SYDNEY GRAMMAR SCHOOL

2014 Half-Yearly Examination

FORM VI
MATHEMATICS EXTENSION 1

Tuesday 4th March 2014

General Instructions

- Writing time — 2 hours
- Write using black or blue pen.
- Board-approved calculators and templates may be used.
- A list of standard integrals is provided at the end of the examination paper.

Total — 70 Marks

- All questions may be attempted.

Section I – 10 Marks

- Questions 1–10 are of equal value.
- Record your solutions to the multiple choice on the sheet provided.

Section II – 60 Marks

- Questions 11–14 are of equal value.
- All necessary working should be shown.
- Start each question in a new booklet.

Collection

- Write your candidate number on each booklet and on your multiple choice answer sheet.
- Hand in the booklets in a single well-ordered pile.
- Hand in a booklet for each question in Section II, even if it has not been attempted.
- If you use a second booklet for a question, place it inside the first.
- Place your multiple choice answer sheet inside the answer booklet for Question Eleven.
- Write your candidate number on this question paper and submit it with your answers.

Checklist

- SGS booklets — 4 per boy
- Multiple choice answer sheet
- Candidature — 130 boys

Examiner
LRP

SECTION I - Multiple Choice

Answers for this section should be recorded on the separate answer sheet handed out with this examination paper.

QUESTION ONE [1]

What is the period of the function $f(x) = 3\tan 2x$?

(A) 4π

(B) 2π

(C) π

(D) $\frac{\pi}{2}$

QUESTION TWO [1]

Which of the following is the derivative of $\sin^{-1} 2x$?

(A) $\dfrac{2}{\sqrt{1-4x^2}}$

(B) $\dfrac{-2}{\sqrt{1-4x^2}}$

(C) $\dfrac{1}{2\sqrt{1-4x^2}}$

(D) $\dfrac{-1}{2\sqrt{1-4x^2}}$

QUESTION THREE [1]

What is the general solution of $2\cos x + 1 = 0$?

(A) $x = 2n\pi \pm \frac{\pi}{3}$, n is an integer

(B) $x = 2n\pi \pm \frac{2\pi}{3}$, n is an integer

(C) $x = 2n\pi \pm \frac{\pi}{6}$, n is an integer

(D) $x = 2n\pi \pm \frac{5\pi}{6}$, n is an integer

Exam continues next page ...

QUESTION FOUR

$\boxed{1}$

What is the range of the function $y = 2\cos^{-1} 3x$?

(A) $-2 \le y \le 2$

(B) $0 \le y \le \frac{\pi}{3}$

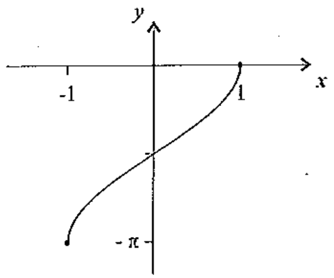
(C) $-\pi \le y \le \pi$

(D) $0 \le y \le 2\pi$

QUESTION FIVE

$\boxed{1}$

Which function does the graph below represent?



(A) $y = \cos^{-1} x - \pi$

(B) $y = \cos^{-1}(-x)$

(C) $y = \sin^{-1} x - \frac{\pi}{2}$

(D) $y = \sin^{-1} x - \pi$

QUESTION SIX

$\boxed{1}$

The angle θ satisfies $\cos\theta = \frac{4}{5}$ and $-\frac{\pi}{2} < \theta < 0$. What is the value of $\sin 2\theta$?

(A) $\frac{24}{25}$

(B) $-\frac{24}{25}$

(C) $\frac{7}{25}$

(D) $-\frac{7}{25}$

Exam continues overleaf ...

QUESTION SEVEN

$\boxed{1}$

Which of the following represents the inverse function of $f(x) = \dfrac{2}{x-3} - 6$?

(A) $f^{-1}(x) = 6 - \dfrac{2}{x-3}$

(B) $f^{-1}(x) = \dfrac{2}{x+6} - 3$

(C) $f^{-1}(x) = \dfrac{2}{x} - 3$

(D) $f^{-1}(x) = \dfrac{2}{x+6} + 3$

QUESTION EIGHT

$\boxed{1}$

How many solutions are there to the equation $\sin 3x = 0$, where $0 \le x \le 2\pi$?

