SYDNEY GRAMMAR SCHOOL

2013 Half-Yearly Examination

FORM VI
MATHEMATICS EXTENSION 1

Tuesday 26th February 2013

General Instructions
- Writing time — 2 hours
- Write using black or blue pen.
- Board-approved calculators and templates may be used.
- A list of standard integrals is provided at the end of the examination paper.

Total — 70 Marks
- All questions may be attempted.

Section I – 10 Marks
- Questions 1–10 are of equal value.
- Record your solutions to the multiple choice on the sheet provided.

Section II – 60 Marks
- Questions 11–14 are of equal value.
- All necessary working should be shown.
- Start each question in a new booklet.

Collection
- Write your candidate number on each booklet and on your multiple choice answer sheet.
- Hand in the booklets in a single well-ordered pile.
- Hand in a booklet for each question in Section II, even if it has not been attempted.
- If you use a second booklet for a question, place it inside the first.
- Place your multiple choice answer sheet inside the answer booklet for Question Eleven.
- Write your candidate number on this question paper and submit it with your answers.

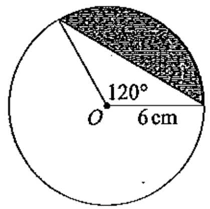
Checklist
- SGS booklets — 4 per boy
- Multiple choice answer sheet
- Candidature — 116 boys

Examiner
BR/DNW

SECTION I - Multiple Choice

Answers for this section should be recorded on the separate answer sheet handed out with this examination paper.

QUESTION ONE 1



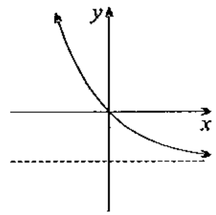
The diagram above shows a shaded segment which subtends an angle of 120° at the centre of a circle with radius 6 cm. The area of the segment correct to one decimal place is:

(A) 22·1 cm^2 (B) 37·7 cm^2 (C) 2144·4 cm^2 (D) 2160·0 cm^2

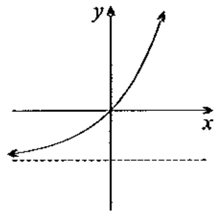
QUESTION TWO

Which of the following graphs best represents $y = 1 - e^x$? 1

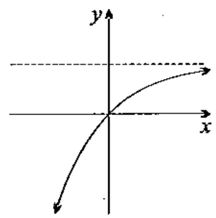
(A)



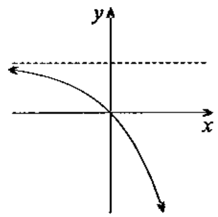
(B)



(C)



(D)



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QUESTION THREE

The derivative of $\log(1+x^2)$ is: [1]

(A) $\dfrac{2}{1+x}$ (B) $\dfrac{1}{1+x^2}$ (C) $\dfrac{2x}{1+x^2}$ (D) $\dfrac{x}{1+x^2}$

QUESTION FOUR

The expression $\cos^2 x$ is equivalent to: [1]

(A) $1+\cos 2x$ (B) $1-\cos 2x$

(C) $\frac{1}{2}(1+\cos 2x)$ (D) $\frac{1}{2}(1-\cos 2x)$

QUESTION FIVE

The focal length of the parabola with equation $y=2x^2$ is: [1]

(A) 8 (B) 2 (C) $\frac{1}{2}$ (D) $\frac{1}{8}$

QUESTION SIX

The volume V cubic centimetres of a sphere with radius r centimetres is $V=\frac{4}{3}\pi r^3$. [1]
The radius is increasing at a rate of 3 cm/s. At what rate is V increasing?

