CANDIDATE NUMBER

SYDNEY GRAMMAR SCHOOL

2015 Half-Yearly Examination

FORM VI

MATHEMATICS EXTENSION 1

Tuesday 24th February 2015

General Instructions

- Writing time — 1 hour and 30 minutes
- Write using black or blue pen.
- Board-approved calculators and templates may be used.
- A list of standard integrals is provided at the end of the examination paper.

Total — 55 Marks

- All questions may be attempted.

Section I – 7 Marks

- Questions 1–7 are of equal value.
- Record your solutions to the multiple choice on the sheet provided.

Section II – 48 Marks

- Questions 8–11 are of equal value.
- All necessary working should be shown.
- Start each question in a new booklet.

Collection

- Write your candidate number on each answer booklet and on your multiple choice answer sheet.
- Staple your answers in a single bundle.
- Hand in the booklets in a single well-ordered pile.
- Hand in a booklet for each question in Section II, even if it has not been attempted.
- If you use a second booklet for a question, place it inside the first.
- Place your multiple choice answer sheet inside the answer booklet for Question Eight.
- Write your candidate number on this question paper and hand it in with your answers.

Checklist

- SGS booklets — 4 per boy
- Multiple choice answer sheet
- Candidature — 114 boys

Examiner
LYL

SECTION I - Multiple Choice

Answers for this section should be recorded on the separate answer sheet handed out with this examination paper.

QUESTION ONE

1

A parabola has a vertex at $(0, 4)$, focal length 2 units and its axis is parallel to one of the coordinate axes. Which equation fits the given conditions?

(A) $(y+4)^2 = -4x$

(B) $(y-4)^2 = 4x$

(C) $x^2 = -8(y-4)$

(D) $x^2 = 8(y+4)$

QUESTION TWO

1

What is the period of $y = -3\sin\frac{1}{2}x$?

(A) $\frac{\pi}{2}$

(B) 2π

(C) π

(D) 4π

QUESTION THREE

1

If θ is the acute angle between the lines $y = -\frac{1}{3}x - 3$ and $y = 2x + 3$, then the value of $\tan\theta$ is:

(A) 7

(B) 1

(C) -7

(D) -1

Exam continues next page ...

QUESTION FOUR

$\boxed{1}$

What is the Cartesian equation of the curve $x = at^2$, $y = 2at$?

(A) $y^2 = 4ax$

(B) $y^2 = 2ax$

(C) $x^2 = 4ay$

(D) $x^2 = 2ay$

QUESTION FIVE

$\boxed{1}$

The angle θ satisfies $\cos\theta = \frac{4}{5}$ and $-\frac{\pi}{2} < \theta < 0$. What is the value of $\sin 2\theta$?

(A) $\frac{24}{25}$

(B) $-\frac{24}{25}$

(C) $\frac{7}{25}$

(D) $-\frac{7}{25}$

QUESTION SIX

$\boxed{1}$

What is the derivative of $\tan^{-1}\dfrac{1}{x}$?

(A) $\dfrac{1}{1+x^2}$

(B) $-\dfrac{1}{x^2+1}$

(C) $-\dfrac{x^2}{x^2+1}$

(D) $\dfrac{x^2}{x^2+1}$

Exam continues overleaf ...

QUESTION SEVEN

$\boxed{1}$

What is the domain and range of $y = 4\cos^{-1} 3x$?

(A) The domain is $-\frac{1}{3} \le x \le \frac{1}{3}$ and the range is $-2\pi \le y \le 2\pi$.

(B) The domain is $-3 \le x \le 3$ and the range is $-2\pi \le y \le 2\pi$.

(C) The domain is $-\frac{1}{3} \le x \le \frac{1}{3}$ and the range is $0 \le y \le 4\pi$.

(D) The domain is $-3 \le x \le 3$ and the range is $-2\pi \le y \le 2\pi$.

——— End of Section I ———

Exam continues next page ...

SECTION II - Written Response

Answers for this section should be recorded in the booklets provided.

Show all necessary working.

Start a new booklet for each question.

