CANDIDATE NUMBER

SYDNEY GRAMMAR SCHOOL

2016 Assessment Examination

FORM VI

MATHEMATICS 2 UNIT

Thursday 19th May 2016

General Instructions

- Writing time — 2 hours
- Write using black pen.
- Board-approved calculators and templates may be used.

Total — 75 Marks

- All questions may be attempted.

Section I – 10 Marks

- Questions 1–10 are of equal value.
- Record your answers to the multiple choice on the sheet provided.

Section II – 65 Marks

- Questions 11–15 are of equal value.
- All necessary working should be shown.
- Start each question in a new booklet.

Collection

- Write your candidate number on each answer booklet and on your multiple choice answer sheet.
- Hand in the booklets in a single well-ordered pile.
- Hand in a booklet for each question in Section II, even if it has not been attempted.
- If you use a second booklet for a question, place it inside the first.
- Write your candidate number on this question paper and hand it in with your answers.
- Place everything inside the answer booklet for Question Eleven.

Checklist

- SGS booklets — 5 per boy
- Multiple choice answer sheet
- Reference sheet
- Candidature — 88 boys

Examiner
LRP

SECTION I - Multiple Choice

Answers for this section should be recorded on the separate answer sheet handed out with this examination paper.

QUESTION ONE

What is $\dfrac{4\pi}{3}$ radians expressed in degrees? [1]

(A) $120°$

(B) $150°$

(C) $210°$

(D) $240°$

QUESTION TWO

What is the period of the function $f(x) = 2\sin 3x$? [1]

(A) $\frac{\pi}{3}$

(B) $\frac{2\pi}{3}$

(C) π

(D) 6π

QUESTION THREE

What is the gradient of the tangent to the curve $y = \tan x$ at $x = \pi$? [1]

(A) 1

(B) 0

(C) -1

(D) undefined

Examination continues next page ...

QUESTION FOUR $\boxed{1}$

Which of the following is a primitive of $3e^{-2x}$?

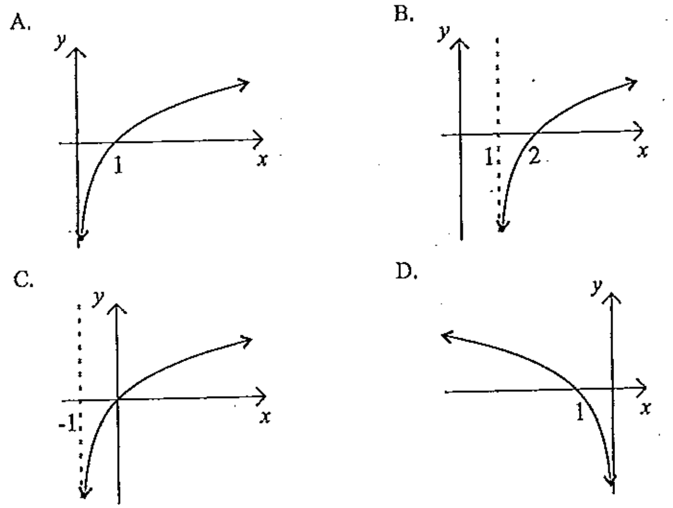
(A) $-6e^{-2x}$

(B) $6e^{-2x}$

(C) $\dfrac{3e^{-2x}}{2}$

(D) $-\dfrac{3e^{-2x}}{2}$

QUESTION FIVE $\boxed{1}$



Which of the above is the graph of $y = \log_e(x-1)$?

(A) A

(B) B

(C) C

(D) D

QUESTION SIX $\boxed{1}$

Which of the following is an expression for x given $y = 2a^{3x} + 5$?

(A) $\dfrac{1}{3}\log_a\left(\dfrac{y-5}{2}\right)$

(B) $\dfrac{1}{6}\log_a(y-5)$

(C) $\dfrac{1}{6}\log_a\left(\dfrac{y}{5}\right)$

(D) $\log_a\left(\dfrac{y-5}{6}\right)$

QUESTION SEVEN $\boxed{1}$

Which of the following statements about $\cos\theta$ is NOT TRUE?

