SYDNEY GRAMMAR SCHOOL

2014 Half-Yearly Examination

FORM VI
MATHEMATICS 2 UNIT

Wednesday 26th February 2014

General Instructions
- Writing time — 2 hours
- Write using black or blue pen.
- Board-approved calculators and templates may be used.
- A list of standard integrals is provided at the end of the examination paper.

Total — 85 Marks
- All questions may be attempted.

Section I – 10 Marks
- Questions 1–10 are of equal value.
- Record your solutions to the multiple choice on the sheet provided.

Section II – 75 Marks
- Questions 11–15 are of equal value.
- All necessary working should be shown.
- Start each question in a new booklet.

Collection
- Write your candidate number on each booklet and on your multiple choice answer sheet.
- Hand in the booklets in a single well-ordered pile.
- Hand in a booklet for each question in Section II, even if it has not been attempted.
- If you use a second booklet for a question, place it inside the first.
- Place your multiple choice answer sheet inside the answer booklet for Question Eleven.
- Write your candidate number on this question paper and submit it with your answers.

Checklist
- SGS booklets — 5 per boy
- Multiple choice answer sheet
- Candidature — 90 boys

Examiner
GMC

SECTION I - Multiple Choice

Answers for this section should be recorded on the separate answer sheet handed out with this examination paper.

QUESTION ONE

Given that $y = \frac{1}{x}$, which of the following statements is true?

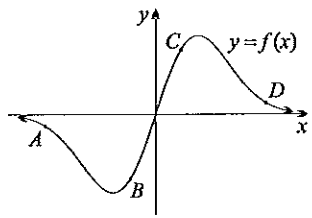
(A) $\frac{dy}{dx} = \frac{1}{x^2}$

(B) $\frac{dy}{dx} = -\frac{1}{x^2}$

(C) $\frac{dy}{dx} = \frac{2}{x^2}$

(D) $\frac{dy}{dx} = -\frac{2}{x^2}$

QUESTION TWO



The graph of $y = f(x)$ is shown above.
Which of the labelled points satisfies $f(x) < 0$ and $f''(x) > 0$?

(A) A

(B) B

(C) C

(D) D

Exam continues next page ...

QUESTION THREE

The value of the limit $\lim_{x \to 3} \dfrac{x-3}{x^2-9}$ is given by:

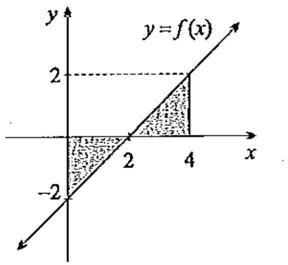
(A) 0

(B) $\dfrac{1}{3}$

(C) $\dfrac{1}{6}$

(D) None of the above.

QUESTION FOUR



A linear function $y = f(x)$ is graphed above.
Which of the following expressions represents the area of the shaded region?

(A) $\displaystyle\int_0^4 f(x)\,dx$

(B) $\displaystyle-\int_0^4 f(x)\,dx$

(C) $\displaystyle 2\int_0^2 f(x)\,dx$

(D) $\displaystyle 2\int_2^4 f(x)\,dx$

QUESTION FIVE

What are the coordinates of the focus of the parabola $x^2 = -4y$?

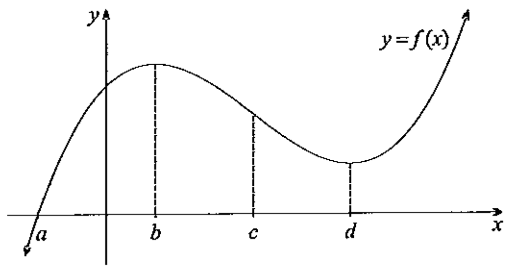
(A) $(0, 1)$

(B) $(1, 0)$

(C) $(0, -1)$

(D) $(-1, 0)$

QUESTION SIX



The diagram above is a graph of $y = f(x)$.
For what values of x is the function $y = f'(x)$ positive?