(A) $\frac{4}{3}\pi r^3$ (B) $4\pi r^3$ (C) $4\pi r^2$ (D) $12\pi r^2$

QUESTION SEVEN

The derivative of $\tan^{-1} 2x$ is: [1]

(A) $\dfrac{2}{4+x^2}$ (B) $\dfrac{2}{1+4x^2}$ (C) $\dfrac{1}{4+x^2}$ (D) $\dfrac{1}{1+4x^2}$

QUESTION EIGHT

The expression $\cos x + \sin x$ is equivalent to: [1]

(A) $\sqrt{2}\cos(x+\frac{\pi}{4})$ (B) $\sqrt{2}\cos(x-\frac{\pi}{4})$

(C) $\sqrt{2}\cos(x+\frac{3\pi}{4})$ (D) $\sqrt{2}\cos(x-\frac{3\pi}{4})$

QUESTION NINE

The directrix of the parabola with equation $(y-1)^2=12(x+1)$ is given by: [1]

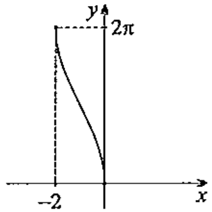
(A) $x=-4$ (B) $x=-2$ (C) $x=2$ (D) $x=4$

Exam continues overleaf ...

QUESTION TEN

Which of the following graphs best represents $y=2\sin^{-1}(x+1)$? [1]

(A)



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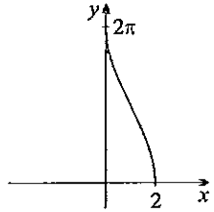

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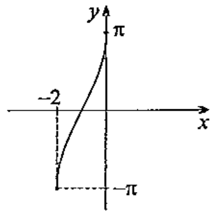

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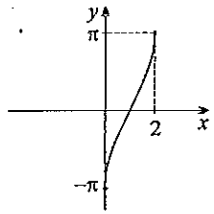


——— End of Section I ———

Exam continues next page ...

SECTION II - Written Response

Answers for this section should be recorded in the booklets provided.

Show all necessary working.

Start a new booklet for each question.

QUESTION ELEVEN (15 marks) Use a separate writing booklet. **Marks**

(a) Simplify:

(i) $\ln(e^3)$ [1]

(ii) $\tan(\frac{5\pi}{6})$ [1]

(iii) $\cos^{-1}(-\frac{\sqrt{3}}{2})$ [1]

(b) Differentiate:

(i) $\tan 3x$ [1]

(ii) $x\sin^{-1}x$ [2]

(c) Evaluate $\displaystyle\lim_{x\to 0}\frac{\tan x}{2x}$. [1]

(d) Prove that $\dfrac{\sin 2A}{\cos 2A - 1} = -\cot A$. [3]

(e) An equilateral triangle has sides of length ℓ.

(i) Show that the area A of the triangle is given by $A = \frac{\sqrt{3}}{4}\ell^2$. [1]

(ii) The area of the triangle is increasing at the rate of $9\,\text{cm}^2/\text{min}$. Determine the rate at which ℓ is increasing when the sides are 6 cm. [2]

(f) Write down the general solution of $\sin x = \frac{1}{2}$. [2]

Exam continues overleaf ...

QUESTION TWELVE (15 marks) Use a separate writing booklet. **Marks**

(a) (i) Differentiate $y = \cos 2x$. [1]

(ii) Hence, or otherwise, evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \sin x\cos x\,dx$. [2]

(b) Evaluate $\displaystyle\int_0^4 \frac{x}{\sqrt{9+x^2}}\,dx$. [2]

(c) Use the substitution $t = \tan\frac{1}{2}\theta$ to show that [2]

$$\frac{1+\sin\theta}{1+\cos\theta} = \tfrac{1}{2}(1+\tan\tfrac{1}{2}\theta)^2.$$

(d) The acute angle between the line $4x + 3y = 8$ and the line $ax + by = 8$ is $45°$. [3]

Find the possible values of the fraction $\dfrac{a}{b}$.