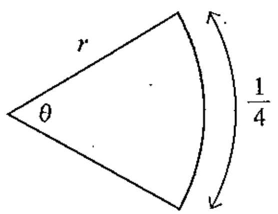
(A) $\cos(\pi - \theta) = -\cos(-\theta)$

(B) $\cos(-\theta) = -\cos\theta$

(C) $\cos(\pi - \theta) = \cos(\pi + \theta)$

(D) $\cos(2\pi - \theta) = \cos\theta$

QUESTION EIGHT $\boxed{1}$



The diagram above shows a sector with radius r and angle θ, where $0 < \theta < 2\pi$.
Given that the arc length is $\frac{1}{4}$, which of the following is the best statement that can be made about r?

(A) $r > \dfrac{1}{8\pi}$

(B) $0 < r < \dfrac{1}{8\pi}$

(C) $r > \dfrac{1}{4\pi}$

(D) $0 < r < \dfrac{1}{4\pi}$

QUESTION NINE $\boxed{1}$

What is the range of the function $y = 2\sin x$ over the domain $\frac{\pi}{4} \leq x \leq \frac{2\pi}{3}$?

(A) $\sqrt{3} \leq y \leq 2$

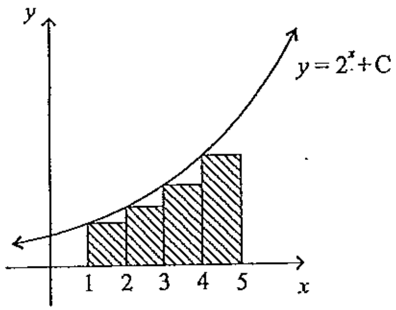
(B) $-2 \leq y \leq 2$

(C) $\sqrt{2} \leq y \leq \sqrt{3}$

(D) $\sqrt{2} \leq y \leq 2$

Examination continues overleaf ...

QUESTION TEN $\boxed{1}$



Consider the graph of $y = 2^x + C$, where C is a constant. The area of the shaded rectangles is used to find an approximation to the area of the region that is bounded by the graph, the x-axis and the lines $x = 1$ and $x = 5$.

What is the value of C if the total area of the shaded rectangles is 32 square units?

(A) 2

(B) $\frac{1}{2}$

(C) -6

(D) -7

End of Section I

Examination continues next page ...

SECTION II - Written Response

Answers for this section should be recorded in the booklets provided.

Show all necessary working.

Start a new booklet for each question.

QUESTION ELEVEN (13 marks) Use a separate writing booklet. **Marks**

(a) Evaluate $6\cos\frac{\pi}{5}$ correct to three significant figures. 1

(b) Solve $4^x = 40$ correct to three decimal places. 2

(c) Simplify $\tan x \cos x$. 1

(d) Find the exact value of $\tan\frac{5\pi}{3}$. 1

(e) Differentiate:

(i) $3e^{4x}$ 1

(ii) $\log_e(3x+2)$ 1

(iii) $\sin\frac{x}{5}$ 1

(f) Find:

(i) $\int \sin 6x\,dx$ 1

(ii) $\int \frac{2}{5x+1}\,dx$ 2

(g) Find the equation of the tangent to the curve $y = \sin x$ at $x = \pi$. 2

QUESTION TWELVE (13 marks) Use a separate writing booklet. **Marks**

(a) Differentiate the following with respect to x:

(i) $x^2 \log_e x$ 2

(ii) $\cos^4 x$ 2

(b) Evaluate:

(i) $\int_0^{\frac{\pi}{6}} \sec^2 2x\,dx$ 2

(ii) $\int_1^3 \frac{3x^2}{x^3+1}\,dx$ 2

(c) Factorise $e^{2x} + 2e^x$. 1

(d) Solve the following equations for $0 \le \theta \le 2\pi$:

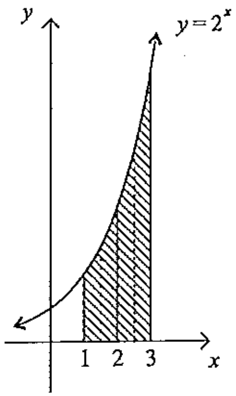
(i) $\cos\theta = 0$ 2

(ii) $2\cos\theta = 1$ 2

QUESTION THIRTEEN (13 marks) Use a separate writing booklet. **Marks**

(a) If $f'(x) = 1 - \dfrac{3}{x}$ and $f(e) = -2$, find $f(x)$. [2]

(b)



In the diagram above, the shaded region is bounded by the curve $y = 2^x$, the x-axis and the lines $x = 1$ and $x = 3$.