(A) $x > 0$

(B) $x < b$ or $x > d$

(C) $x > a$

(D) $b < x < d$

QUESTION SEVEN

What is the value of the definite integral $\int_0^1 e^{-x}\,dx$?

(A) $\frac{e-1}{e}$

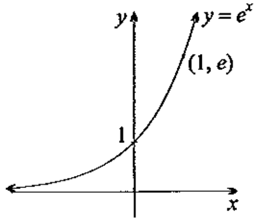
(B) $\frac{1-e}{e}$

(C) $\frac{e+1}{e}$

(D) $\frac{1}{e}$

QUESTION EIGHT

The graph of $f(x) = e^x$ is shown below.



Which one of the following statements is false?

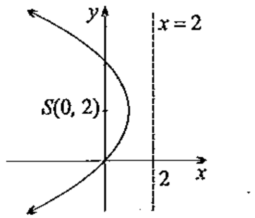
(A) $f(x) > 0$

(B) $f'(x) > 0$

(C) $f(x) = f'(x)$

(D) $f(-x) = -f(x)$

QUESTION NINE



A parabola with focus $S(0,2)$ and directrix $x = 2$ is shown above.
Which of the following is the equation of the parabola?

(A) $(y-2)^2 = 4(x-1)$

(B) $(y-2)^2 = -4(x-1)$

(C) $(y-2)^2 = 8x$

(D) $(y-2)^2 = -8x$

QUESTION TEN

The continuous function $y = f(x)$ has the properties that $\int_a^c f(x)\,dx = 7$ and $\int_b^c f(x)\,dx = -4$. Given that $a < b < c$, what is the value of $\int_a^b f(x)\,dx$?

(A) 11

(B) -11

(C) 3

(D) -3

End of Section I

SECTION II - Written Response

Answers for this section should be recorded in the booklets provided.

Show all necessary working.

Start a new booklet for each question.

QUESTION ELEVEN (15 marks) Use a separate writing booklet.

Marks

(a) Calculate $3e^{-2}$ correct to 2 decimal places. [1]

(b) Differentiate the following with respect to x:

(i) $2x^2+5$ [1]

(ii) $x^{\frac{1}{3}}$ [1]

(iii) $(4x+1)^6$ [2]

(c) Find a primitive for each of the following:

(i) $3x^5$ [1]

(ii) e^{-2x} [1]

(iii) $\sqrt{x}$ [2]

(d) Write down the equation of the locus of the point $P(x,y)$ that is:

(i) 3 units from the point $(-2,1)$, [1]

(ii) 4 units below the line $y=1$. [1]

(e) Sketch a graph of $y=e^{-x}+2$ clearly showing the asymptote and y-intercept. [2]

(f) Consider a curve whose first derivative is given by $y'=3x+2$. For what value of x is the curve stationary? [1]

(g) Consider a curve whose second derivative is given by $y''=2x+4$. For what values of x is the curve concave up? [1]

Exam continues overleaf ...

QUESTION TWELVE (15 marks) Use a separate writing booklet.

Marks

(a) Simplify $\dfrac{e^{3x+2}}{e^x}$. [1]

(b) Differentiate the following with respect to x:

(i) $y=(2x-1)(3x+2)$ [2]

(ii) $y=\dfrac{x^2-2}{x}$ [2]

(iii) $y=\sqrt{x^2+1}$ [2]

(c) Evaluate $\displaystyle\int_{-1}^{2}(4-x^2)\,dx$. [2]

(d) A parabola has equation $x^2=16y$.

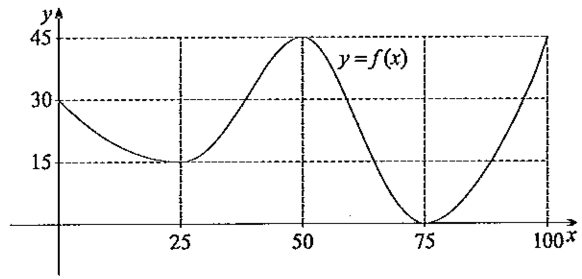
(i) Write down the coordinates of the vertex. [1]

(ii) Find the coordinates of the focus. [1]

(iii) Find the equation of the directrix. [1]

(iv) Sketch the parabola clearly showing the vertex, focus and directrix. [1]

(e)



The diagram above shows the graph of $y=f(x)$.

