CANDIDATE NUMBER

SYDNEY GRAMMAR SCHOOL

2016 Assessment Examination

FORM VI

MATHEMATICS EXTENSION 1

Monday 23rd May 2016

General Instructions

- Writing time — 2 hours
- Write using black pen.
- Board-approved calculators and templates may be used.

Total — 70 Marks

- All questions may be attempted.

Section I – 10 Marks

- Questions 1–10 are of equal value.
- Record your answers to the multiple choice on the sheet provided.

Section II – 60 Marks

- Questions 11–14 are of equal value.
- All necessary working should be shown.
- Start each question in a new booklet.

Collection

- Write your candidate number on each answer booklet and on your multiple choice answer sheet.
- Hand in the booklets in a single well-ordered pile.
- Hand in a booklet for each question in Section II, even if it has not been attempted.
- If you use a second booklet for a question, place it inside the first.
- Write your candidate number on this question paper and hand it in with your answers.
- Place everything inside the answer booklet for Question Eleven.

Checklist

- SGS booklets — 4 per boy
- Multiple choice answer sheet
- Reference sheet
- Candidature — 111 boys

Examiner
SG

SECTION I - Multiple Choice

Answers for this section should be recorded on the separate answer sheet handed out with this examination paper.

QUESTION ONE

1

Evaluate $\dfrac{d}{dx}\sin^{-1}(2x)$.

(A) $\dfrac{2}{\sqrt{1-x^2}}$

(B) $\dfrac{2}{\sqrt{1-4x^2}}$

(C) $\dfrac{1}{\sqrt{1-4x^2}}$

(D) $\dfrac{1}{2\sqrt{1-4x^2}}$

QUESTION TWO

1

A particle undergoes simple harmonic motion according to the equation $x = 3\cos(2t)$.

What is the period of the motion?

(A) $\dfrac{1}{\pi}$

(B) π

(C) 2π

(D) $\dfrac{\pi}{2}$

QUESTION THREE

1

If $(x+2)$ is a factor of $3x^3 + kx^2 - 31x - 54$, what is the value of k?

(A) 4

(B) −4

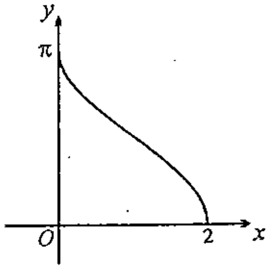
(C) 23

(D) −23

Examination continues next page ...

QUESTION FOUR

[1]



The diagram above shows the graph of:

(A) $y = \sin^{-1}(x-1)$

(B) $y = \sin^{-1}(1-x)$

(C) $y = \cos^{-1}(x-1)$

(D) $y = \cos^{-1}(1-x)$

QUESTION FIVE

[1]

What is the natural domain of the function $\log_e(7-x)$?

(A) $x > 7$

(B) $x < 7$

(C) $x \geq 7$

(D) $x \leq 7$

QUESTION SIX

[1]

The integral $\int xe^{x^2}\,dx$ is:

(A) $e^{x^2} + C$

(B) $2e^{x^2} + C$

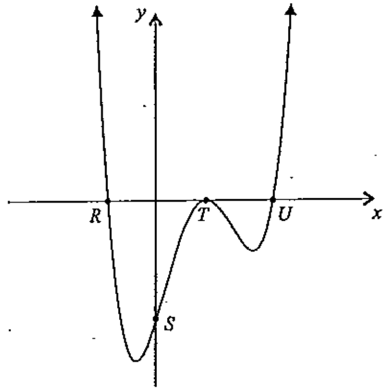
(C) $\frac{1}{2}e^{x^2} + C$

(D) $xe^{x^2} + C$

Examination continues overleaf ...

QUESTION SEVEN

[1]



The diagram above shows the graph of $y = P(x)$ where $P(x) = ax^4 + bx^3 + cx^2 + dx + e$.

A pupil makes the following statements:

I) The y-coordinate of the point S is e.

II) $a > 0$.

III) There is a double zero at the point T.

Which of the above statements are correct?

(A) I only.

(B) I and II only.

(C) I and III only.

(D) I, II and III are correct.

QUESTION EIGHT

[1]

Evaluate $\lim\limits_{h \to 0} \dfrac{\sin(\pi + h) - \sin \pi}{h}$.