(i) Copy and complete the table below, giving your answers in simplified surd form. [2]

x	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3
y		$2\sqrt{2}$			

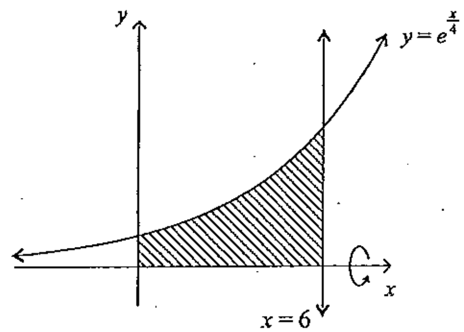
(ii) Use Simpson's rule with five function values to estimate the area of the shaded region. Express your answer in surd form. [2]

(c) Consider the curve $y = \log_e x - x$.

(i) Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$. [2]

(ii) Hence find the x-coordinate of the stationary point and determine its nature. [2]

(d)

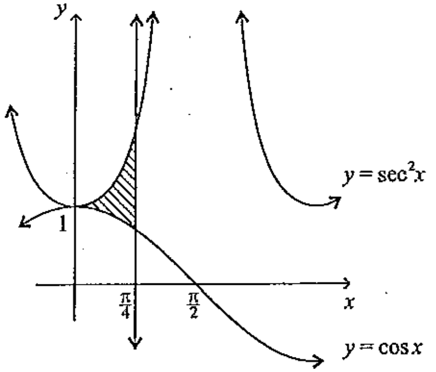


The diagram shows the region bounded by the curve $y = e^{\frac{x}{4}}$, the x-axis, the y-axis and the line $x = 6$. Find the exact volume of the solid formed when this region is rotated about the x-axis. [3]

QUESTION FOURTEEN (13 marks) Use a separate writing booklet. **Marks**

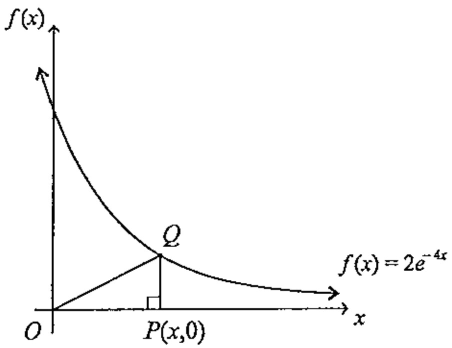
(a) Solve $\log_e x + \log_e x^2 = 3$. **2**

(b)



The diagram above shows the shaded region bounded by the graphs of $y = \sec^2 x$, $y = \cos x$ and $x = \frac{\pi}{4}$. Find the area of the shaded region. **3**

(c)



A right-angled triangle OPQ has vertex O at the origin, vertex P on the x-axis and vertex Q on the graph of $f(x) = 2e^{-4x}$, as shown.

(i) Show that the area A of triangle OPQ is given by $A = xe^{-4x}$. **1**

(ii) Find the maximum area of triangle OPQ as P varies. You must justify that it is a maximum. **3**

(d) (i) Show that $2 + \sin x - 2\cos^2 x = 2\sin^2 x + \sin x$. **1**

(ii) Hence find all solutions of $2 + \sin x - 2\cos^2 x = 0$, where $0 \le x \le 2\pi$. **3**

Examination continues overleaf ...