(i) Copy and complete the table below. [1]

x	0	25	50	75	100
$f(x)$					

(ii) Hence estimate $\displaystyle\int_{0}^{100} f(x)\,dx$ using Simpson's rule with five function values. [1]

Exam continues next page ...

QUESTION THIRTEEN (15 marks) Use a separate writing booklet. **Marks**

(a) Consider the function $y = x^3 - 6x^2 + 9x - 1$.

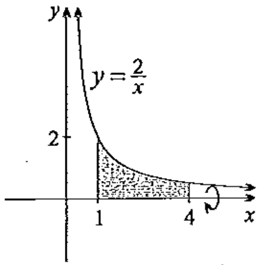
(i) Show that $\dfrac{dy}{dx} = 3(x-1)(x-3)$ and find $\dfrac{d^2y}{dx^2}$. $\boxed{2}$

(ii) Find the coordinates of any stationary points and determine their nature. $\boxed{2}$

(iii) Find the coordinates of the point of inflexion. You must show that it is a point of inflexion. $\boxed{2}$

(iv) Sketch the graph of the function, clearly showing all stationary points, the point of inflexion and the y-intercept. Do NOT attempt to find any x-intercepts. $\boxed{2}$

(b) $\boxed{2}$



The region bounded by $y = \dfrac{2}{x}$, the x-axis and the lines $x = 1$ and $x = 4$ is shown above.

Find the volume of the solid generated when this region is rotated about the x-axis.

(c) A curve has gradient function $\dfrac{dy}{dx} = 6x^2 - 3$ and passes through the point $(1, 5)$. $\boxed{2}$

Find the equation of the curve.

(d) Find the value of k if $\displaystyle\int_2^k (x-1)\,dx = 4$ and $k > 2$. $\boxed{3}$

Exam continues overleaf ...

QUESTION FOURTEEN (15 marks) Use a separate writing booklet. **Marks**

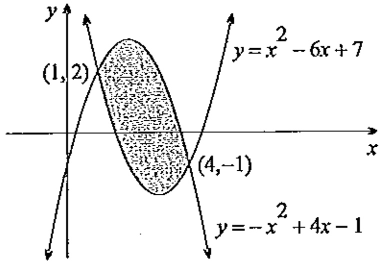
(a) Find the equation of the tangent to the curve $y = 1 - e^{-x}$ when $x = 1$. $\boxed{3}$

(b) (i) Express the equation $y^2 + 4y - 3x - 5 = 0$ in the form $(y-k)^2 = 4a(x-h)$. $\boxed{2}$

(ii) Hence find the coordinates of the focus and the equation of the directrix of the parabola $y^2 + 4y - 3x - 5 = 0$. $\boxed{2}$

(c) Using the quotient rule, or otherwise, find the derivative of $y = \dfrac{2x^2}{e^{3x}}$. Express your answer in simplest form. $\boxed{2}$

(d)



The diagram above shows the curves $y = -x^2 + 4x - 1$ and $y = x^2 - 6x + 7$.

(i) By solving a pair of simultaneous equations, show that the points of intersection of the two curves are $(1, 2)$ and $(4, -1)$. $\boxed{1}$

(ii) Hence calculate the shaded area between the curves. $\boxed{2}$

(e) (i) Differentiate $y = e^{x^2}$. $\boxed{1}$

(ii) Hence evaluate $\displaystyle\int_0^1 6xe^{x^2}\,dx$. $\boxed{2}$

Exam continues next page ...

QUESTION FIFTEEN (15 marks) Use a separate writing booklet. **Marks**

(a) Consider the exponential function $y = e^{-3x}$.

