Student Name: ______________________________

Class Teacher: ______________________________

St George Girls High School

Year 12

Mid-HSC Course Examination

2016

Mathematics Extension 2

General Instructions

- Working time – 90 minutes
- Reading time – 5 minutes
- Write using blue or black pen.
- Board-approved calculators may be used.
- A multiple choice answer sheet is provided for Section I
- A reference sheet is provided at the end of this paper.
- All drawn graphs should take up at least one third of a page.

Total marks – 60

Section I
Total marks (5)
Attempt Questions 1 -5
Use the answer sheet provided

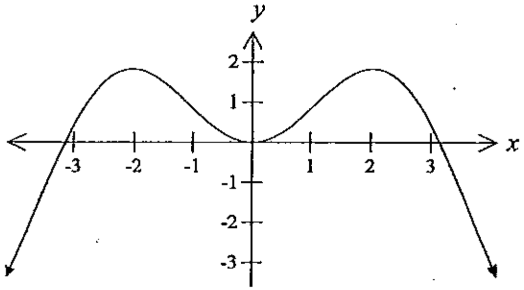
Section II
Total marks (55)
Attempt Questions 6 – 7
Start each question in a new booklet

Section I:

5 marks
Allow 10 minutes for this section

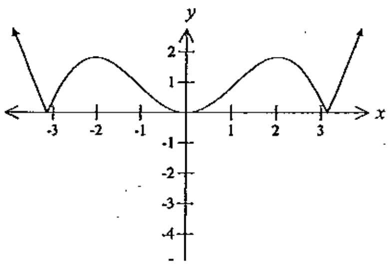
Use the multiple choice answer sheet to record your answers for Questions 1 - 5

1. The diagram below shows the graph of the function $y = f(x)$.

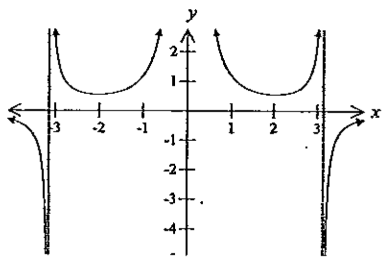


Which of the following is the graph of $y = |f(x)|$?

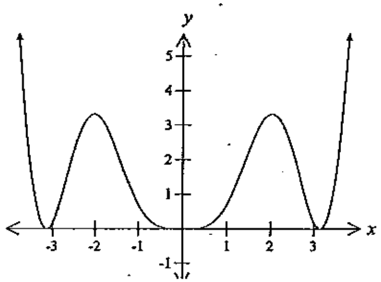
A



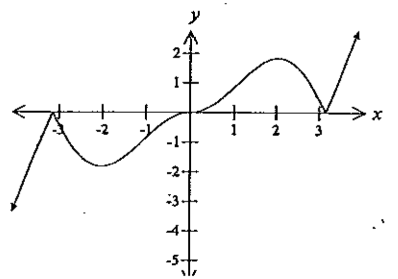
B



C



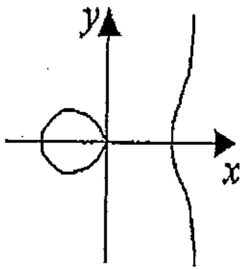
D



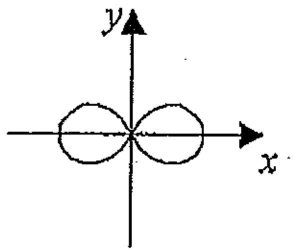
Section I (cont'd)

2. Which sketch best represents the graph of the equation $y^2 = x(x^2 - 1)$?

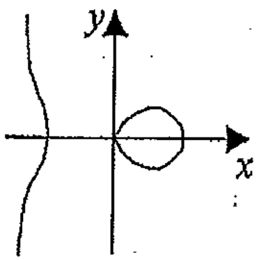
A



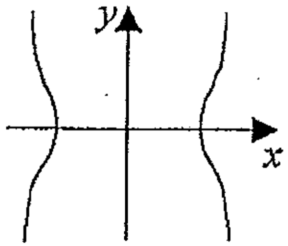
B



C



D



3. An ellipse with equation $\frac{x^2}{9} + \frac{y^2}{b^2} = 1$ has its foci at $S\left(\sqrt{5}, 0\right)$ *and* $S'(-\sqrt{5}, 0)$. The value of b is:

A 1

B 2

C $\sqrt{5}$

D 4

Section I (cont'd)

4. The coordinates of the foci of the hyperbola with equation $4x^2 - 9y^2 + 8x + 18y - 41 = 0$ are:

A $S\left(\sqrt{13} - 1, -1\right)$, $S'(-\sqrt{13} - 1, -1)$

B $S\left(\sqrt{13}, 0\right)$, $S'(-\sqrt{13}, 0)$

C $S\left(\sqrt{13} + 1, -1\right)$, $S'(-\sqrt{13} + 1, -1)$

D $S\left(\sqrt{13} - 1, 1\right)$, $S'(-\sqrt{13} - 1, 1)$

5. The equation of the chord of contact to the hyperbola $xy = 4$ from the point $T(2, 6)$ is:

A $x + 3y = 4$

B $2x + 6y = 1$

C $3x + y = 4$

D $2x - 6y = 1$

Section II:

Total marks (55)
Allow about 1 hour and 20 minutes for this section.

Answer all questions, starting each question in a new booklet with your name and question number.
All necessary working should be shown in every question.

Marks

Question 6 (27 marks) Use a separate writing booklet

a) Let $f(x) = \frac{1-x}{x}$. On separate diagrams sketch the graphs of the following functions. For each graph label the asymptotes.

(i) $y = f(x)$ 2

(ii) $y = f(|x|)$ 2

(iii) $y = e^{f(x)}$ 2

(iv) $y^2 = f(x)$ 2

Discuss the behaviour of the curve of (iv) at $x = 1$. 1

b) $\frac{x^2}{100} + \frac{y^2}{16} = 1$ is the equation of the ellipse E.

(i) Find:

I. The eccentricity of E. 2

II. The coordinates of the foci, S and S'. 1

III. The equations of the directrices. 1

(ii) If Q is an arbitrary point on E show that $SQ + S'Q$ is a constant. 3

Question 6 (cont't) **Marks**

c) (i) Show that the normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $(a^2 > b^2)$ at the point $P(x_1, y_1)$ has equation

$$a^2 y_1 x - b^2 x_1 y = (a^2 - b^2) x_1 y_1.$$

2

(ii) This normal meets the major axis of the ellipse at G. S is one focus of the ellipse. Show that G has coordinates $(e^2 x, 0)$ and hence show that $GS = ePS$ (where e is the eccentricity of the ellipse). 4

(iii) Using the result in c) (i) above, or otherwise, show that the normal to the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$ at the point $P(5\cos\theta, 3\sin\theta)$ has equation:

$$5x\sin\theta - 3y\cos\theta = 16\sin\theta\cos\theta$$

1

(iv) This normal cuts the major and minor axes of the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$ at G and H respectively. Show that as $P(5\cos\theta, 3\sin\theta)$ moves on the ellipse, the locus of the midpoint of GH describes another ellipse with the same eccentricity as the ellipse in c) (iii). 4

Marks

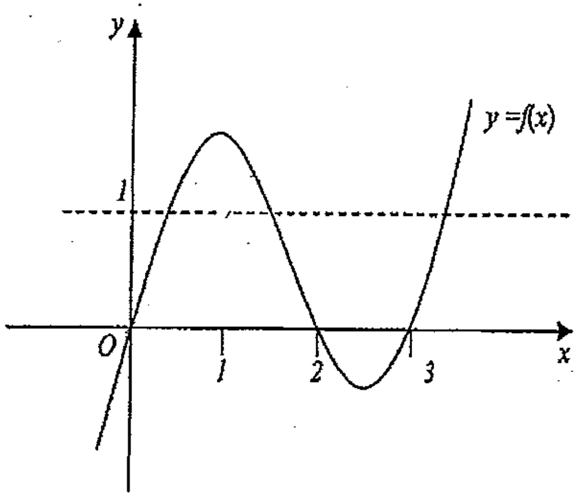
Question 7 (28 marks) Use a separate writing booklet

a) (i) On the same set of axes, sketch and label clearly the graphs of the functions $y = x^{\frac{1}{3}}$ and $y = e^x$. 2

(ii) Hence, on a different set of axes, without using calculus, sketch and label clearly the graph of the function $y = x^{\frac{1}{3}}e^x$. 3

(iii) Use your sketch to determine for which values of m the equation $x^{\frac{1}{3}}e^x = mx + 1$ has exactly one solution. 2

b) The diagram shows the graph of a function $f(x)$.