QUESTION FIFTEEN (13 marks) Use a separate writing booklet. **Marks**

(a) Luke the gardener works in a temperature-controlled greenhouse. During a particular 24-hour time interval, the temperature T degrees Celsius after t hours was given by:

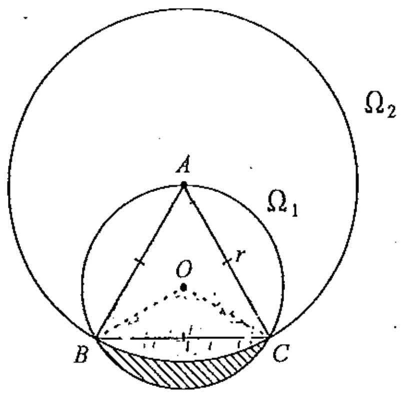
$$T = 21 + 3\cos\frac{\pi t}{8} \quad \text{for } 0 \le t \le 24.$$

(i) What is the period of the function T? **1**

(ii) Find the maximum temperature in the greenhouse and the values of t when this occurred. [HINT: a sketch would be useful.] **3**

(iii) Find the first time when the temperature was 22·5°C. **2**

(b)



Consider two circles, Ω_1 and Ω_2. Circle Ω_1 has centre O and circumscribes equilateral triangle ABC of side r cm. Circle Ω_2 has centre A and passes through both B and C, as shown.

(i) Find the area of minor sector ABC of Ω_2. **1**

(ii) Using $\triangle OBC$, show that the radius of Ω_1 is $\frac{r}{\sqrt{3}}$ cm. **2**

(iii) Find the area of $\triangle OBC$ in terms of r. **1**

(iv) Find the area of minor sector OBC of Ω_1. **1**

(v) Hence find the area of the region shaded in the diagram. **2**

End of Section II

END OF EXAMINATION

FORM VI MATHEMATICS

May Assessment - Solutions

Q1. $\frac{4\pi}{3} \times \frac{180^\circ}{\pi} = 240^\circ$ D

Q2. Period $= \frac{2\pi}{3}$ B

Q3. $\frac{dy}{dx} = \sec^2 x$

when $x = \pi$, $\frac{dy}{dx} = \sec^2 \pi$

$= \frac{1}{(\cos\pi)^2}$

$= \frac{1}{(-1)^2}$

$= 1$ A

Q4. $\int 3e^{-2x}\,dx = \frac{3e^{-2x}}{-2} + C$ D

Q5. $y = \log_e(x-1)$

→ shift 1 unit right

/ $x \neq 1$

/ when $y=0$ $x = e^0 + 1$

$= 2$ B

Q6. $y = 2a^{3x} + 5$

$a^{3x} = \frac{y-5}{2}$

$3x = \log_a\left(\frac{y-5}{2}\right)$

$x = \frac{1}{3}\log_a\left(\frac{y-5}{2}\right)$ A

Q7. $\cos(-\theta) = \cos\theta$

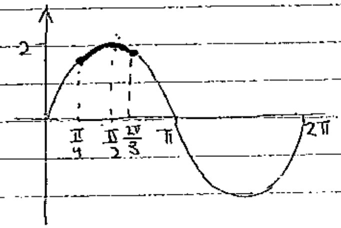
$\therefore$ B is not true B

Q8. $l < 2\pi r$

$\frac{1}{4} < 2\pi r$

$\therefore r > \frac{1}{8\pi}$ A

Q9.



$2\sin\frac{\pi}{4} = \frac{2}{\sqrt{2}} = \sqrt{2}$

$2\sin\frac{2\pi}{3} = 2 \times \frac{\sqrt{3}}{2} = \sqrt{3}$

$\sqrt{2} \leq y \leq 2$ D

Q10. $(2^1+c)\times 1 + (2^2+c)\times 1 + (2^3+c)\times 1 + (2^4+c)\times 1 = 32$

$2 + 4 + 8 + 16 + 4c = 32$

$c = \frac{1}{2}$ B

QUESTION 11:

a) $6\cos\frac{\pi}{5} = 4.85410\ldots$ ✓

$= 4.85$ (to 3 sig. fig)

b) $4^x = 40$

$x = \log_4 40$ ✓

$= \frac{\log 40}{\log 4}$

$= 2.66096\ldots$

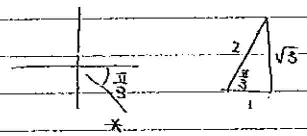
$= 2.661$ (to 3 d.p.) ✓ ROUNDING Q.