(A) 1

(B) 5

(C) 7

(D) 9

QUESTION NINE

$\boxed{1}$

The volume, $V\,\text{cm}^3$, of water in a container is given by $V = \frac{1}{3}\pi h^3$, where h cm is the depth of water in the container at time t minutes. Water is draining from the container at a constant rate of $100\,\text{cm}^3/\text{min}$. What is the rate of decrease of h, in cm/min, when $h = 5$?

(A) $\dfrac{\pi}{4}$

(B) $\dfrac{4}{\pi}$

(C) 25π

(D) 2500π

Exam continues next page ...

QUESTION TEN

$\boxed{1}$

A trigonometric function has the properties $f(\pi - x) = -f(x)$ and $f(\pi - x) = -f(-x)$ for all real values of x. Which of the following is a possible equation of this function?

(A) $f(x) = \sin x$

(B) $f(x) = \cos x$

(C) $f(x) = \tan x$

(D) $f(x) = \operatorname{cosec} x$

End of Section I

Exam continues overleaf ...

SECTION II - Written Response

Answers for this section should be recorded in the booklets provided.

Show all necessary working.

Start a new booklet for each question.

QUESTION ELEVEN (15 marks) Use a separate writing booklet. **Marks**

(a) Find the acute angle between the lines $y = 2x + 3$ and $y = -3x + 1$. $\boxed{2}$

(b) Write down the exact value of:

(i) $\sin \dfrac{5\pi}{3}$ $\boxed{1}$

(ii) $\tan^{-1}\left(-\dfrac{1}{\sqrt{3}}\right)$ $\boxed{1}$

(c) Differentiate:

(i) $\sin \dfrac{x}{2}$ $\boxed{1}$

(ii) $(\cos^{-1} x)^2$ $\boxed{2}$

(iii) $x \tan 2x$ $\boxed{2}$

(d) Find:

(i) $\displaystyle\int \cos(2x+1)\, dx$ $\boxed{1}$

(ii) $\displaystyle\int \frac{3}{9+x^2}\, dx$ $\boxed{1}$

(e) (i) Show that $\tan x = \dfrac{\sin 2x}{1 + \cos 2x}$. $\boxed{2}$

(ii) Hence evaluate $\tan \frac{\pi}{8}$ in simplest exact form. $\boxed{2}$

Exam continues next page ...

QUESTION TWELVE (15 marks) Use a separate writing booklet. **Marks**

(a) Find the exact value of $\tan\left(2\cos^{-1}\frac{3}{4}\right)$. **2**

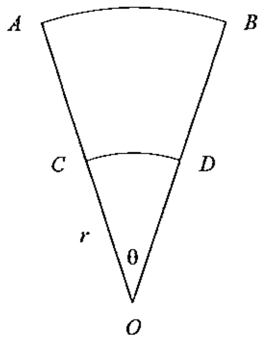
(b) The two points $P\left(2ap,\, ap^2\right)$ and $Q\left(2aq,\, aq^2\right)$ are on the parabola $x^2 = 4ay$.
The equation of the tangent to $x^2 = 4ay$ at an arbitrary point $\left(2at,\, at^2\right)$ is $y = tx - at^2$.
(Do NOT prove this.)

(i) Show that the tangents at the points P and Q meet at $R\left(a\,(p+q)\,,\, apq\right)$. **2**

(ii) Show that R lies on the directrix if the tangents at P and Q are perpendicular. **2**

(c) Solve $\cos 2\theta + 3\sin\theta - 2 = 0$ for $0 \le \theta \le 2\pi$. **3**

(d)



The diagram above shows a sector OAB and an arc CD, both with centre O. The figure is formed by pieces of wire. The area of sector OAB is four times the area of sector OCD.

The length of OC is r cm and $\angle AOB$ is θ.

(i) Show that $AC = r$. **2**

(ii) If the total length of wire forming the figure is 48 cm, show that $\theta = \dfrac{48 - 4r}{3r}$. **1**

(iii) Hence find the value of r that maximises the area of sector OCD. **3**

Exam continues overleaf ...