Sketch the following curves on separate diagrams:

(i) $y = \sqrt{f(x)}$ 3

(ii) $y = xf(x)$ 2

(iii) $y = \frac{1}{1-f(x)}$ 4

Marks

Question 7 (cont'd)

c) The hyperbola $\frac{x^2}{a^2} - \frac{y^2}{16} = 1$ has eccentricity $\sqrt{5}$

(i) Find the value of a, if $a > b$. 2

(ii) Find the equations of the asymptotes. 1

d) The hyperbola H has Cartesian equation $xy = 16$.

(i) Prove that the equation of the tangent to H at the point $P\left(4p, \frac{4}{p}\right)$ is $x + p^2y = 8p$. 3

(ii) If point Q has coordinates $\left(4q, \frac{4}{q}\right)$, show that the equation of the chord PQ is $x + pqy = 4(p + q)$. 2

(iii) The tangent at the point P and the tangent at the point Q intersect at T. Find the coordinates of T. 2

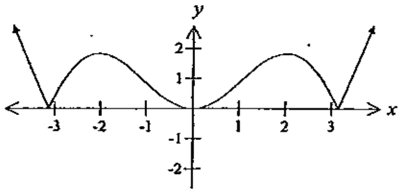
(iv) If the chord PQ passes through the point $R(8, 8)$, show that the locus of T lies on the line $x + y = 4$. 2

END OF EXAMINATION

Yr 12 Mid-Course Exam Mathematics Ext 2 2016 (1)

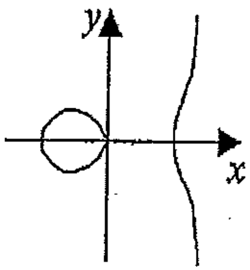
Solutions Section I

1.



A

2.



A

3. $ae = \sqrt{5}$

$3e = \sqrt{5}$

$e = \frac{\sqrt{5}}{3}$

$e^2 = \frac{5}{9}$

$b^2 = a^2(1-e^2)$

$b^2 = 9\left(1 - \frac{5}{9}\right)$

$= 9 \times \frac{4}{9}$

$= 4$

$b = 2 \quad (b > 0)$

B

4. $4x^2 + 8x + 4 - 9y^2 + 18y - 9 - 41 - 4 + 9 = 0$

$4(x+1)^2 - 9(y-1)^2 - 36 = 0$

$\frac{(x+1)^2}{9} - \frac{(y-1)^2}{4} = 1 \qquad a^2 = 9, \; b^2 = 4$

$a = 3 \qquad e^2 = 1 + \frac{4}{9}$

$= \frac{13}{9} \qquad e = \frac{\sqrt{13}}{3}$

$ae = 3 \times \frac{\sqrt{13}}{3}$

$= \sqrt{13}$

$(-1 \pm \sqrt{13}, 1)$

D

5. $6x + 2y = 2 \times 4$

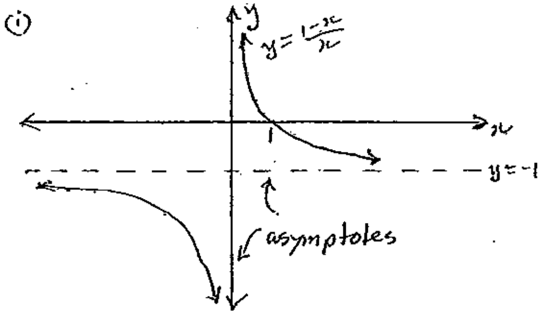
$3x + y = 4$

C

Solutions Section II	Marks / Comments

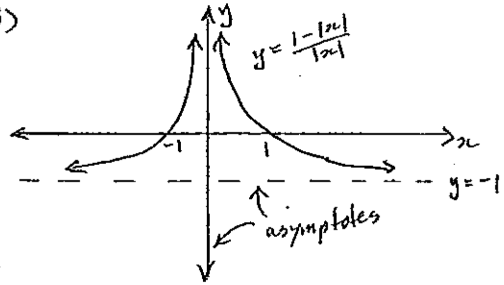
Question 6

a) (i)