(i) Find y' and y''. $\boxed{2}$

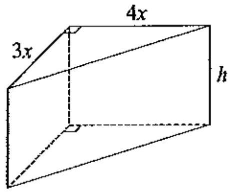
(ii) Show that $y = e^{-3x}$ satisfies the equation $3y = 5y' + 2y''$. $\boxed{1}$

(b) The locus of a point $P(x, y)$ is a circle. The distance of P from $A(8, -16)$ is three times the distance of P from the origin.

(i) Find the equation of the locus of P. $\boxed{2}$

(ii) Hence write down the centre and radius of the circle described by P. $\boxed{1}$

(c)



A closed metal box is in the shape of a prism with a right-angled triangular cross section. The surface area of the box is $240\,\text{cm}^2$ and the perpendicular sides of the triangular cross section are in the ratio $3 : 4$. Let the dimensions of the box be $3x$, $4x$ and h as shown in the diagram above.

(i) Show that the surface area, S, of the box is given by $S = 12x^2 + 12xh$. $\boxed{1}$

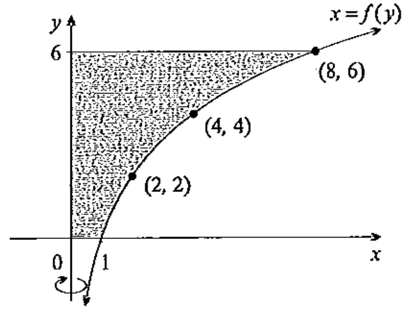
(ii) Show that the volume of the box, V, is given by $V = 120x - 6x^3$. $\boxed{2}$

(iii) Hence find the greatest possible volume of the box in exact form. $\boxed{2}$

QUESTION CONTINUES ON THE NEXT PAGE

QUESTION FIFTEEN (Continued)

(d)



The curve $x = f(y)$ passes through the points $(1, 0)$, $(2, 2)$, $(4, 4)$ and $(8, 6)$ as shown in the diagram above.

(i) Using the trapezoidal rule with four function values, estimate the volume formed by rotating the region bounded by $x = f(y)$, the x-axis, the y-axis and $y = 6$ about the y-axis. $\boxed{3}$

(ii) Does the trapezoidal rule under-estimate or over-estimate the volume of the solid formed in part (i)? Justify your answer. $\boxed{1}$

End of Section II

END OF EXAMINATION

The following list of standard integrals may be used:

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x,\ x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\ a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\ a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\ a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\ a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax,\ a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\ a > 0,\ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x + \sqrt{x^2-a^2}\right),\ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x + \sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x,\ x > 0$

FORM VI HALF YEARLY 2014 2 UNIT.

1.	B	6.	B
2.	B	7.	A
3.	C	8.	D
4.	D	9.	B
5.	C	10.	A

/10

11. a) $3e^{-2} = 0.41$ (2 d.p.) ✓ — must show correct to 2 d.p.

b) i) $\frac{d}{dx}(2x^2+5) = 4x$ ✓

ii) $\frac{d}{dx}\left(x^{1/3}\right) = \frac{1}{3}x^{-2/3}$ ✓ — or $\frac{1}{3x^{2/3}}$

iii) $\frac{d}{dx}\left((4x+1)^6\right) = 24(4x+1)^5$ ✓✓ — one mark for $6(4x+1)^5$

c) i) $\int 3x^5\,dx = \frac{x^6}{2} + C$ ✓

ii) $\int e^{-2x}\,dx = -\frac{1}{2}e^{-2x} + C$ ✓

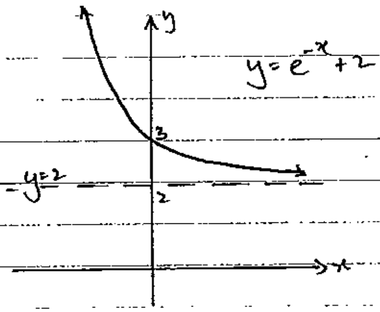
iii) $\int x^{1/2}\,dx = \frac{2}{3}x^{3/2} + C$. ✓✓

NB: Constant of integration not required here.