QUESTION EIGHT (12 marks) Use a separate writing booklet.

Marks

(a) A sector has arc length 6 units and radius 4 units. Find the exact area of the sector. **2**

(b) Write down the exact value of:

(i) $\cos^{-1}\left(\frac{1}{2}\right)$ **1**

(ii) $\tan\frac{5\pi}{3}$ **1**

(c) Differentiate the following:

(i) $\tan\frac{x}{3}$ **1**

(ii) $e^x \sin x$ **2**

(iii) $\cos^3 x$ **1**

(d) Find:

(i) $\int \sec^2\frac{x}{3}\,dx$ **1**

(ii) $\int \frac{4}{25+x^2}\,dx$ **1**

(e) Find the exact value of $\cos\left(\sin^{-1}\frac{1}{3}\right)$. **2**

Exam continues overleaf ...

QUESTION NINE (12 marks) Use a separate writing booklet.

Marks

(a) The radius r of a circle is increasing such that the rate of increase of the area of the circle is $\pi^2 r\ \text{cm}^2/\text{s}$. Calculate the rate of increase of the radius. **2**

(b) Consider the function defined by $f(x) = x^2 - 4$, for $x \le 0$.

(i) Draw a neat sketch of the function $y = f(x)$, for $x \le 0$, clearly showing any intercepts with the axes. **1**

(ii) Sketch the graph of the inverse function $y = f^{-1}(x)$. **1**

(iii) State the domain of the inverse function $y = f^{-1}(x)$. **1**

(c) Write down the general solution of $\cos x = \frac{\sqrt{3}}{2}$. Leave your answer in radians. **2**

(d) Evaluate $\int_0^{\frac{\pi}{2}} \cos^2 x\,dx$. **2**

(e) Use mathematical induction to prove $13^n - 1$ is divisible by 3 for all positive integers n. **3**

Exam continues next page ...

QUESTION TEN (12 marks) Use a separate writing booklet. Marks

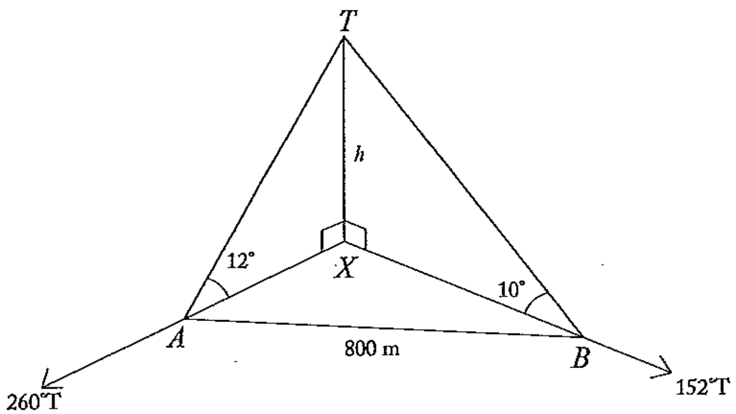
(a) Evaluate $\lim_{x \to 0} \dfrac{x}{\tan 3x}$. You must show working. [1]

(b) Find the equation of the normal to $x^2 = 12y$ at the point $(6p, 3p^2)$. Leave your answer in general form. [2]

(c) Evaluate $\displaystyle\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{dx}{\sqrt{1-2x^2}}$. [2]

(d) Solve the equation $3\tan 2\theta = 2\tan\theta$, for $0 \leq \theta \leq 2\pi$. [3]

(e)



In diagram above TX represents a vertical tower of height h metres standing on the horizontal plane AXB. Two men 800 metres apart on the same plane observe the top of the tower. One man at point A is on a bearing of 260°T from the tower and the angle of elevation to the top of the tower is 12°. The second man at point B is on a bearing of 152° T from the tower and the angle of elevation to the top of the tower is 10°.

(i) Using a diagram, or otherwise, explain why $\angle AXB = 108°$. [1]

(ii) Express AX in terms of h. [1]

(iii) Find the height of the tower to the nearest metre. [2]

Exam continues overleaf ...