QUESTION THIRTEEN (15 marks) Use a separate writing booklet. **Marks**

(a) Use the substitution $t = \tan\dfrac{\theta}{2}$ to show that $\operatorname{cosec}\theta + \cot\theta = \cot\dfrac{\theta}{2}$. **2**

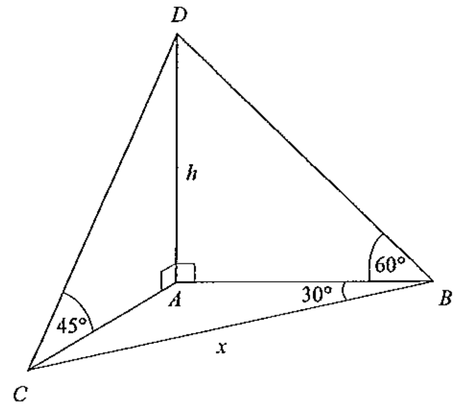
(b) (i) Express $2\sin x + 4\cos x$ in the form $R\sin(x+\alpha)$, where $R > 0$ and $0 \le \alpha < 360°$. Give α correct to the nearest minute. **2**

(ii) Hence, or otherwise, solve the equation $2\sin x + 4\cos x = 3$, for $0 \le x \le 360°$. Give your solutions correct to the nearest minute. **3**

(c) Use mathematical induction to prove that for all integers $n \ge 1$, **3**

$$\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \cdots + \frac{n}{2^n} = 2 - \frac{n+2}{2^n}.$$

(d)



In the diagram above, $ABCD$ is a triangular pyramid with base ABC and perpendicular height $AD = h$.

It is given that $\angle ABC = 30°$, $\angle ACD = 45°$ and $\angle ABD = 60°$. Let BC be x.

(i) Show that $AB = \dfrac{h}{\sqrt{3}}$ and $AC = h$. **1**

(ii) Use the cosine rule to show that $2h^2 + 3xh - 3x^2 = 0$. **2**

(iii) Hence show that $\dfrac{h}{x} = \dfrac{\sqrt{33} - 3}{4}$. **2**

Exam continues next page ...

QUESTION FOURTEEN (15 marks) Use a separate writing booklet. **Marks**

(a) Consider the function $y = 3\cos^{-1}(x-1)$.

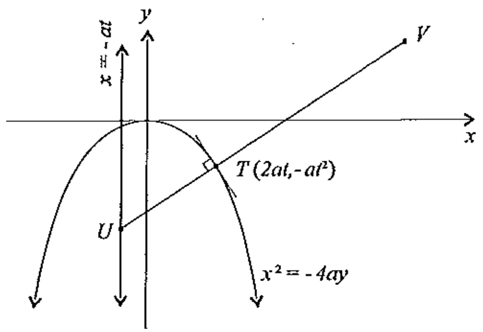
(i) Draw a neat sketch of $y = 3\cos^{-1}(x-1)$, showing the intercepts with the axes. $\boxed{2}$

(ii) Find the exact area enclosed by $y = 3\cos^{-1}(x-1)$ and the coordinate axes. $\boxed{2}$

(b) A function is defined by $f(x) = \sin(x - \frac{\pi}{6})$, where $-a \le x \le a$. $\boxed{2}$

If the inverse of $f(x)$ is a function, what is the maximum possible value of a?

(c)



The point $T(2at, -at^2)$ is a point on the parabola $x^2 = -4ay$. The normal at T meets the line $x = -at$ at point U. Point V lies on the normal and divides TU externally in the ratio $2:3$.

(i) Show that the equation of the normal to the parabola at T is $x - ty = 2at + at^3$. $\boxed{2}$

(ii) Find the coordinates of U in terms of a and t. $\boxed{2}$

(iii) Find the coordinates of V in terms of a and t. $\boxed{2}$

(iv) The locus of V is another parabola. Find the coordinates of its focus. $\boxed{3}$

End of Section II

END OF EXAMINATION

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The following list of standard integrals may be used:

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

$$\text{NOTE}: \ \ln x = \log_e x, \quad x > 0$$

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- Record your multiple choice answers by filling in the circle corresponding to your choice for each question.
- Fill in the circle completely.
- Each question has only one correct answer.