d) i) $(x+2)^2 + (y-1)^2 = 9$ ✓

ii) $y = -3$ ✓

e)



✓✓ −1 for incorrect shape, asymptote or missing y-intercept. Do not need a separate point.

f) let $y' = 0$

$3x + 2 = 0$

$x = -2/3$

So the curve is stationary when $x = -2/3$ ✓

g) let $y'' > 0$

$2x + 4 > 0$

$2x > -4$

$x > -2$

So the curve is concave up when $x > -2$ ✓

12. a) $\dfrac{e^{3x+2}}{e^{x}} = e^{2x+2}$ ✓

b) i) $\frac{d}{dx}\left((2x-1)(3x+2)\right)$

$= \frac{d}{dx}\left(6x^2 + x - 2\right)$ ✓

$= 12x + 1$ ✓

ii) $\frac{d}{dx}\left(\dfrac{x^2-2}{x}\right)$

$= \frac{d}{dx}\left(x - \frac{2}{x}\right)$ ✓

$= 1 + \dfrac{2}{x^2}$ ✓ or $1 + 2x^{-2}$

iii) $\frac{d}{dx}\left(\sqrt{x^2+1}\right)$

$= \frac{d}{dx}\left((x^2+1)^{1/2}\right)$ ✓

$= \frac{1}{2}(x^2+1)^{-1/2} \times 2x$ ✓

$= \dfrac{x}{\sqrt{x^2+1}}$ or $x(x^2+1)^{-1/2}$

c) $\displaystyle\int_{-1}^{2}(4 - x^2)\,dx = \left[4x - \frac{x^3}{3}\right]_{-1}^{2}$ ✓

$= \left(8 - \frac{8}{3}\right) - \left(-4 + \frac{1}{3}\right)$

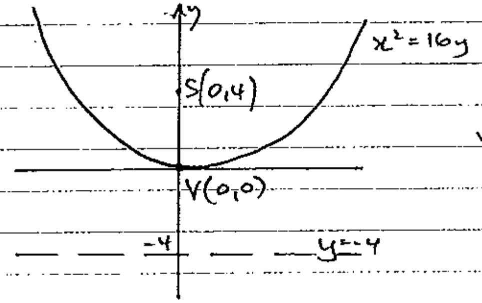
$= 9$ ✓

d) i) Vertex $(0,0)$ ✓

ii) Focus $(0,4)$ ✓

iii) Directrix : $y = -4$ ✓

iv)



✓

e) i)

x	0	25	50	75	100
$f(x)$	30	15	45	0	45

✓

ii) $\displaystyle\int_{0}^{100} f(x)\,dx \doteqdot \frac{50-0}{6}\left\{30 + 4\times15 + 45\right\} + \frac{100-50}{6}\left\{45 + 4\times0 + 45\right\}$

$\doteqdot 1875$ ✓

13. a) $y = x^3 - 6x^2 + 9x - 1$

i) $\frac{dy}{dx} = 3x^2 - 12x + 9$

$= 3(x^2 - 4x + 3)$

$= 3(x-1)(x-3)$ as reqd. ✓

$\frac{d^2y}{dx^2} = 6x - 12$ ✓

ii) let $\frac{dy}{dx} = 0$

$3(x-1)(x-3) = 0$

$\left.\begin{array}{l} x = 1, \ x = 3 \\ y = 3, \ y = -1 \end{array}\right\}$ Stat pts at $(1,3)$ and $(3,-1)$ ✓

when $x = 1$, $\frac{d^2y}{dx^2} = -6$
< 0 so $(1,3)$ is a max

when $x = 3$, $\frac{d^2y}{dx^2} = 6$
> 0 so $(3,-1)$ is a min ✓

iii) let $\frac{d^2y}{dx^2} = 0$

$6x - 12 = 0$

$x = 2$, $y = 1$

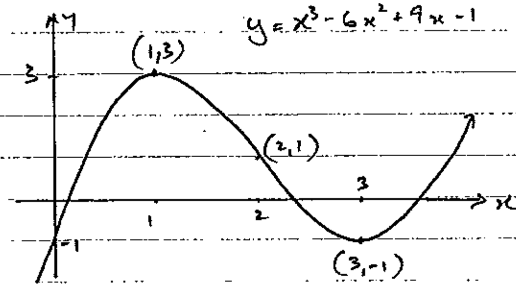
test:

x	1	2	3
$\frac{d^2y}{dx^2}$	-6	0	6

change of sign ✓

So $(2,1)$ is a point of inflexion ✓

iv)