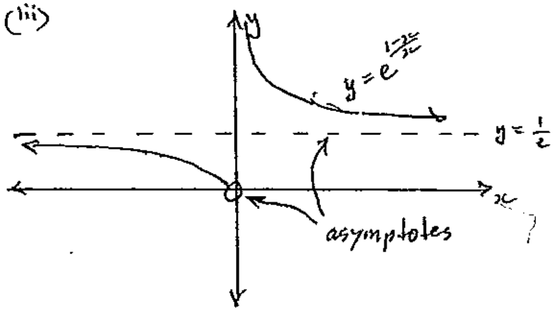
1 for graph
1 for asymptotes

(ii)



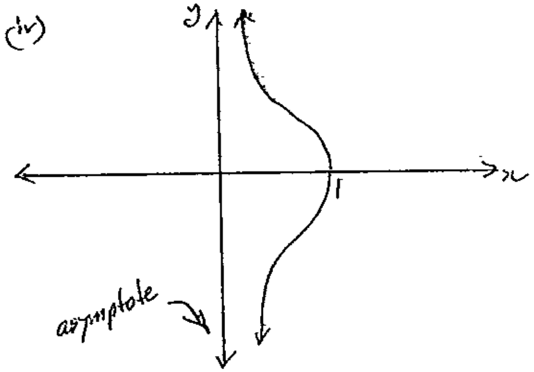
1 for graph
1 for asymptotes

(iii)



1 for graph
1 for asymptotes

(iv)



1 for graph
1 for asymptote

Solutions	Marks/Comments
With graph (iv) there is a vertical tangent at (1,0)	1 mark

b)(i) I. Let e be the eccentricty of E

$$b^2 = a^2(1-e^2)$$

$$\frac{b^2}{a^2} = 1 - e^2$$

$$e^2 = 1 - \frac{b^2}{a^2} \qquad b^2 = 16,\ a^2 = 100$$ 1

$$= 1 - \frac{16}{100}$$

$$= \frac{21}{25}$$

$$e = \frac{\sqrt{21}}{5} \quad (e > 0)$$ 1 (2)

II. $S(ae, 0)$ $S'(-ae, 0)$

$$S\left(10 \times \frac{\sqrt{21}}{5}, 0\right) \quad S'\left(-10, \frac{\sqrt{21}}{5}, 0\right)$$

$$= S(2\sqrt{21}, 0),\ S'(-2\sqrt{21}, 0)$$ 1 (1)

III. $x = \pm \frac{a}{e}$

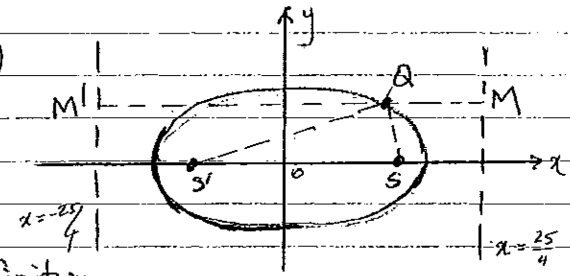
$$= \pm \frac{10}{\frac{\sqrt{21}}{5}}$$

$$= \pm 10 \times \frac{5}{\sqrt{21}}$$

$$= \pm \frac{50}{\sqrt{21}} \times \frac{\sqrt{21}}{\sqrt{21}}$$

$$= \pm \frac{50\sqrt{21}}{21}$$ 1 (1)

Q6b)ii)