CANDIDATE NUMBER: ....

Question One
A ○ B ○ C ○ D ○

Question Two
A ○ B ○ C ○ D ○

Question Three
A ○ B ○ C ○ D ○

Question Four
A ○ B ○ C ○ D ○

Question Five
A ○ B ○ C ○ D ○

Question Six
A ○ B ○ C ○ D ○

Question Seven
A ○ B ○ C ○ D ○

Question Eight
A ○ B ○ C ○ D ○

Question Nine
A ○ B ○ C ○ D ○

Question Ten
A ○ B ○ C ○ D ○

2014 HALF-YEARLY → SOLUTIONS

Form VI - Mathematics Extension 1

Q1 D
Q2 A
Q3 B
Q4 D
Q5 C
Q6 B
Q7 D
Q8 C
Q9 B
Q10 B

Multiple Choice Working

Q1. $f(x) = 3\tan 2x$

Period $= \frac{\pi}{2}$

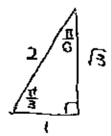
Q2. $\frac{d}{dx}\left(\sin^{-1}2x\right) = \frac{1}{\sqrt{1-(2x)^2}} \times 2$

$= \frac{2}{\sqrt{1-4x^2}}$

Q3. $\cos x = -\frac{1}{2}$

$\therefore x = 2n\pi \pm \cos^{-1}\left(-\frac{1}{2}\right)$

$= 2n\pi \pm \frac{2\pi}{3}$



$\alpha = \pi - \frac{\pi}{3}$
$= \frac{2\pi}{3}$

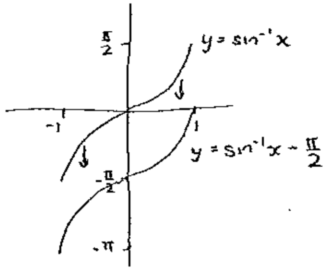
Q4. $y = 2\cos^{-1}3x$

$0 \le \cos^{-1}3x \le \pi$

$2\times 0 \le 2\cos^{-1}3x \le 2\pi$

$\therefore 0 \le y \le 2\pi$

Q5.

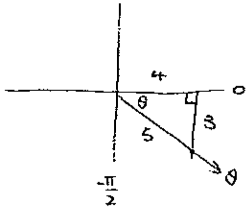


Q6. $\sin 2\theta$

$= 2\sin\theta\cos\theta$

$= 2 \times -\frac{3}{5} \times \frac{4}{5}$

$= -\frac{24}{25}$

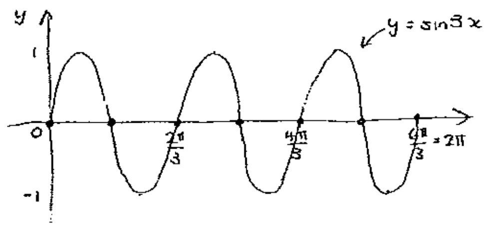


Q7. $x = \frac{2}{y-3} - 6$

$y - 3 = \frac{2}{x+6}$

$\therefore y = 3 + \frac{2}{x+6}$

Q8.



Period $= \frac{2\pi}{3}$

Q9. $\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$

$100 = \pi h^2 \times \frac{dh}{dt}$

$\therefore \frac{dh}{dt} = \frac{100}{\pi h^2}$

when $h = 5$, $\frac{dh}{dt} = \frac{100}{\pi \times 25}$

$= \frac{4}{\pi}$ cm/min

Q10. $f(x) = \cos x$ is positive in the 1st & 4th quadrants and negative in the 2nd quadrant

$\therefore f(\pi - \theta) = -f(\theta)$

$\& f(\pi - \theta) = -f(-\theta)$

also $f(x) = f(-x) \rightarrow$ even function

$\therefore f(x) = \cos x$ is the only option.