c) $\tan x \cos x = \frac{\sin x}{\cos x} \times \cos x$

$= \sin x$ ✓

d) $\tan\frac{5\pi}{3} = -\tan\frac{\pi}{3}$

$= -\sqrt{3}$ ✓



e) i) $\frac{d}{dx}\left(3e^{4x}\right) = 12e^{4x}$ ✓

ii) $\frac{d}{dx}\left(\log_e(3x+2)\right) = \frac{3}{3x+2}$ ✓

iii) $\frac{d}{dx}\left(\sin\frac{x}{5}\right) = \frac{1}{5}\cos\frac{x}{5}$ ✓

f) i) $\int \sin 6x \, dx = -\frac{1}{6}\cos 6x + C$ ✓

f) ii) $\int \frac{2}{5x+1}\,dx = \frac{2}{5}\int \frac{5}{5x+1}\,dx$ ✓

$= \frac{2}{5}\ln|5x+1| + C$ ✓

g) $\frac{dy}{dx} = \cos x$

when $x = \pi$, $\frac{dy}{dx} = \cos\pi$

$= -1$ ✓

$y = \sin\pi$

$= 0$

$y - 0 = -1(x - \pi)$

$\therefore x + y - \pi = 0$ ✓

(or $y = -x + \pi$)

/13

QUESTION 12:

a) i) Let $y = x^2 \log_e x$

$u = x^2 \qquad v = \log_e x$

$u' = 2x \qquad v' = \frac{1}{x}$

$y' = \log_e x \times 2x + x^2 \times \frac{1}{x}$

$= x(2\log_e x + 1)$

ii) Let $y = (\cos x)^4$

$y' = 4(\cos x)^3 \times -\sin x$

$= -4\sin x \cos^3 x$

b) i) $\int_0^{\frac{\pi}{6}} \sec^2 2x \, dx = \left[\frac{\tan 2x}{2}\right]_0^{\frac{\pi}{6}}$

$= \frac{1}{2}\left(\tan\frac{\pi}{3} - \tan 0\right)$

$= \frac{\sqrt{3}}{2}$

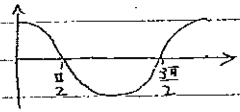
ii) $\int_1^3 \frac{3x^2}{x^3+1} \, dx = \left[\log_e(x^3+1)\right]_1^3$

$= \log_e 28 - \log_e 2$

$= \log_e 14$

c) $e^{2x} + 2e^x = e^x(e^x + 2)$

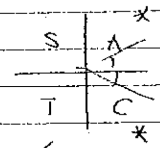
d) i) $\cos\theta = 0$

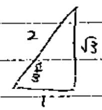


$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$

ii) $2\cos\theta = 1$

$\cos\theta = \frac{1}{2}$





Related $\angle = \frac{\pi}{3}$

$\therefore \theta = \frac{\pi}{3}, \frac{5\pi}{3}$

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QUESTION 13:

a) $f'(x) = 1 - \frac{3}{x}$

$f(x) = x - 3\ln x + c$ ✓

$f(e) = -2$:

$-2 = e - 3\ln e + c$

$-2 = e - 3 + c$

$\therefore c = 1 - e$ ✓

$\therefore f(x) = x - 3\ln x + 1 - e$

b) i)

x	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3
y	2^1 $= 2$	$2^{\frac{3}{2}}$ $2\sqrt{2}$	2^2 $= 4$	$2^{\frac{5}{2}}$ $= 4\sqrt{2}$	2^3 $= 8$

✓ ✓ (rest)

ii) $A \doteqdot \frac{2-1}{6}\left(2 + 4 \times 2\sqrt{2} + 4\right) + \frac{3-2}{6}\left(4 + 4 \times 4\sqrt{2} + 8\right)$ ✓

$= \frac{1}{6}\left(6 + 8\sqrt{2} + 12 + 16\sqrt{2}\right)$

$= \frac{1}{6}\left(18 + 24\sqrt{2}\right)$

$= 3 + 4\sqrt{2}$ ✓

c) $y = \log_e x - x$

i) $\frac{dy}{dx} = \frac{1}{x} - 1$ ✓

$= x^{-1} - 1$

$\frac{d^2y}{dx^2} = -x^{-2}$ ✓

$= -\frac{1}{x^2}$

ii) $\frac{dy}{dx} = 0$:

$\frac{1}{x} - 1 = 0$

$\frac{1}{x} = 1$

$\therefore x = 1$ ✓

when $x = 1$:

$\frac{d^2y}{dx^2} = -\frac{1}{1^2}$

$= -1$

< 0 $\therefore \frown$ $\therefore$ local maximum turning point when $x = 1$ ✓

d) $V = \pi \int_0^6 \left(e^{\frac{x}{4}}\right)^2 dx$ ✓

$= \pi \int_0^6 e^{\frac{x}{2}} \, dx$

$= \pi \left[2e^{\frac{x}{2}} \right]_0^6$ ✓

$= \pi \left(2e^3 - 2e^0 \right)$

$= 2\pi \left(e^3 - 1 \right)$ ✓ cubic units

/13

QUESTION 14:

a) $\log_e x + \log_e x^2 = 3$

$$\log_e x + 2\log_e x = 3$$

$$3\log_e x = 3$$

$$\log_e x = 1$$

$$x = e^1$$
$$= e$$

b) $A = \int_0^{\frac{\pi}{4}} (\sec^2 x - \cos x)\,dx$

$$= \left[\tan x - \sin x\right]_0^{\frac{\pi}{4}}$$

$$= \tan\tfrac{\pi}{4} - \sin\tfrac{\pi}{4} - (\cancel{\tan 0} - \sin 0)$$

$$= 1 - \frac{1}{\sqrt{2}} \text{ square units}$$

$$\left(= \frac{2-\sqrt{2}}{2} \text{ sq. units} \right)$$

c) i) $A = \frac{1}{2} \times x \times 2e^{-4x}$ (show that)

$$= xe^{-4x}$$

ii) Let $u = x$, $v = e^{-4x}$
$u' = 1$, $v' = -4e^{-4x}$

$$\frac{dA}{dx} = e^{-4x} \times 1 + x \times -4e^{-4x}$$

$$= e^{-4x}(1-4x)$$

$$= 0 \text{ when } x = \frac{1}{4} \qquad (e^{-4x} \neq 0)$$

c) ii) cont...

x	$\frac{1}{8}$	$\frac{1}{4}$	1
$\frac{dA}{dx}$	$e^{-\frac{1}{2}}(1-\frac{1}{2})$ $= \frac{1}{2}e^{-\frac{1}{2}}$ > 0	0	$e^{-4}(1-4)$ $= -3e^{-4}$ < 0
	$+$	$-$	$-$

$\therefore$ max area occurs when $x = \frac{1}{4}$

$$A_{max} = \frac{1}{4} e^{-4 \times \frac{1}{4}}$$

$$= \frac{1}{4e} \text{ square units}$$

OR// to gain the second mark, a much less efficient approach would be to investigate the second derivative...

Let $u = e^{-4x}$, $v = 1 - 4x$
$u' = -4e^{-4x}$, $v' = -4$

$$\frac{d^2A}{dx^2} = (1-4x) \times -4e^{-4x} + e^{-4x} \times -4$$

$$= -8e^{-4x} + 16xe^{-4x}$$

$$= 8e^{-4x}(2x-1)$$

when $x = \frac{1}{4}$, $\frac{d^2A}{dx^2} = 8e^{-1}\left(-\frac{1}{2}\right)$

$$= -\frac{4}{e}$$

< 0 $\therefore \cap$ $\therefore$ max occurs when $x = \frac{1}{4}$.

d) i) $LHS = 2 + \sin x - 2\cos^2 x$

$= 2 + \sin x - 2(1 - \sin^2 x)$ ✓

$= 2 + \sin x - 2 + 2\sin^2 x$

$= 2\sin^2 x + \sin x$

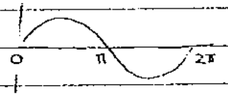
$= RHS$

ii) $2\sin^2 x + \sin x = 0$

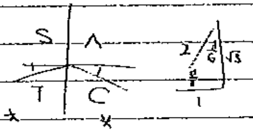
$\sin x(2\sin x + 1) = 0$ ✓

$\therefore \sin x = 0$ or $\sin x = -\frac{1}{2}$

$x = 0, \pi, 2\pi$ ✓



Related $\angle = \frac{\pi}{6}$



$\therefore x = \frac{7\pi}{6}, \frac{11\pi}{6}$ ✓

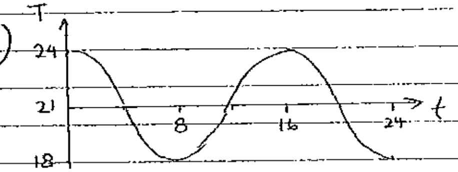
/13

QUESTION 15:

a) i) Period $= \dfrac{2\pi}{\frac{\pi}{8}}$

$= 16$ ✓ (hours)

ii)