✓✓ -1 for any missing point or y-intercept.

b) $V = \pi \int_1^4 y^2 \, dx$

$= \pi \int_1^4 4x^{-2} \, dx$ ✓

$= -4\pi \left[x^{-1} \right]_1^4$

$= -4\pi \left(\frac{1}{4} - 1 \right)$

$= 3\pi$ ✓

So volume is 3π units^3

c) $\frac{dy}{dx} = 6x^2 - 3$

$y = 2x^3 - 3x + C$ ✓ must have $+C$

when $x = 1$, $y = 5$

$5 = 2 - 3 + C$

$C = 6$

$y = 2x^3 - 3x + 6$ ✓

d) $\int_2^k (x-1) \, dx = 4$

$\left[\frac{x^2}{2} - x \right]_2^k = 4$ ✓

$\left(\frac{k^2}{2} - k \right) - \left(\frac{4}{2} - 2 \right) = 4$

$k^2 - 2k - 8 = 0$ ✓

$(k-4)(k+2) = 0$

$k = 4$, $k = -2$

but $k > 2$

So $k = 4$ ✓

14. a) i) $y = 1 - e^{-x}$

$\frac{dy}{dx} = e^{-x}$

when $x = 1$, $\frac{dy}{dx} = \frac{1}{e}$ ✓

tangent has grad $\frac{1}{e}$, passes through $\left(1, 1-\frac{1}{e}\right)$ ✓

$y - y_1 = m(x - x_1)$

$y - \left(1 - \frac{1}{e}\right) = \frac{1}{e}(x - 1)$ ✓

$y - 1 + \frac{1}{e} = \frac{1}{e}x - \frac{1}{e}$

$y = \frac{1}{e}x + 1 - \frac{2}{e}$ or $x - ey + e - 2 = 0$

b) i) $y^2 + 4y - 3x - 5 = 0$

$y^2 + 4y + 4 = 3x + 5 + 4$ ✓ for completing the square

$(y+2)^2 = 3(x+3)$ ✓ either of last two lines ok

$(y+2)^2 = 4 \times \frac{3}{4}(x+3)$

ii) focal length : $\frac{3}{4}$

focus : $\left(-2\frac{1}{4}, -2\right)$ ✓

directrix : $x = -3\frac{3}{4}$ ✓

c) $y = \frac{2x^2}{e^{3x}}$

$\frac{dy}{dx} = \frac{e^{3x} \cdot 4x - 2x^2 \cdot 3e^{3x}}{(e^{3x})^2}$ ✓ ok to use product rule.