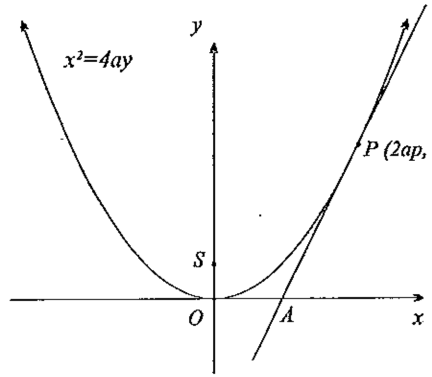
QUESTION ELEVEN (12 marks) Use a separate writing booklet. Marks

(a) Express $8\cos x + 15\sin x$ in the form $R\cos(x-\phi)$ where $R > 0$ and $0° < \phi < 360°$. In your answer, give the angle ϕ correct to the nearest degree. [2]

(b) Prove the identity below using the substitution $t = \tan\frac{\theta}{2}$. [2]

$$\frac{\sin\theta - 1 + \cos\theta}{\sin\theta + 1 - \cos\theta} = \frac{1-\tan\frac{\theta}{2}}{1+\tan\frac{\theta}{2}}$$

(c)



In the diagram above the tangent to the parabola $x^2 = 4ay$ at the point $P(2ap, ap^2)$ meets the tangent to the vertex at the point A. The equation of the tangent at P is $y = px - ap^2$. (Do not prove this.)

(i) If S is the focus, prove that SA is perpendicular to PA. [2]

(ii) It is given that R is the centre of a circle which passes through P, S and A. Determine the equation of the locus of R as P varies. [2]

(d) Consider the curves $y = \sin x$ and $y = \sin^2 x$, where $0 \leq x \leq \frac{\pi}{2}$.

(i) Explain why $\sin^2 x \leq \sin x$, for $0 \leq x \leq \frac{\pi}{2}$. [1]

(ii) Find the volume of revolution generated when the area between the two curves is rotated about the x-axis. [3]

End of Section II

END OF EXAMINATION

The following list of standard integrals may be used:

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x,\ x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\ a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\ a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\ a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\ a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax,\ a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\ a > 0,\ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x + \sqrt{x^2-a^2}\right),\ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x + \sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x,\ x > 0$

FORM VI Extension I HY 2015 (7)

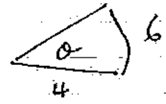
Section I Q1 C, Q2 D, Q3 A, Q4 A, Q5 B, Q6 B, Q7 C.

Section 2

Q8 a) $l = r\theta$

$6 = 4\theta$

$\theta = \frac{3}{2}$ radians ✓



$A = \frac{1}{2}r^2\theta$

$= \frac{1}{2} \times 4^2 \times \frac{3}{2}$

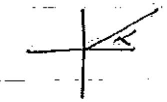
$= 12u^2$ ✓

b) i) Let $\alpha = \cos^{-1}\frac{1}{2}$ where $0 < \alpha \leq \pi$

$\cos\alpha = \frac{1}{2}$

α in 1st quad

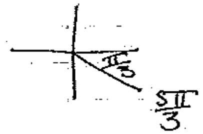
$\alpha = \frac{\pi}{3}$ ✓



ii) Let $\alpha = \tan\frac{5\pi}{3}$

$= -\tan\frac{\pi}{3}$

$= -\sqrt{3}$ ✓



c) i) $y = \tan\frac{x}{3}$

$y' = \frac{1}{3}\sec^2\left(\frac{x}{3}\right)$ ✓

ii) $y = e^x \sin x$ $u = e^x$ $v = \sin x$

$= uv$ $u' = e^x$ $v' = \cos x$ ✓

$y' = vu' + uv'$

$= e^x \sin x + e^x \cos x$

$= e^x(\sin x + \cos x)$ ✓

iii) $y = \cos^3 x$

$y' = -3\cos^2 x \sin x$ ✓

d) i) $\int \sec^2\frac{x}{3}\,dx = 3\tan\frac{x}{3} + c$ ✓

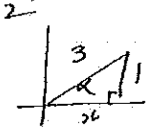
ii) $\int \frac{4}{25+x^2}\,dx = \frac{4}{5}\tan^{-1}\frac{x}{5} + c$ ✓

e) Let $\alpha = \sin^{-1}\frac{1}{3}$ $-\frac{\pi}{2} < \alpha \leq \frac{\pi}{2}$