(e) (i) Sketch $y = \cos x + 1$ for $-\pi \le x \le \pi$. [1]

(ii) The region bounded by $y = \cos x + 1$ and the x-axis, where $-\pi \le x \le \pi$, is rotated about the x-axis to generate a solid. Find the volume of this solid. [4]

QUESTION THIRTEEN (15 marks) Use a separate writing booklet. **Marks**

(a) Determine $\displaystyle\int \frac{1}{\sqrt{3-4x^2}}\,dx$. [2]

(b) The point $P(6t, 3t^2)$ lies on the parabola $x^2 = 12y$.

(i) Show that the equation of the normal at P is $x + ty = 6t + 3t^3$. [2]

(ii) The normal at P cuts the y-axis at A.

(α) The mid-point of PA is R. Find the coordinates of R in terms of t. [1]

(β) The locus of R is another parabola. Find its vertex and focal length. [2]

(c) Solve $2\cos^2 x + \sqrt{3}\sin 2x = 0$ for $0 \le x \le 2\pi$. [3]

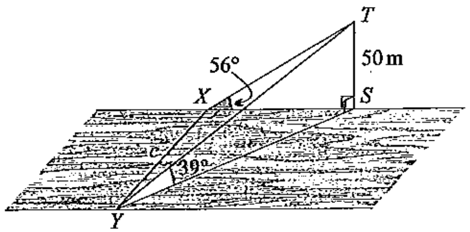
(d) (i) Express $\sqrt{3}\sin\theta - \cos\theta$ in the form $R\sin(\theta - \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$. [2]

(ii) Hence solve $\sqrt{3}\sin\theta - \cos\theta \ge 1$ for $0 \le \theta \le 2\pi$. [3]

Exam continues next page ...

QUESTION FOURTEEN (15 marks) Use a separate writing booklet. Marks

(a) [3]



From a point X on a straight canal bank a surveyor measures the angle of elevation to T, the top of a 50 m tower with base S on the same bank. This angle is 56°. Directly opposite X on the other side of the canal, another surveyor at Y finds the angle of elevation to T is 39°. The points X, Y and S are on level ground.

Find c, the width of the canal, correct to three significant figures.

(b) (i) Prove by mathematical induction that for all integers $n \geq 1$ [3]

$$\frac{1}{1\times 5}+\frac{1}{5\times 9}+\frac{1}{9\times 13}+\ldots+\frac{1}{(4n-3)(4n+1)}=\frac{n}{4n+1}.$$

(ii) Hence evaluate [1]

$$\lim_{n\to\infty}\left(\frac{1}{1\times 5}+\frac{1}{5\times 9}+\frac{1}{9\times 13}+\ldots+\frac{1}{(4n-3)(4n+1)}\right).$$

(c) The function $\sin^{-1}x$ is odd. By finding x as a function of y or otherwise, show algebraically that [2]

$$y=4\cos^{-1}x-2\pi$$

is also odd.

(d) Suppose that

$$\int_{-2h}^{2h} f(x)\,dx = A\times f(-h)+B\times f(0)+C\times f(h) \qquad (**)$$

for $f(x)=1$, $f(x)=x$ and $f(x)=x^2$.

(i) Determine the values of A, B and C in terms of h. [4]

(ii) Show that equation (**) is also valid when $f(x)=x^3$. [1]

(iii) Equation (**) may be used to approximate the integrals of other functions. Use it to show that [1]

$$\int_{-1}^{1} 2^x\,dx \doteqdot \tfrac{2}{3}(3\sqrt{2}-1).$$

End of Section II

END OF EXAMINATION

The following list of standard integrals may be used:

$$\int x^n\,dx=\frac{1}{n+1}x^{n+1},\quad n\neq -1;\ x\neq 0,\text{ if } n<0$$

$$\int \frac{1}{x}\,dx=\ln x,\quad x>0$$

$$\int e^{ax}\,dx=\frac{1}{a}e^{ax},\quad a\neq 0$$

$$\int \cos ax\,dx=\frac{1}{a}\sin ax,\quad a\neq 0$$

$$\int \sin ax\,dx=-\frac{1}{a}\cos ax,\quad a\neq 0$$

$$\int \sec^2 ax\,dx=\frac{1}{a}\tan ax,\quad a\neq 0$$

$$\int \sec ax\tan ax\,dx=\frac{1}{a}\sec ax,\quad a\neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx=\frac{1}{a}\tan^{-1}\frac{x}{a},\quad a\neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx=\sin^{-1}\frac{x}{a},\quad a>0,\ -a<x<a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx=\ln\left(x+\sqrt{x^2-a^2}\right),\quad x>a>0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx=\ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x=\log_e x,\ x>0$

FORM VI MATHEMATICS EXTENSION 1

Q1/ A ✓

Q2/ D ✓

Q3/ C ✓

Q4/ C ✓

Q5/ D ✓

Q6/ D ✓

Q7/ B ✓

Q8/ B ✓

Q9/ A ✓

Q10/ C ✓

Q11/

a) i) 3 ✓

ii) $\frac{-1}{\sqrt{3}}$ ✓

iii) $\frac{5\pi}{6}$ ✓

b) i) $3\sec^2 3x$ ✓

ii) $\sin^{-1}x + \frac{x}{\sqrt{1-x^2}}$ ✓ ✓

c) $\frac{1}{2}$ ✓

d) $\text{LHS} = \frac{2\sin A\cos A}{1-2\sin^2 A-1}$ ✓ ✓

$= \frac{\cancel{2}\cancel{\sin A}\cos A}{-\cancel{2}\sin^{\cancel{2}} A}$

$= -\frac{\cos A}{\sin A}$

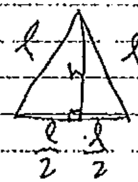
$= -\cot A$ ✓

$= \text{RHS}$

e) i) $A = \frac{1}{2}\ell^2 \sin\frac{\pi}{3}$

$= \frac{\sqrt{3}}{4}\ell^2$ ✓

OR



$h^2 = \ell^2 - \left(\frac{\ell}{2}\right)^2$

$h^2 = \frac{3\ell^2}{4}$

$h = \frac{\ell\sqrt{3}}{2}$

$\therefore A = \frac{1}{2}\cdot\ell\cdot\frac{\ell\sqrt{3}}{2} = \frac{\sqrt{3}}{4}\ell^2$

ii) $\frac{dA}{dt} = 9$, $\frac{dA}{dl} = \frac{\sqrt{3}}{2} l$, $\frac{dl}{dt} = ?$

$$\frac{dA}{dt} = \frac{dA}{dl} \cdot \frac{dl}{dt}$$

$$\therefore 9 = \frac{\sqrt{3}}{2} l \cdot \frac{dl}{dt}$$ ✓

$l = 6$, $$9 = \frac{\sqrt{3}}{\cancel{2}} \cdot \cancel{6}3 \frac{dl}{dt}$$

$$\frac{dl}{dt} = \frac{9}{3\sqrt{3}}$$

$$= \frac{3}{\sqrt{3}}$$

$$= \sqrt{3} \text{ cm/min}$$ ✓

f) $\sin x = \frac{1}{2}$ ✓

$\therefore x = \frac{\pi}{6} + 2n\pi$ or $\frac{5\pi}{6} + 2n\pi$ ✓

for integers n.

OR

$$x = (-1)^n \frac{\pi}{6} + n\pi$$

for integers n.