(A) 0

(B) 1

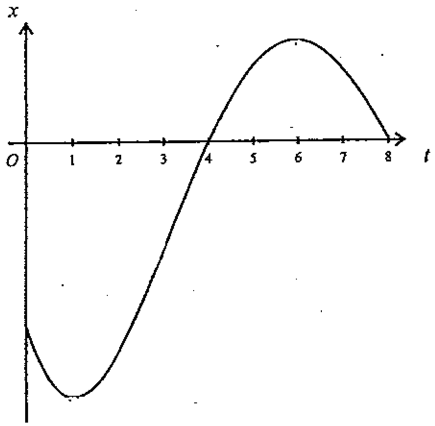
(C) -1

(D) ∞

Examination continues next page ...

QUESTION NINE

1



The graph above shows the position of a particle x along a straight line at a time t, where $0 \leq t < 8$. There are horizontal tangents when $t = 1$ and $t = 6$, and a point of inflexion when $t = 3$.

For what values of t is the particle's velocity increasing?

(A) $0 \leq t < 3$

(B) $1 < t < 6$

(C) $4 < t < 8$

(D) $4 < t < 6$ only.

QUESTION TEN

1

If the length of a rectangle decreases at a rate of 29 cm/s while its width increases at a rate of 29 cm/s, which of the following describes the change in area while the length is greater than the width?

(A) The area is increasing.

(B) The area is decreasing.

(C) The area increases and then decreases.

(D) The area does not change.

End of Section I

Examination continues overleaf ...

SECTION II - Written Response

Answers for this section should be recorded in the booklets provided.

Show all necessary working.

Start a new booklet for each question.

QUESTION ELEVEN (15 marks) Use a separate writing booklet. **Marks**

(a) Differentiate $y = e^x \sin x$ with respect to x. **2**

(b) What is the remainder when $P(x) = x^3 - 3x^2 + 2x - 1$ is divided by $x - 2$? **2**

(c) A particle's displacement x along a straight line at time t is given by the equation $x = e^{2t} + 4t^2 + 7$. Find the acceleration of the particle in terms of t. **2**

(d) Using the identity $\tan^2 x + 1 = \sec^2 x$, evaluate $\int_0^{\frac{\pi}{4}} \tan^2 x \, dx$. **3**

(e) State the domain and range of the function $f(x) = 2\sin^{-1} x$. **2**

(f) Consider the polynomial $P(x) = 2x^3 - 3x^2 - 3x + 2$.

(i) Show that $x + 1$ is a factor of $P(x)$. **1**

(ii) Write $P(x)$ as a product of its factors and hence solve $P(x) = 0$. **3**

Examination continues next page ...

QUESTION TWELVE (15 marks) Use a separate writing booklet. **Marks**

(a) Show that the normal to the parabola $x = 4t$, $y = 2t^2$ has equation **2**

$$yt - 2t^3 = 4t - x.$$

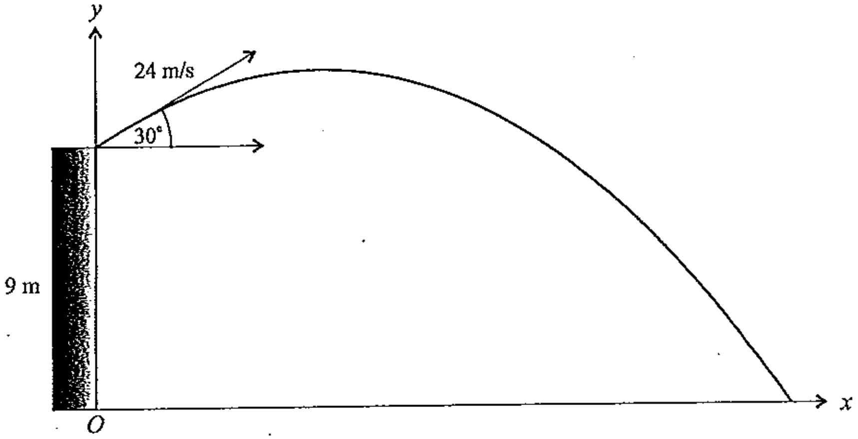
(b) A particle moves along a straight line according to the equation

$$x = te^{-t}.$$

(i) Find the velocity of the particle as a function of time. **2**

(ii) At what time does the particle change direction? **2**

(c)



The diagram above shows an object launched from a cliff nine metres above the ground, at an angle of elevation of 30° and at a speed of 24 m/s. Assume that the acceleration of the object is due only to gravity, with $g = 10\,\text{m/s}^2$. Assume also that the origin lies at the base of the cliff, as shown in the diagram.