By definition

$SQ = e\,QM$ and $S'Q = e\,QM'$ — 1 mark

$$= \frac{\sqrt{21}}{5}\left(\frac{50}{\sqrt{21}} - x_2\right) \qquad = \frac{\sqrt{21}}{5}\left(x_2 + \frac{50}{\sqrt{21}}\right)$$

$$SQ + S'Q = \frac{\sqrt{21}}{5}\left(\frac{50}{\sqrt{21}} - x_2\right) + \frac{\sqrt{21}}{5}\left(\frac{50}{\sqrt{21}} + x_2\right)$$ 1 mark

$$= \frac{\sqrt{21}}{5}\left(\frac{50}{\sqrt{21}} - x_2 + \frac{50}{\sqrt{21}} + x_2\right)$$

$$= \frac{\cancel{\sqrt{21}}}{5}\left(\frac{100}{\cancel{\sqrt{21}}}\right)$$

$$= 20$$ 1 mark (3)

$\Rightarrow$ which is a constant.

Alternative method

$SQ = e\,QM$ $\qquad S'Q = e\,QM'$

$= e\left(\frac{a}{e} - x_2\right)$ $\qquad = e\left(\frac{a}{e} + x_2\right)$ 1

$$SQ + S'Q = e\left(\frac{a}{e} - x_2\right) + e\left(\frac{a}{e} + x_2\right)$$ 1

$$= e\left(\frac{a}{e} + \frac{a}{e} - x_2 + x_2\right)$$

$$= e\left(\frac{2a}{e}\right)$$

$$= 2a$$ 1

$\therefore$ constant as $a = 10$. (3)

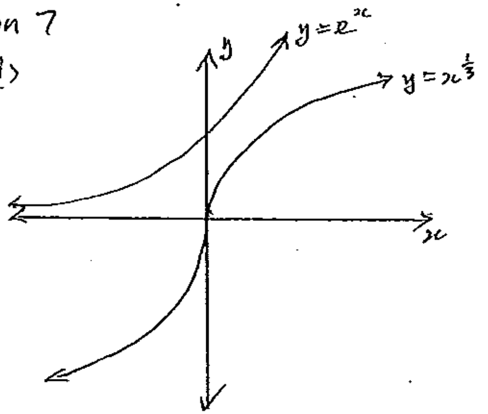
SOLUTIONS	MARKS / COMMENTS
c) (i) $\frac{d}{dx}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)=\frac{d}{dx}(1)$	
$\frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0$	
$\frac{dy}{dx}=\frac{-b^2x}{a^2y}$ ∴ $M_N=\frac{a^2y_1}{b^2x_1}$	1 mark
at (x_1,y_1) the equation of the normal is:	
$y-y_1=\frac{a^2y_1}{b^2x_1}(x-x_1)$	1 mark (2)
$a^2y_1x-b^2x_1y=(a^2-b^2)x_1y_1$	
(ii) substitute $y=0$	
$a^2y_1x=(a^2-b^2)x_1y_1$	1 mark
$x=\frac{a^2-b^2}{a^2}x_1$	
$=\frac{a^2-a^2(1-e^2)}{a^2}x_1$	since $b^2=a^2(1-e^2)$, $b^2=a^2-a^2e^2$
$=\frac{a^2e^2}{a^2}x_1$	
$=e^2x_1$	1 mark
$G(e^2x_1,0)$, $S(ae,0)$	
$PS=ePM$	
$=e\left(\frac{a}{e}-x_1\right)$	
$=a-ex_1$	1 mark
$GS=ae-e^2x_1$	
$=e(a-ex_1)$	1 mark
$=ePS$	(4)
(iii) $x_1=5\cos\theta$, $y_1=3\sin\theta$, $a=5$, $b=3$	
$25\times3\sin\theta.x-9\times5\cos\theta.y=(25-9)5\cos\theta.3\sin\theta$	
$5\times3\sin\theta.x-9\times\cos\theta.y=16\cos\theta3\sin\theta$	
$5\sin\theta.x-3\cos\theta.y=16\sin\theta\cos\theta$	1 mark
$5x\sin\theta-3y\cos\theta=16\sin\theta\cos\theta$	(1)