$= \frac{4x - 6x^2}{e^{3x}}$ ✓ or $(4x - 6x^2)e^{-3x}$

$= \frac{2x(2-3x)}{e^{3x}}$

d) i) let $-x^2 + 4x - 1 = x^2 - 6x + 7$

$2x^2 - 10x + 8 = 0$

$x^2 - 5x + 4 = 0$

$(x-4)(x-1) = 0$

$x = 1, \quad x = 4$ ✓

$y = 2, \quad y = -1$

so points of intersection are $(1,2)$ and $(4,-1)$

ii) Area $= \int_1^4 \left((-x^2+4x-1) - (x^2-6x+7)\right) dx$

$= \int_1^4 (-2x^2 + 10x - 8)\, dx$ ✓

$= \left[\frac{-2x^3}{3} + 5x^2 - 8x\right]_1^4$

$= \left(\frac{-128}{3} + 80 - 32\right) - \left(-\frac{2}{3} + 5 - 8\right)$

$= 9$ ✓

So Area $= 9$ units^2

e) i) $\frac{d}{dx}\left(e^{x^2}\right) = 2xe^{x^2}$ ✓

ii) $\int_0^1 6xe^{x^2}\, dx = 3\int_0^1 2xe^{x^2}\, dx$

$= 3\left[e^{x^2}\right]_0^1$ ✓

$= 3(e - 1)$ ✓

15 a) i) $y = e^{-3x}$

$y' = -3e^{-3x}$ ✓

$y'' = 9e^{-3x}$ ✓

ii) Show $3y = 5y' + 2y''$

$LHS = 3 \times e^{-3x}$

$= 3e^{-3x}$

$RHS = 5 \times (-3e^{-3x}) + 2 \times 9e^{-3x}$

$= 3e^{-3x}$

So $LHS = RHS$

So $3y = 5y' + 2y''$ as req'd. ✓

b) $P(x, y)$, $A(8, -16)$

i) $PA = 3PO$

$PA^2 = 9PO^2$

$(x-8)^2 + (y+16)^2 = 9[x^2 + y^2]$ ✓

$x^2 - 16x + 64 + y^2 + 32y + 256 = 9x^2 + 9y^2$

$8x^2 + 16x + 8y^2 - 32y = 320$

$x^2 + 2x + y^2 - 4y = 40$ ✓

$x + 2x + 1 + y^2 - 4y + 4 = 45$

$(x+1)^2 + (y-2)^2 = (\sqrt{45})^2$

ii) So centre is $(-1, 2)$ radius $3\sqrt{5}$ units ✓ ($\sqrt{45}$ ok)

c) i) By Pythagoras, hypotenuse of triangular cross section is $5x$

So $S = 2 \times (\frac{1}{2} \times 3x \times 4x) + (3x + 4x + 5x) \times h$

$= 12x^2 + 12xh$ as req'd. ✓

ii) $V = \frac{1}{2} \times 3x \times 4x \times h$

$= 6x^2h$

from i) $240 = 12x^2 + 12xh$

$20 = x^2 + xh$

$h = \frac{20 - x^2}{x}$ ✓

So $V = 6x^2 \times \frac{20 - x^2}{x}$

$= 6x(20 - x^2)$

$= 120x - 6x^3$ as required ✓

iii) $\frac{dV}{dx} = 120 - 18x^2$

let $\frac{dV}{dx} = 0$

$120 - 18x^2 = 0$

$x^2 = \frac{20}{3}$

$x = \pm \frac{2\sqrt{5}}{\sqrt{3}}$ but $x > 0$ ✓

$\frac{d^2V}{dx^2} = -36x$

when $x = \frac{2\sqrt{5}}{\sqrt{3}}$, $\frac{d^2V}{dx^2} < 0$

So maximum occurs when $x = \frac{2\sqrt{5}}{3}$

−1 for not testing nature

$V = 120 \times \frac{2\sqrt{5}}{\sqrt{3}} - 6 \times \left(\frac{2\sqrt{5}}{\sqrt{3}}\right)^3$

$= \frac{240\sqrt{5}}{\sqrt{3}} - \frac{240\sqrt{5}}{3\sqrt{3}}$

$= \frac{160\sqrt{5}}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{160\sqrt{15}}{3}$

So maximum volume is $\frac{160\sqrt{15}}{3}$ cm^3. ✓

d) i) $V = \pi \int_0^6 x^2 \, dy$

y	0	2	4	6
x	1	2	4	8
x^2	1	4	16	64

✓

$$\int_0^6 x^2 \, dy \doteq \frac{2-0}{2}\left\{1 + 2\times4 + 2\times16 + 64\right\}$$ ✓

$$= 105$$ ✓

So Volume $\doteq 105\pi$ units^3

ii) The estimation in part i) is an over-estimation as from the perspective of the y-axis, the curve is closer to the y axis than the straight line segments formed by joining the points. This relationship is preserved when the x values are squared in order to estimate the volume integral as the x values are larger than or equal to one.

✓ or similar.