The equations of motion are

$$\ddot{x} = 0\text{m/s}^2 \quad \text{and} \quad \ddot{y} = -10\text{m/s}^2.$$

(i) Show that the horizontal and vertical components of the initial velocity are **1**

$$\dot{x} = 12\sqrt{3}\,\text{m/s} \quad \text{and} \quad \dot{y} = 12\,\text{m/s}.$$

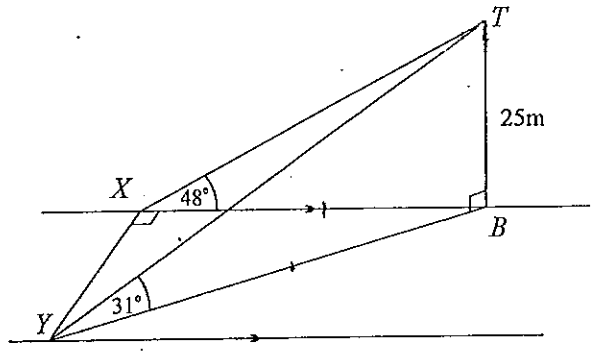
(ii) Hence show that the equations for horizontal and vertical displacement are **2**

$$x = 12t\sqrt{3} \quad \text{and} \quad y = 9 + 12t - 5t^2.$$

(iii) How long is the object in flight? **1**

(iv) At what distance from the base of the cliff does the object hit the ground? **1**

(d)



The base B of a vertical tower BT sits on the edge of a straight river. From a point X on the edge of the same bank, the angle of elevation to the top of the tower is $\angle BXT = 48°$. From point Y, directly opposite X on the other bank, the angle of elevation to the top of the tower is $\angle BYT = 31°$. The height of the tower BT is 25 metres.

(e) Show that the width of the river is given by **3**

$$XY^2 = 625\left(\tan^2 59° - \tan^2 42°\right).$$

(f) Hence find the width of the river, correct to 2 significant figures. **1**

QUESTION THIRTEEN (15 marks) Use a separate writing booklet. **Marks**

(a) Suppose α, β and γ are the roots of the polynomial $2x^3 + 3x^2 + 8x - 4 = 0$.

Find:

(i) $\alpha + \beta + \gamma$ $\boxed{1}$

(ii) $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma}$ $\boxed{2}$

(iii) $\alpha^2 + \beta^2 + \gamma^2$ $\boxed{2}$

(b) Use mathematical induction to show that $3^n - 1$ is divisible by 2 for all integers $n \geq 1$. $\boxed{3}$

(c) Use the substitution $t = \tan\theta$ to prove the identity $\tan 2\theta \cot\theta = 1 + \sec 2\theta$. $\boxed{2}$

(d) A particle moves along a straight line with velocity v given as a function of its displacement x by the equation

$$v^2 = -4x^2 + 8x + 12.$$

(i) Prove that the motion is simple harmonic motion by showing $\ddot{x} = -4(x - 1)$. $\boxed{2}$

(ii) Write down the period of the motion. $\boxed{1}$

(iii) Find the amplitude of the motion. $\boxed{2}$

QUESTION FOURTEEN (15 marks) Use a separate writing booklet. **Marks**

(a) Let $f(x) = x - \sin^{-1}(\cos x)$.

(i) Show that $\boxed{2}$

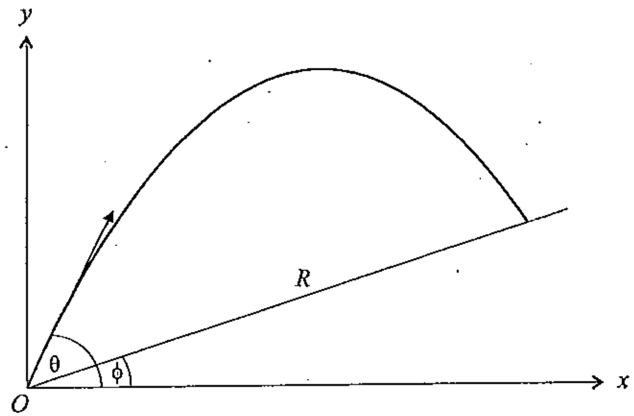
$$f'(x) = 1 + \frac{\sin x}{|\sin x|}$$

where $|\sin x| = \sqrt{\sin^2 x}$.