Q12

a) i) $y = \cos 2x$

$$\frac{dy}{dx} = -2\sin 2x$$ ✓

$$= -4\sin x \cos x$$

ii) $$\int_0^{\pi/2} \sin x \cos x \, dx = \frac{-1}{4}\int_0^{\pi/2} -4\sin x \cos x \, dx$$

$$= \frac{-1}{4}\Big[\cos 2x\Big]_0^{\pi/2}$$ ✓

$$= \frac{-1}{4}(\cos\pi - \cos 0)$$

$$= \frac{-1}{4}(-1-1)$$

$$= \frac{1}{2}$$ ✓

b) $$\frac{d}{dx}(9+x^2)^{1/2} = \frac{1}{\cancel{2}}(9+x^2)^{-1/2} \cdot \cancel{2}x$$

$$= \frac{x}{\sqrt{9+x^2}}$$

$$\therefore \int_0^4 \frac{x}{\sqrt{9+x^2}} dx = \Big[(9+x^2)^{1/2}\Big]_0^4$$ ✓

$$= \sqrt{25} - \sqrt{9}$$

$$= 2$$ ✓

c) $t = \tan\frac{\theta}{2}$

so, $\sin\theta = \dfrac{2t}{1+t^2}$, $\cos\theta = \dfrac{1-t^2}{1+t^2}$

$$LHS = \frac{1 + \frac{2t}{1+t^2}}{1 + \frac{1-t^2}{1+t^2}}$$

$$= \frac{1+t^2+2t}{1+t^2+1-t^2}$$

$$= \frac{(1+t)^2}{2}$$

$$= \frac{1}{2}\left(1+\tan\frac{\theta}{2}\right)^2$$

$= RHS$

d) $m_1 = \dfrac{-4}{3}$, $m_2 = \dfrac{-a}{b}$

$$\therefore \tan 45^\circ = \left| \frac{-\frac{4}{3} + \frac{a}{b}}{1 + \frac{4a}{3b}} \right|$$

$$1 = \left| \frac{-4b+3a}{3b+4a} \right|$$

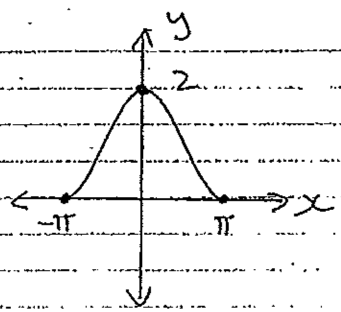
$\dfrac{-4b+3a}{3b+4a} = 1$ or $\dfrac{-4b+3a}{3b+4a} = -1$

$-4b+3a = 3b+4a$ $\qquad$ $-4b+3a = -3b-4a$

$-a = 7b$ $\qquad$ $7a = b$

$\dfrac{a}{b} = -7$ OR $\dfrac{a}{b} = \dfrac{1}{7}$

e) i)



ii) $V = \pi \int_{-\pi}^{\pi} (\cos x + 1)^2 \, dx$

$= 2\pi \int_0^{\pi} (\cos^2 x + 2\cos x + 1)\, dx$

$= \cancel{2}\pi \int_0^{\pi} \frac{1}{\cancel{2}}(1+\cos 2x)\,dx + 2\pi \int_0^{\pi} (2\cos x + 1)\, d$

$= \pi \left[x + \frac{1}{2}\sin 2x \right]_0^{\pi} + 2\pi \left[\sin x + x \right]_0^{\pi}$

$= \pi(\pi + 0) + 2\pi(0 + \pi)$

$= \pi^2 + 2\pi^2$

$= 3\pi^2$ units^3

Q13

a) $\displaystyle\int \frac{1}{\sqrt{3-4x^2}}\,dx = \frac{1}{2}\int \frac{1}{\sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 - x^2}}\,dx$ ✓