$\sin\alpha = \frac{1}{3}$

$x^2 + 1^2 = 3^2$

$x = 2\sqrt{2}$ ✓



$\cos\alpha = \frac{2\sqrt{2}}{3}$ ✓

(12)

Q 9 a) $A = \pi r^2$

$\frac{dA}{dr} = 2\pi r$ $\qquad \frac{dA}{dt} = \pi^2 r$

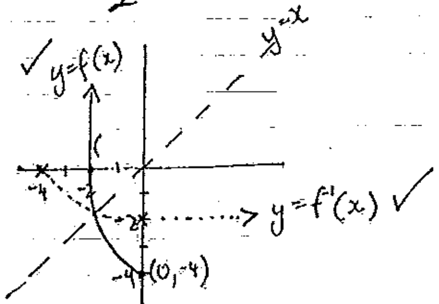
$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$

$\pi^2 r = 2\pi r \times \frac{dr}{dt}$ ✓

$\frac{dr}{dt} = \frac{\pi^{\cancel{2}} \cancel{r}}{2\cancel{\pi}\cancel{r}}$

$= \frac{\pi}{2}$ cm/s ✓ units required.

b) $f(x) = x^2 - 4 \quad x \leq 0$



Domain of $y = f^{-1}(x)$ $\quad x \geq -4$ ✓

c) $\cos x = \frac{\sqrt{3}}{2}$

$x = \cos^{-1}\frac{\sqrt{3}}{2} + 2n\pi$ or $-\cos^{-1}\frac{\sqrt{3}}{2} + 2n\pi \quad n \in \mathbb{Z}$

$= 2n\pi \pm \frac{\pi}{6} \quad n \in \mathbb{Z}$ ✓✓

one mark for $\frac{\pi}{6}$

d) $\int_0^{\frac{\pi}{2}} \cos^2 x \, dx$

$= \int_0^{\frac{\pi}{2}} \frac{1}{2}(1 + \cos 2x) \, dx$

$= \frac{1}{2}\left[x + \frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{2}}$ ✓

$= \frac{1}{2}\left[\left(\frac{\pi}{2} + \frac{1}{2}\sin\pi\right) - (0+0)\right]$

$= \frac{\pi}{4}$ ✓

Q9e) Prove $13^n - 1$ div. by 3 for all positive integers n

A: When $n=1$

$13 - 1 = 12$ which is div. by 3

$\therefore$ Statement true for $n = 1$ ✓

B: Assume the statement holds true for $n=k$ where k is a positive integer.

ie. $13^k - 1 = 3M$ where M is a positive integer.

$13^k = 3M + 1 \quad *$

Must prove true for $n = k+1$

ie. $13^{k+1} - 1 = 3N$ where N is a positive integer. ✓

$LHS = 13^{k+1} - 1$

$= 13^k \cdot 13^1 - 1$

$= (3M+1)13 - 1$ $\quad$ using the induction hypothesis

$= 13 \times 3M + 13 - 1$

$= 13 \times 3M + 12$

$= 3(13M + 4)$ ✓

C. It follows from parts A and B by M I that the statement holds true for all positive integers n.

(12)

Q10 a) $\lim_{x \to 0} \frac{x}{\tan 3x} = \frac{1}{3} \lim_{x \to 0} \frac{3x}{\tan 3x}$ ✓ must have working.