(ii) Explain why $f'(x) = 2$ for $0 < x < \pi$ and $f'(x) = 0$ for $\pi < x < 2\pi$. $\boxed{1}$

(iii) Hence sketch $y = f(x)$ for $-2\pi \leq x \leq 2\pi$. $\boxed{2}$

(b)



The diagram above shows a projectile fired from the foot of a ramp. The projectile is launched at a speed of v m/s and at an angle of θ to the horizontal. The ramp is inclined at an angle of ϕ to the horizontal such that $\phi < \theta < 90°$.

You may assume that the equation of the projectile's path is

$$y = x\tan\theta - \frac{gx^2\sec^2\theta}{2v^2}.$$

(i) The equation of the ramp is $y = (\tan\phi)x$. Show that the projectile lands at a distance R upon the ramp, where $\boxed{3}$

$$R = \frac{2v^2}{g}(\tan\theta - \tan\phi)\cos^2\theta\sec\phi.$$

(ii) Show that $\dfrac{dR}{d\theta} = \dfrac{2v^2}{g}\sec^2\phi\cos(2\theta - \phi)$, for fixed v and ϕ. $\boxed{2}$

(iii) Hence find the value of θ for which R is a maximum. $\boxed{1}$

(c) Let α, β, γ be the roots of the monic polynomial $P(x)$ so that

$$P(x) = x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha\beta + \alpha\gamma + \beta\gamma)x - \alpha\beta\gamma.$$

(i) By considering $P(\alpha)$, $P(\beta)$ and $P(\gamma)$, show that **[2]**

$$\alpha^3 + \beta^3 + \gamma^3 - 3\alpha\beta\gamma = (\alpha + \beta + \gamma)(\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta - \alpha\gamma - \beta\gamma).$$

(ii) Hence prove that **[2]**

$$\sqrt[3]{p-q} + \sqrt[3]{q-r} + \sqrt[3]{r-p} \neq 0$$

for all distinct real numbers p, q and r.

End of Section II

END OF EXAMINATION

Form VI 2016 Extension I May Assessment: Solutions

Multiple choice

1 B
2 B
3 A
4 C
5 B
6 C
7 D
8 C
9 A
10 A

Question Eleven

(a) $y' = e^x \cos x + e^x \sin x$ ✓ ✓

$= e^x(\cos x + \sin x)$

(b) By remainder theorem,

$R(2) = P(2)$ ✓

$\to \quad R(2) = (2)^3 - 3(2)^2 + 2(2) - 1$

$= -1$ ✓

(c) $x = e^{2t} + 4t^2 + 7$

$\to \dot{x} = 2e^{2t} + 8t$ ✓

$\to \ddot{x} = 4e^{2t} + 8$ ✓

(d) $\int_0^{\pi/4} \tan^2 x \, dx = \int_0^{\pi/4} \sec^2 x - 1 \, dx$ ✓

$= \tan x - x \Big|_0^{\pi/4}$ ✓

$= \left(\tan\frac{\pi}{4} - \frac{\pi}{4}\right) - \left(\tan 0 - 0\right)$

$= 1 - \frac{\pi}{4}$ ✓

(e) Domain of $2\sin^{-1}x$: $-1 \le x \le 1$ ✓

Range of $2\sin^{-1}x$: $-\pi \le 2\sin^{-1}x \le \pi$ ✓

(f) (i) $P(-1) = 2(-1)^3 - 3(-1)^2 - 3(-1) + 2$

$= 0$ ✓

$\to$ by factor theorem, $(x+1)$ divides $P(x)$.

(ii)
$$\begin{array}{r l}
 & 2x^2 - 5x + 2 \\
x+1 \,\big)\!\! & 2x^3 - 3x^2 - 3x + 2 \\
 & \underline{2x^3 + 2x^2} \\
 & \quad -5x^2 - 3x + 2 \\
 & \quad \underline{-5x^2 - 5x} \\
 & \qquad\quad 2x + 2 \\
 & \qquad\quad \underline{2x + 2} \\
 & \qquad\qquad 0
\end{array}$$

✓ OR any other valid method of finding quotient.