$\displaystyle = \frac{1}{2}\sin^{-1}\left(\frac{2x}{\sqrt{3}}\right) + c$ ✓

b) i) $y = \dfrac{x^2}{12}$

$\dfrac{dy}{dx} = \dfrac{x}{6}$

$x = 6t,\quad \dfrac{dy}{dx} = t$

$\therefore$ gradient of normal is $-\dfrac{1}{t}$ ✓

$m = -\dfrac{1}{t}\qquad P(6t, 3t^2)$

$\therefore y - 3t^2 = -\dfrac{1}{t}(x - 6t)$

$ty - 3t^3 = -x + 6t$ ✓

$x + ty = 6t + 3t^3$

ii) α) $A(0, 6+3t^2)\qquad P(6t, 3t^2)$

$\therefore R(3t, 3+3t^2)$ ✓

β) $x = 3t\qquad y = 3+3t^2$

$3y = 9 + 9t^2$

$3y = 9 + x^2$

$x^2 = 3(y-3)$ ✓

$\therefore V(0,3),\ a = \dfrac{3}{4}$ ✓ for both

c) $2\cos^2 x + \sqrt{3}\sin 2x = 0$ for $0 \le x \le 2\pi$

$2\cos^2 x + 2\sqrt{3}\sin x\cos x = 0$ ✓

$2\cos x(\cos x + \sqrt{3}\sin x) = 0$

$\therefore \cos x = 0$ or $\cos x + \sqrt{3}\sin x = 0$

$x = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$ ✓

$\sqrt{3}\sin x = -\cos x$

$\tan x = \dfrac{-1}{\sqrt{3}}$

Quad II, IV

$[x = \pi/6]$

$x = \dfrac{5\pi}{6}, \dfrac{11\pi}{6}$ ✓

$\therefore x = \dfrac{\pi}{2}, \dfrac{5\pi}{6}, \dfrac{3\pi}{2}, \dfrac{11\pi}{6}$ for $0 \le x \le 2\pi$

d) i) $\sqrt{3}\sin\theta - \cos\theta \equiv R\sin(\theta - \alpha)$

$= R\sin\theta\cos\alpha - R\sin\alpha\cos\theta$

By equating coeff.,

$R\cos\alpha = \sqrt{3}$ and $R\sin\alpha = 1$ ✓

$\therefore R = 2,\ \alpha = \dfrac{\pi}{6}$ ✓

$\therefore \sqrt{3}\sin\theta - \cos\theta \equiv 2\sin(\theta - \pi/6)$

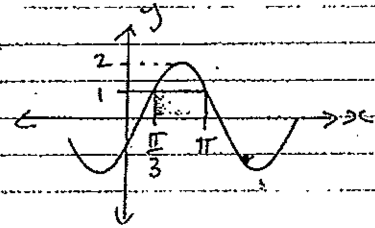
ii) $\sqrt{3}\sin\theta - \cos\theta \ge 1$ ✓

$2\sin(\theta - \frac{\pi}{6}) \ge 1$

$\sin(\theta - \pi/6) \ge \frac{1}{2}$

for equality, $\theta - \dfrac{\pi}{6} = \dfrac{\pi}{6}$ or $\dfrac{5\pi}{6}$

$\therefore \theta = \dfrac{\pi}{3}$ or π ✓



$\therefore \dfrac{\pi}{3} \le \theta \le \pi$ ✓

Q14/

a) $XS = \frac{50}{\tan 56^\circ}$ $YS = \frac{50}{\tan 39^\circ}$ ✓

$\therefore c^2 = \left(\frac{50}{\tan 39^\circ}\right)^2 - \left(\frac{50}{\tan 56^\circ}\right)^2$ ✓

$= 2675.023\ldots$ ✓

$\therefore c = 51.7$ (1 d.p.) ✓

b) i) ① when $n=1$

$LHS = \frac{1}{1\times5}$

$= \frac{1}{5}$

$RHS = \frac{1}{4\times1+1}$

$= \frac{1}{5}$

$= LHS$

$\therefore$ true when $n=1$ ✓

② Assume true for $n=k$

ie. $\frac{1}{1\times5} + \frac{1}{5\times9} + \ldots + \frac{1}{(4k-3)(4k+1)} = \frac{k}{4k+1}$