✓

$T_{max} = 21 + 3$

$= 24$ (°C) ✓

occurs when $t = 0, 16$ (hours) ✓

Alternatively:

$24 = 21 + 3\cos\frac{\pi t}{8}$ $\qquad 0 \le t \le 24$

$0 \le \frac{\pi t}{8} \le \frac{\pi}{8} \times 24$

$3\cos\frac{\pi t}{8} = 3$

$0 \le \frac{\pi t}{8} \le 3\pi$

$\cos\frac{\pi t}{8} = 1$

$\frac{\pi t}{8} = 0, 2\pi$

$t = 0, 16$ (hours)

iii) $22.5 = 21 + 3\cos\frac{\pi t}{8}$

$1.5 = 3\cos\frac{\pi t}{8}$

$\cos\frac{\pi t}{8} = \frac{1}{2}$ ✓

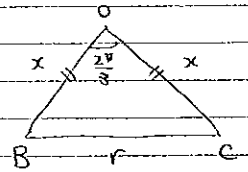
$\frac{\pi t}{8} = \frac{\pi}{3}$ ~~(smallest value only...)~~

$\therefore t = \frac{8}{3}$ ✓ (hours) / $2\frac{2}{3}$ (hours)

b) i) $A_{\text{sector ABC}} = \frac{1}{2} r^2 \times \frac{\pi}{3}$

$= \frac{\pi r^2}{6}$ ✓

ii)



Let the radius of Ω_1 be x

Using the cosine rule: ✓ for angle /diagram

$r^2 = x^2 + x^2 - 2 \times x \times x \times \cos \frac{2\pi}{3}$

$= x^2 \left(2 - 2 \times -\frac{1}{2}\right)$

$= 3x^2$

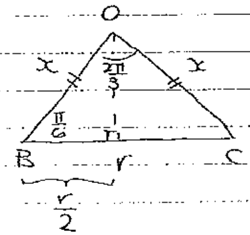
$\therefore x^2 = \frac{r^2}{3}$

$x = \frac{r}{\sqrt{3}} \quad (x > 0)$

✓ show that

$\therefore$ the radius of Ω_1 is $\frac{r}{\sqrt{3}}$

Alternatively:



Let the radius of Ω_1 be x

$\cos \frac{\pi}{6} = \frac{\frac{r}{2}}{x}$ (← or sine ratio equivalent...)

$x = \frac{r}{2 \cos \frac{\pi}{6}}$

$= \frac{r}{2 \times \frac{\sqrt{3}}{2}}$

$= \frac{r}{\sqrt{3}}$

iii) $A_{\triangle OBC} = \frac{1}{2} \times \frac{r}{\sqrt{3}} \times \frac{r}{\sqrt{3}} \times \sin \frac{2\pi}{3}$

$= \frac{r^2}{6} \times \frac{\sqrt{3}}{2}$

$= \frac{r^2\sqrt{3}}{12}$ ✓

iv) $A_{\text{sector OBC}} = \frac{1}{2} \times \left(\frac{r}{\sqrt{3}}\right)^2 \times \frac{2\pi}{3}$

$= \frac{\pi r^2}{9}$ ✓

v) Shaded area $= A_{\text{sector OBC}} + 2 \times A_{\triangle OBC} - A_{\text{sector ABC}}$ ✓

$= \frac{\pi r^2}{9} + \frac{2r^2\sqrt{3}}{12} - \frac{\pi r^2}{6}$

$= r^2 \left(\frac{\pi}{9} - \frac{\pi}{6} + \frac{\sqrt{3}}{6}\right)$

$= \frac{r^2(3\sqrt{3} - \pi)}{18}$ ✓ square units

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