$= \frac{1}{3} \times 1$

$= \frac{1}{3}$

b) $x^2 = 12y \quad (6p, 3p^2)$

$x = 6p \qquad y = 3p^2$

$\frac{dx}{dp} = 6 \qquad \frac{dy}{dp} = 6p$

$\frac{dy}{dx} = \frac{dy}{dp} \div \frac{dx}{dp}$

$= 6p \times \frac{1}{6}$

$= p$

gradient of the normal $= -\frac{1}{p}$ ✓

$y - 3p^2 = -\frac{1}{p}(x - 6p)$

$-py + 3p^3 = x - 6p$ ✓ asked for general form

$x + py - 6p - 3p^3 = 0$

c) $\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{dx}{\sqrt{1-2x^2}} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{dx}{\sqrt{2}\sqrt{\frac{1}{2} - x^2}}$

$= \frac{1}{\sqrt{2}} \left[\sin^{-1}\sqrt{2}x\right]_{-\frac{1}{2}}^{\frac{1}{2}}$ ✓

$= \frac{2}{\sqrt{2}} \left[\sin^{-1}\sqrt{2}x\right]_{0}^{\frac{1}{2}}$

$= \frac{2}{\sqrt{2}} \times \frac{\pi}{4}$

$= \frac{\pi}{2\sqrt{2}}$ ✓

$= \frac{\pi\sqrt{2}}{4}$

Q10 d) $3\tan 2\theta = 2\tan\theta \qquad 0 \le \theta \le 2\pi$

$\frac{3(\tan\theta + \tan\theta)}{1 - \tan^2\theta} = 2\tan\theta$

$6\tan\theta = 2\tan\theta(1 - \tan^2\theta)$ ✓

$= 2\tan\theta - 2\tan^3\theta$

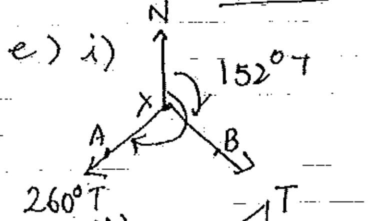
$2\tan^3\theta + 4\tan\theta = 0$

$2\tan\theta(\tan^2\theta + 2) = 0$ ✓

$\tan\theta = 0 \qquad \tan^2\theta = -2$

$\theta = 0, \pi, 2\pi$ ✓ $\qquad$ no solution

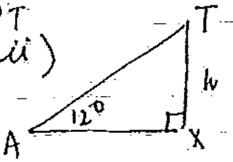
e) i)



$\angle AXB = 260 - 152$

$= 108°$ ✓

ii)



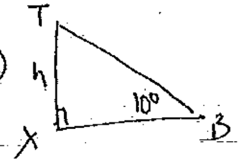
$\tan 12° = \frac{h}{AX}$

$AX = \frac{h}{\tan 12°}$ ✓ or similar

$= h\cot 12°$

$= h\tan 78°$

iii)



$\tan 12° = \frac{h}{BX}$

$BX = \frac{h}{\tan 10°}$

$= h\tan 80°$

$AB^2 = AX^2 + BX^2 - 2AX \times BX \times \cos 108$ ✓

$800^2 = h^2\tan^2 78 + h^2\tan^2 80 - 2h^2\tan 78\tan 80\cos 108$

$h^2 = \frac{800^2}{(\tan^2 78 + \tan^2 80 - 2\tan 78\tan 80\cos 108)}$

$h^2 \doteqdot 9041.218$

$h \doteqdot 95\text{m}$ ✓ (nearest metre)

12

Q11 a) $8\cos x + 15\sin x = R\cos(x-\phi)$

$8\cos x + 15\sin x = R\cos x\cos\phi + R\sin x\sin\phi$

$= \underline{R\cos\phi}\cos x + \underline{R\sin\phi}\sin x$

equating coefficients

$R\cos\phi = 8$

$R\sin\phi = 15$

$R = \sqrt{8^2+15^2}$

$= 17$ ✓

$\cos\phi = \frac{8}{17}$

$\sin\phi = \frac{15}{17}$

$\phi = \sin^{-1}\left(\frac{15}{17}\right)$

$\doteqdot 61.97°$ ✓

$\therefore\ 8\cos x + 15\sin x = 17\cos(x-62)$

b) LHS $= \dfrac{\sin\theta - 1 + \cos\theta}{\sin\theta + 1 - \cos\theta}$

$= \dfrac{\frac{2t}{1+t^2} - 1 + \frac{1-t^2}{1+t^2}}{\frac{2t}{1+t^2} + 1 - \frac{(1-t^2)}{1+t^2}}$ ✓