③ Prove true for $n=k+1$

R.T.P. $\frac{1}{1\times5} + \frac{1}{5\times9} + \ldots + \frac{1}{(4k-3)(4k+1)} + \frac{1}{(4k+1)(4k+5)}$

$= \frac{k+1}{4k+5}$

$LHS = \left(\frac{1}{1\times5} + \frac{1}{5\times9} + \ldots + \frac{1}{(4k-3)(4k+1)}\right) + \frac{1}{(4k+1)(4k+5)}$

$= \frac{k}{4k+1} + \frac{1}{(4k+1)(4k+5)}$ ✓

$= \frac{4k^2+5k+1}{(4k+1)(4k+5)}$

$= \frac{\cancel{(4k+1)}(k+1)}{\cancel{(4k+1)}(4k+5)}$

$= \frac{k+1}{4k+5}$

$= RHS$ ✓

As true for $n=k+1$ and true for $n=1$ it follows that it is true for $n=2,3,4\ldots$ and all integers of n greater or equal to 1

ii) $\lim_{n\to\infty} \frac{n}{4n+1} = \lim_{n\to\infty} \frac{1}{4+\frac{1}{n}}$

$= \frac{1}{4}$ ✓

c) $y = 4\cos^{-1}x - 2\pi$

$4\cos^{-1}x = y + 2\pi$

$\cos^{-1}x = \frac{y}{4} + \frac{\pi}{2}$

$x = \cos\left(\frac{y}{4} + \frac{\pi}{2}\right)$ ✓

$= -\sin\left(\frac{y}{4}\right)$

So $\sin\left(\frac{y}{4}\right) = -x$

$y = 4\sin^{-1}(-x)$

$= -4\sin^{-1}x$ ✓

hence y is odd as $\sin^{-1}x$ is odd

d) i) * $f(x) = 1$ gives

$$\int_{-2h}^{2h} 1\,dx = A + B + C$$

$$\therefore A + B + C = 4h$$

* $f(x) = x$ gives

$$\int_{-2h}^{2h} x\,dx = -Ah + Ch$$

$$-Ah + Ch = \left[\frac{x^2}{2}\right]_{-2h}^{2h}$$

$$= 0$$

$$\therefore A = C$$

* $f(x) = x^2$ gives

$$\int_{-2h}^{2h} x^2\,dx = Ah^2 + Ch^2$$

$$\left[\frac{x^3}{3}\right]_{-2h}^{2h} = Ah^2 + Ch^2$$

$$\frac{16h^3}{3} = Ah^2 + Ch^2$$

Using $A = C$, $\quad 2Ah^2 = \frac{16h^3}{3}$

$$A = \frac{8h}{3}$$

$$C = \frac{8h}{3}$$

and $B + \frac{16h}{3} = 4h$

$$B = -\frac{4h}{3}$$

$$\therefore A = \frac{8h}{3}, \quad B = -\frac{4h}{3}, \quad C = \frac{8h}{3}$$

ii) $f(x) = x^3$

$$LHS = \int_{-2h}^{2h} x^3\,dx$$

$$= \left[\frac{x^4}{4}\right]_{-2h}^{2h}$$

$$= 0$$

$$RHS = -Ah^3 + Ch^3$$

$$= h^3(-A + C)$$

$$= 0$$

$$= LHS$$

iii) $h = \frac{1}{2}$, $f(x) = 2^x$

$$\therefore \int_{-1}^{1} 2^x\,dx \doteqdot A.2^{-1/2} + B.2^0 + C.2^{1/2}$$

$$\doteqdot \frac{4}{3}\frac{1}{\sqrt{2}} - \frac{2}{3} + \frac{4}{3}\sqrt{2}$$

$$\doteqdot \frac{2}{3}\left(\frac{2}{\sqrt{2}} - 1 + 2\sqrt{2}\right)$$

$$\doteqdot \frac{2}{3}\left(3\sqrt{2} - 1\right)$$