Solutions	Marks/comments
Q6 c) iv)	
From part iii)	
when $y=0$	
$5x\sin\theta=16\sin\theta\cos\theta$	
$x=\frac{16\sin\theta\cos\theta}{5\sin\theta}$	
$=\frac{16\cos\theta}{5}$ ∴ G is $\left(\frac{16}{5}\cos\theta,0\right)$	
when $x=0$	
$-3y\cos\theta=16\sin\theta\cos\theta$	
$y=\frac{-16}{3}\sin\theta$	
∴ H is $\left(0,-\frac{16}{3}\sin\theta\right)$	1
∴ Midpoint of GH is:	
$\left(\frac{16\cos\theta}{10},\frac{-16\sin\theta}{6}\right)$	
$=\left(\frac{8\cos\theta}{5},\frac{-8\sin\theta}{3}\right)$	
∴ $x=\frac{8\cos\theta}{5}$ and $y=\frac{-8\sin\theta}{3}$	
$\cos\theta=\frac{5x}{8}$ $\quad\sin\theta=\frac{-3y}{8}$	
$\sin^2\theta+\cos^2\theta=\left(\frac{5x}{8}\right)^2+\left(\frac{3y}{8}\right)^2$	
$=\frac{25}{64}x^2+\frac{9}{64}y^2$	1
∴ $\frac{25}{64}x^2+\frac{9}{64}y^2=1$ ---- (1)	
Eccentricity for (1): $a^2=b^2(1-e^2)$; $\frac{64}{25}=\frac{64}{9}(1-e^2)$; $e^2=1-\frac{9}{25}$; $=\frac{16}{25}$; $e=\frac{4}{5}$ $(e>0)$	
Eccentricity for $\frac{x^2}{25}+\frac{y^2}{9}=1$: $e^2=1-\frac{9}{25}$; $=\frac{16}{25}$; $e=\frac{4}{5}$ $(e>0)$	2
∴ the eccentricity is the same for both ellipses	(4)

SOLUTIONS | MARKS / COMMENTS

Question 7

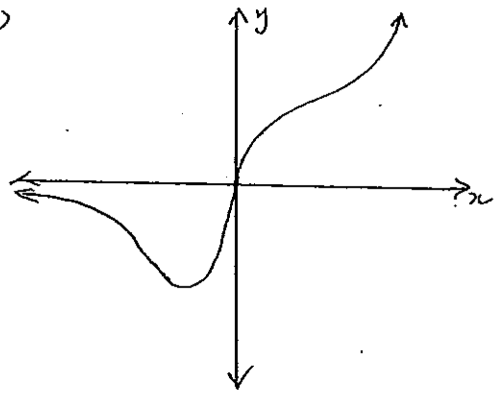
a) (i)



1 mark for each graph

(1)

(ii)



1 mark for shape about origin

1 mark for shape to right of origin

1 mark for shape to left of origin

(3)

(iii) If $m > 0$ the line will intersect the graph more than once so $m \leq 0$

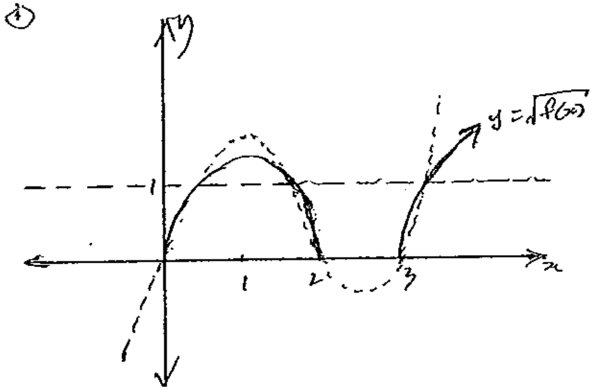
1 mark for ~~for~~ using the diagram

1 mark for answer

(2)

SOLUTIONS | MARKS / COMMENTS

7 b) (i)



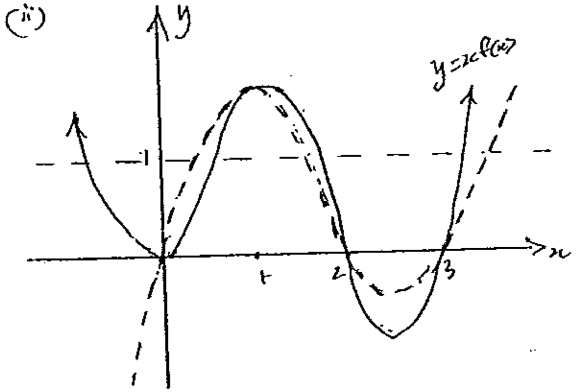
1 mark for matching points at $x = 0$ and $x = 1$