So $P(x) = (x+1)(2x^2 - 5x + 2)$

$= (x+1)(2x^2 - 4x - x + 2)$

$= (x+1)(2x(x-2) - (x-2))$

$= (x+1)(x-2)(2x-1)$ ✓

$\therefore\ P(x) = 0 \rightarrow x = -1, \frac{1}{2}, 2.$ ✓

Question Twelve

(a) $x = 4t \quad y = 2t^2$

So $y = 2\left(\frac{x}{4}\right)^2$

$= \frac{x^2}{8}$

$\rightarrow y' = \frac{x}{4}$

$= t$, the gradient at $(4t, 2t^2)$.

$\therefore$ gradient of normal at $(4t, 2t^2)$ is $-\frac{1}{t}$ $(t \neq 0)$. ✓

Equation of normal at $(4t, 2t^2)$:

$y - 2t^2 = -\frac{1}{t}(x - 4t)$

$\Leftrightarrow \quad yt - 2t^3 = 4t - x$ as required. ✓

(b) (i) $x = te^{-t}$

$\rightarrow \dot{x} = t(-e^{-t}) + e^{-t}$ ✓ 1 mark for identifying velocity as derivative of displacement wrt time.

$= e^{-t}(1-t)$ ✓ 1 mark for the correct derivative.

(ii) Directional change occurs when sign of $\dot{x}$ changes; i.e. at t such that $\dot{x} = 0$. ✓

Since $e^{-t} > 0$ for all t, any change in sign will be due to the factor $(1-t)$.

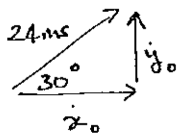
When $t = 1$, $1 - t = 0$.

For $0 \leq t < 1$, $1 - t > 0$ & for $t > 1$, $1 - t < 0$.

Hence the direction changes only once, when $t = 1$. ✓

– Note, $t = 1$ need only be given – no need to justify uniqueness.

(c) (i)



$v_y = 24\sin 30^\circ$ m/s

$= 12$ m/s.

$v_x = 24\cos 30^\circ$ m/s

$= 12\sqrt{3}$ m/s.

(ii) Horizontal displacement:

$\ddot{x} = 0$

$\rightarrow \quad \dot{x} = c_1$ for c_1 constant for all $t \geq 0$.

$\therefore \; c_1 = \dot{x}_0$ for all $t \geq 0$.

So $\dot{x} = 12\sqrt{3}$ m/s

$\rightarrow \quad x = 12\sqrt{3}t + c_2$ for c_2 constant.

When $t=0$, $x=0$ so

$0 = c_2$

$\therefore \; x = 12\sqrt{3}t$ m. ✓

Vertical displacement:

$\ddot{y} = -10$ m/s^2

$\rightarrow \quad \dot{y} = -10t + c_3$ m/s.

When $t=0$, $\dot{y} = c_3 = 12$ m/s.

Hence $\dot{y} = -10t + 12$ m/s.

$\rightarrow \quad y = -5t^2 + 12t + c_4$ m.

When $t=0$ $y = 9$ m, hence

$y = -5t^2 + 12t + 9$ m. ✓

(iii) Time of flight: from $t=0$ to t such that $y=0$.

So

$0 = -5t^2 + 12t + 9$

$= -5t^2 + 15t - 3t + 9$

$= -5t(t-3) - 3(t-3)$

$= -(t-3)(5t+3)$

$\rightarrow \; t = 3, \; -\frac{3}{5}$

Reject $-\frac{3}{5}$ as $t \geq 0$.

$\therefore$ time of flight is 3 seconds. ✓

(iv) Horizontal range:

$x(t) = 12\sqrt{3}t$

$\rightarrow \; x(3) = (12\sqrt{3})(3)$

$= 36\sqrt{3}$ metres. ✓

(d) (i) $\angle BTY = 59^\circ$ (complement of $\angle BYT$)

$\angle BTX = 42^\circ$ (complement of $\angle BXT$)

and $\tan 59^\circ = \frac{BY}{25}$, $\tan 42^\circ = \frac{BX}{25}$. ✓

Now, $XY^2 = BY^2 - BX^2$ (Pythagoras) ✓

So $XY^2 = 25^2\tan^2 59^\circ - 25^2\tan^2 42^\circ$

$= 625(\tan^2 59^\circ - \tan^2 42^\circ)$ ✓

as required.

