac{p}{2} + \frac{q}{2} + \frac{r}{2} = 0} \quad \text{QED}$$