1 mark for no graph below x axis

1 mark for shapes of curve compared to original

(3)

(ii)

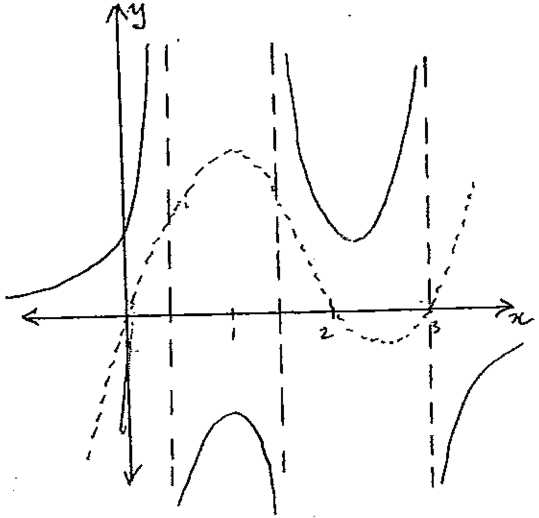


1 mark for x intercepts

1 mark for general shape compared to original

(2)

(iii)



1 mark for three asymptotes

1 mark for shape above the x axis

1 mark for shape below the x axis

1 mark for overall shape

(4)

SOLUTIONS	MARKS / COMMENTS
7 c) (i) $a^2 = b^2(e^2-1)$	
$a^2 = b^2(5-1)$	1
$= 16 \times 4$	
$= 64$	
$a = 8$	1 (2)
(ii) asymptotes $y = \pm \frac{b}{a}x$	1 mark
$= \pm \frac{4}{8}x$	
$= \pm \frac{1}{2}x$	(1)
7 d) (i) $y = \frac{16}{x}$ $(x \neq 0)$	
$\frac{dy}{dx} = -\frac{16}{x^2}$	1 mark
at P $\frac{dy}{dx} = \frac{-16}{16p^2}$	
$= \frac{-1}{p^2}$	1 mark
Equation of tangent at P is $y - \frac{4}{p} = -\frac{1}{p^2}(x - 4p)$	
$p^2y - 4p = -x + 4p$	
$x + p^2y = 8p$	1 mark (3)
(ii) Two point form: $\frac{y - \frac{4}{p}}{\frac{4}{q} - \frac{4}{p}} = \frac{x - 4p}{4q - 4p}$	1 mark
$\frac{pqy - 4q}{4p - 4q} = \frac{x - 4p}{4q - 4p}$	
$pqy - 4q = -(x - 4p)$ $(p \neq q)$	
$x + pqy = 4(p+q)$	1 mark (2)

SOLUTIONS	MARKS / COMMENTS
(iii) Tangent at Q	
$x + q^2y = 8q$ ——①	
$x + p^2y = 8p$ ——②	
① − ② $(q^2 - p^2)y = 8(q-p)$	
$(q+p)y = 8$	
$y = \frac{8}{q+p}$	1 mark
substitute into ① $x = 8q - \frac{8q^2}{p+q}$	
$= \frac{8pq + 8q^2 - 8q^2}{p+q}$	
$= \frac{8pq}{p+q}$	1 mark
$\therefore$ T is $\left(\frac{8pq}{p+q}, \frac{8}{p+q}\right)$	(2)
(iv) The chord in (ii) goes through $(8, 8)$	
$\therefore 8 + 8pq = 4(p+q)$	
$2(1+pq) = p+q$	
$1 + pq = \frac{p+q}{2}$	1 mark
Substitute T into $x + y = 4$	
LHS $= x + y$	
$= \frac{8pq}{p+q} + \frac{8}{p+q}$	
$= \frac{8(pq+1)}{p+q}$	
$= \frac{8\left(\frac{p+q}{2}\right)}{p+q}$	
$= 4$ $(p \neq q)$	
$=$ RHS	1 mark
$\therefore$ T lies on the line $x + y = 4$	(2)