(ii) $XY = 34.9919\ldots$ m

$\rightarrow \; XY = 35$ m to two significant figures. ✓

Question Thirteen

(a) $2x^3 + 3x^2 + 8x - 4 = 0$

(i) $\alpha + \beta + \gamma = -\frac{3}{2}$ ✓

(ii) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\alpha\beta + \beta\gamma + \gamma\alpha}{\alpha\beta\gamma}$ ✓

$= \frac{4}{2}$

$= 2$ ✓

(iii) $\alpha^2 + \beta^2 + \gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$ ✓

$= \left(-\frac{3}{2}\right)^2 - 2(4)$

$= \frac{9}{4} - 8$

$= -\frac{23}{4}$ ✓

(b) $S(n)$: $3^n - 1 = 2m$ for m some integer.

(I) $S(1)$: $3^1 - 1 = 2$

$= 2 \cdot 1$

So statement is true for $n=1$. ✓

(II) Suppose it's the case that $S(k)$ is true; that is,

$3^k - 1 = 2m$ for m some integer.

(III) $S(k+1)$: $3^{k+1} - 1 = 3 \cdot 3^k - 1$

$= 3(2m+1) - 1$ (by $S(k)$) ✓

$= 2(3m) + 2$

$= 2(3m+1)$

$= 2m'$ where $m' = 3m+1$.

Since m is an integer, $3m+1$ is an integer.

Hence $3^{k+1} - 1$ is divisible by 2

if $3^k - 1$ is divisible by 2. ✓

By the principle of mathematical induction,

$3^n - 1$ is divisible by 2

for all integers $n \geq 1$.

(c) Let $t = \tan\theta$.

Then $\tan 2\theta \cot\theta = \frac{2t}{1-t^2} \cdot \frac{1}{t}$ ✓

$= \frac{2}{1-t^2}$

$= \frac{1-t^2+1+t^2}{1-t^2}$

$= 1 + \frac{1+t^2}{1-t^2}$

$= 1 + \sec 2\theta$ // ✓

(d)

(i) $v^2 = -4x^2 + 8x + 12$

Now, $\ddot{x} = \frac{d}{dx}\frac{1}{2}v^2$

so $\ddot{x} = \frac{d}{dx}\frac{1}{2}(-4x^2 + 8x + 12)$ ✓

$= \frac{d}{dx}(-2x^2 + 4x + 6)$

$= -4x + 4$

$= -4(x-1)$

(ii) The period is $T = \frac{2\pi}{n}$

$= \frac{2\pi}{2}$

$= \pi$

(iii) Amplitude: extremeties of motion occur when $v=0$.

Solving $0 = -4x^2 + 8x + 12$

we have $0 = x^2 - 2x - 3$

$0 = (x-3)(x+1)$

So max. displacement at $x=3$

min. displacement at $x=-1$

$\rightarrow$ [amplitude] $= \frac{3-(-1)}{2}$

$= 2$.

Question Fourteen

(a)

(i) $f(x) = x - \sin^{-1}(\cos x)$

$\rightarrow f'(x) = 1 - \frac{1}{\sqrt{1-\cos^2 x}} \cdot (-\sin x)$

$= 1 + \frac{\sin x}{\sqrt{\sin^2 x}}$

$= 1 + \frac{\sin x}{|\sin x|}$

(ii) In quadrants one & two, $\sin x > 0$. Hence

$f'(x) = 1 + \frac{|\sin x|}{|\sin x|}$

$= 1 + 1$

$= 2$

In quadrants three & four, $\sin x < 0$. So

$f'(x) = 1 - \frac{|\sin x|}{|\sin x|}$

$= 1 - 1$

$= 0$

(iii) Boundary: $f(-2\pi) = -2\pi - \sin^{-1}(\cos(-2\pi))$

$= -2\pi - \frac{\pi}{2}$

$= -\frac{5\pi}{2}$

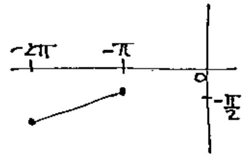
For $-2\pi < x < -\pi$ (quadrants I, II), gradient is 2.