$= \dfrac{\frac{2t-(1+t^2)+1-t^2}{1+t^2}}{\frac{2t+(1+t^2)-1+t^2}{1+t^2}}$

$= \dfrac{2t \cancel{-1} - t^2 \cancel{+1} - t^2}{1+t^2} \div \dfrac{2t \cancel{+1} + t^2 \cancel{-1} + t^2}{1+t^2}$

$= \dfrac{2t-2t^2}{\cancel{1+t^2}} \times \dfrac{\cancel{1+t^2}}{2t+2t^2}$ ✓

$= \dfrac{\cancel{2t}(1-t)}{1} \times \dfrac{1}{\cancel{2t}(1+t)}$

$= \dfrac{1-t}{1+t}$ $\left(t = \tan\frac{\theta}{2}\right)$

$=$ RHS as required.

Q11 c) i) Given $y = px - ap^2$ eq'n of the tgt.

when $y=0$

$px - ap^2 = 0$

$px = ap^2$

$\underline{x = ap}$

Coordinates of A $(ap, 0)$ ✓

$m_{PA} = \dfrac{ap^2}{ap}$

$= p$

$m_{SA} = \dfrac{a-0}{0-ap}$

$= -\dfrac{1}{p}$

$m_{PA} \times m_{SA} = p \times -\dfrac{1}{p}$

$= -1$ ✓

$\therefore\ PA \perp SA$

ii) Since $\angle PAS$ is a right angle

PS is a diameter of a circle

R is the midpoint of PS.

which is the centre C.

$C\left(\dfrac{0+2ap}{2}, \dfrac{a+ap^2}{2}\right)$ ✓

$= C\left(ap, \dfrac{a(1+p^2)}{2}\right)$

$x = ap$

$p = \dfrac{x}{a}$

$y = \dfrac{a\left(1+\frac{x^2}{a^2}\right)}{2}$

$2y = a + \dfrac{x^2}{a}$ ✓

$x^2 = a(2y-a)$

Q11 d) i) At $x=0$ and $x=\frac{\pi}{2}$ $\sin^2 x = \sin x$

For $0 < x < \frac{\pi}{2}$ $0 < \sin x < 1$

so $\sin^2 x$ will always have a smaller value than $\sin x$.

ii) $V = \pi \int_0^{\frac{\pi}{2}} (y_1{}^2 - y_2{}^2)\, dx$ $\quad y_1 = \sin x$, $y_2 = \sin^2 x$

$= \pi \int_0^{\frac{\pi}{2}} (\sin^2 x - \sin^4 x)\, dx$

$= \pi \int_0^{\frac{\pi}{2}} \sin^2 x(1 - \sin^2 x)\, dx$

$= \pi \int_0^{\frac{\pi}{2}} \sin^2 x \cos^2 x\, dx$

$= \frac{\pi}{4} \int_0^{\frac{\pi}{2}} (2\sin x \cos x)^2\, dx$

$= \frac{\pi}{4} \int_0^{\frac{\pi}{2}} \sin^2 2x\, dx$

$= \frac{\pi}{4} \int_0^{\frac{\pi}{2}} \left(\frac{1}{2} - \frac{1}{2}\cos 4x\right) dx$ ✓

$= \frac{\pi}{8} \int_0^{\frac{\pi}{2}} (1 - \cos 4x)\, dx$

$= \frac{\pi}{8} \left[x - \frac{1}{4}\sin 4x \right]_0^{\frac{\pi}{2}}$ ✓

$= \frac{\pi}{8}\left(\left(\frac{\pi}{2} - \frac{1}{4}\sin 2\pi\right) - (0-0) \right)$

$= \frac{\pi^2}{16}\ u^3$ ✓

12