Boundary: $f(-\pi) = -\pi - \sin^{-1}(\cos(-\pi))$

$= -\pi - \sin^{-1}(-1)$

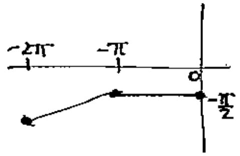
$= -\pi + \pi/2$

$= -\frac{\pi}{2}$



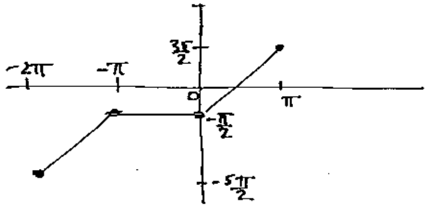
For $-\pi < x < 0$ (quadrants III, IV), gradient is zero.

Boundary: $f(0) = 0 - \sin^{-1}(\cos(0))$
$= 0 - \pi/2$
$= -\pi/2$



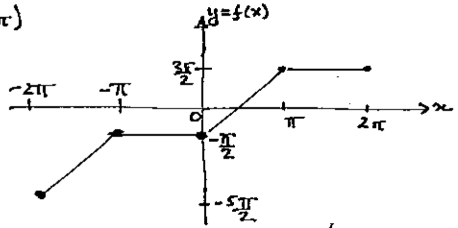
For $0 < x < \pi$ (quad's I, II), gradient is 2.

Boundary: $f(\pi) = \pi - \sin^{-1}(\cos\pi)$
$= \pi - \sin^{-1}(-1)$
$= \pi + \pi/2$
$= \frac{3\pi}{2}$



For $\pi < x < 2\pi$ (quad's III, IV), gradient is 0.

Boundary: $f(2\pi) = 2\pi - \sin^{-1}(\cos 2\pi)$
$= \frac{3\pi}{2}$



√√

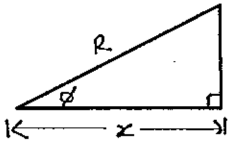
(b) (i) $y = (\tan\phi)x$ &
$y = (\tan\theta)x - \frac{g\sec^2\theta}{2v^2}x^2$

So $x(\tan\phi) = x(\tan\theta) - \frac{g\sec^2\theta}{2v^2}x^2$

$x \neq 0 \rightarrow \tan\phi = \tan\theta - \frac{g\sec^2\theta}{2v^2}.x$

so $x = \frac{2v^2}{g}\cos^2\theta\,(\tan\theta - \tan\phi)$ —— (I) √

Now, by construction, $R = x\sec\phi$, √ (for some valid relation between R & x



so

$R = \frac{2v^2}{g}(\tan\theta - \tan\phi)\cos^2\theta\sec\phi$ // √

(ii) $\frac{dR}{d\theta} = \frac{2v^2}{g}\sec\phi\left[\frac{d}{d\theta}\left(\cos^2\theta\,(\tan\theta - \tan\phi)\right)\right]$ √

$= \frac{2v^2}{g}\sec\phi\left[\cos^2\theta\cdot\sec^2\theta - 2\cos\theta\sin\theta\,(\tan\theta - \tan\phi)\right]$

$= \frac{2v^2}{g}\sec\phi\left[1 - 2\cancel{\cos\theta}\sin\theta\cdot\frac{\sin\theta}{\cancel{\cos\theta}} + 2\cos\theta\sin\theta\tan\phi\right]$

$= \frac{2v^2}{g}\sec\phi\left[\cos 2\theta + \sin 2\theta\cdot\frac{\sin\phi}{\cos\phi}\right]$

$= \frac{2v^2}{g}\sec\phi\left[\frac{1}{\cos\phi}\left(\cos 2\theta\cos\phi + \sin 2\theta\sin\phi\right)\right]$ √

$= \frac{2v^2}{g}\sec^2\phi\cos(2\theta - \phi)$ //

(iii) $\frac{dR}{d\theta} = 0$ at extrema. Hence we have extrema here for

$$\cos(2\theta - \phi) = 0 \rightarrow 2\theta - \phi = \frac{\pi}{2}$$

$$\text{So} \quad \theta = \frac{\phi}{2} + \frac{\pi}{4}. \quad \checkmark$$

For completion (but not necessary for mark):

Note that this result gives a maximum : for $\theta + h$ ($h > 0$ and h 'small'),

$$2(\theta+h) - \phi > 2\theta - \phi, \quad \text{so} \quad \cos(2(\theta+h) - \phi) < 0$$

Also, for $\theta - h$,

$$2(\theta - h) - \phi < 2\theta - \phi, \quad \text{so} \quad \cos(2(\theta - h) - \phi) > 0.$$

Next value for $2\theta - \phi$ that stands a chance of yielding θ within restricted domain such that $\frac{dR}{d\theta} = 0$ is $\frac{3\pi}{2}$. But then $2\theta - \phi = \frac{3\pi}{2} \Leftrightarrow \theta = \frac{3\pi}{4} + \frac{\phi}{2}$. Assume $0 \le \theta \le \frac{\pi}{2}$ & $\theta = \frac{3\pi}{4} + \frac{\phi}{2}$.

Then $0 \le \frac{3\pi}{4} + \frac{\phi}{2} \le \frac{\pi}{2}$

$$\Leftrightarrow \quad -\frac{3\pi}{2} \le \phi \le -\frac{\pi}{2}$$

which is false since $0 \le \phi \le \frac{\pi}{2}$.

So $\theta = \frac{\pi}{4} + \frac{\phi}{2}$ is the unique angle that maximises R here.

(c) $P(x) = x^3 - (\alpha+\beta+\gamma)x^2 + (\alpha\beta + \beta\gamma + \gamma\alpha)x - \alpha\beta\gamma$

(i) Given α, β, γ roots of $P(x)$, we have

$$P(\alpha) = P(\beta) = P(\gamma) = 0. \quad \checkmark$$

So $P(\alpha) + P(\beta) + P(\gamma) = 0$

$$\rightarrow \quad 0 = (\alpha^3+\beta^3+\gamma^3) - (\alpha+\beta+\gamma)(\alpha^2+\beta^2+\gamma^2) + (\alpha\beta+\beta\gamma+\gamma\alpha)(\alpha+\beta+\gamma) - 3\alpha\beta\gamma$$

$$= (\alpha^3+\beta^3+\gamma^3) - 3\alpha\beta\gamma - (\alpha+\beta+\gamma)(\alpha^2+\beta^2+\gamma^2 - \alpha\beta - \beta\gamma - \gamma\alpha)$$

So $(\alpha^3+\beta^3+\gamma^3) - 3\alpha\beta\gamma = (\alpha+\beta+\gamma)(\alpha^2+\beta^2+\gamma^2 - \alpha\beta - \beta\gamma - \gamma\alpha)$ — (I) $\checkmark$

(ii) Assume there exist distinct real numbers p, q, r such that

$$\sqrt[3]{p-q} + \sqrt[3]{q-r} + \sqrt[3]{r-p} = 0 \quad \text{—— (II)}$$

Let $\alpha = \sqrt[3]{p-q}$, $\beta = \sqrt[3]{q-r}$, $\gamma = \sqrt[3]{r-p}$.

Then $\alpha + \beta + \gamma = 0$, forcing RHS of (I) to be zero. $\checkmark$

Now, LHS of (I) gives

$$\alpha^3 + \beta^3 + \gamma^3 - 3\alpha\beta\gamma = (p-q) + (q-r) + (r-p) - 3\sqrt[3]{(p-q)(q-r)(r-p)}$$

$$= 0 - 3\sqrt[3]{(p-q)(q-r)(r-p)}$$

$$\neq 0$$

since p, q, r distinct $\Rightarrow p-q \neq 0,\ q-r \neq 0,\ r-p \neq 0$.

So in (I): LHS $= 0$ while RHS $\neq 0$ – a contradiction. $\checkmark$ #

$\therefore$ There are no distinct real numbers p, q, r such that $\sqrt[3]{p-q} + \sqrt[3]{q-r} + \sqrt[3]{r-p} = 0$

$\Leftrightarrow$ for all distinct real numbers p, q, r,

$$\sqrt[3]{p-q} + \sqrt[3]{q-r} + \sqrt[3]{r-p} \neq 0 \quad //$$