QUESTION 11:

a) $m_1 = 2$
$m_2 = -3$

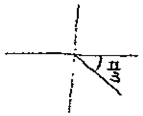
$$\alpha = \tan^{-1}\left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

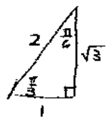
$$= \tan^{-1}\left|\frac{2-(-3)}{1+2\times(-3)}\right| \checkmark$$

$$= \tan^{-1} 1$$
$$= 45^\circ \quad (\text{or } \tfrac{\pi}{4}) \checkmark$$

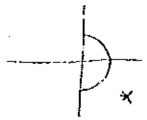
b) i) $\sin\frac{5\pi}{3} = -\sin\frac{\pi}{3}$

$$= -\frac{\sqrt{3}}{2} \checkmark$$





ii) $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6} \checkmark$
(or -30°)



c) i) $\frac{d}{dx}\left(\sin\frac{x}{2}\right) = \cos\frac{x}{2} \times \frac{1}{2}$

$$= \frac{1}{2}\cos\frac{x}{2} \checkmark$$

ii) $\frac{d}{dx}\left((\cos^{-1}x)^2\right) = 2\cos^{-1}x \checkmark \times -\frac{1}{\sqrt{1-x^2}} \checkmark$

$$= -\frac{2\cos^{-1}x}{\sqrt{1-x^2}}$$

iii) $\frac{d}{dx}(x\tan 2x)$

Let $u = x$ $\quad v = \tan 2x$
$u' = 1$ $\quad v' = 2\sec^2 2x$

$$= \tan 2x \times 1 + x \times 2\sec^2 2x \checkmark$$

$$= \tan 2x + 2x\sec^2 2x \checkmark$$

d) i) $\int \cos(2x+1)\,dx = \frac{1}{2}\sin(2x+1) + C \checkmark$

ii) $\int \frac{3}{9+x^2}\,dx = \int \frac{3}{3^2+x^2}\,dx$

$$= \frac{3}{3}\tan^{-1}\frac{x}{3} + C$$

$$= \tan^{-1}\frac{x}{3} + C \checkmark$$

e) i) $\text{RHS} = \frac{\sin 2x}{1+\cos 2x}$

$$= \frac{2\sin x\cos x}{2\cos^2 x} \checkmark \text{ (for either one)}$$

$$\left.\begin{aligned} &= \frac{\sin x}{\cos x} \\ &= \tan x \end{aligned}\right\} \checkmark$$

$= \text{LHS}$ as required.

ii) $\tan\frac{\pi}{8} = \frac{\sin\left(2\times\frac{\pi}{8}\right)}{1+\cos\left(2\times\frac{\pi}{8}\right)}$

$$= \frac{\sin\frac{\pi}{4}}{1+\cos\frac{\pi}{4}} \checkmark$$

$$= \frac{\frac{1}{\sqrt{2}}}{1+\frac{1}{\sqrt{2}}} \quad \left(\times\frac{\sqrt{2}}{\sqrt{2}}\right)$$

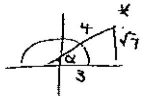
$$= \frac{1}{\sqrt{2}+1} \quad (\text{or } \sqrt{2}-1) \checkmark \text{ (either)}$$

QUESTION 12:

a) $\tan\left(2\cos^{-1}\frac{3}{4}\right)$

Let $\cos^{-1}\frac{3}{4} = \alpha$

$\therefore \cos\alpha = \frac{3}{4} \quad 0 \le \alpha \le \pi$



$\tan\alpha = \frac{\sqrt{7}}{3}$ ✓

$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}$

$= \frac{2 \times \frac{\sqrt{7}}{3}}{1 - \frac{7}{9}}$

$= 3\sqrt{7}$ ✓

b) i) Tangent at P: $y = px - ap^2$ ①

" Q: $y = qx - aq^2$ ②

① − ②: $(p-q)x - a(p^2-q^2) = 0$ ✓

$x = \frac{a(p+q)(p-q)}{(p-q)}$ [$p \ne q$ assuming P,Q distinct]

$= a(p+q)$

Sub into ①:

$y = p(ap+aq) - ap^2$

$= apq$ ✓

$\therefore R\,(a(p+q),\ apq)$

ii) Gradient of tangent at P: $m_P = p$

" Q: $m_Q = q$

If perpendicular then $m_P \times m_Q = -1$

$\therefore pq = -1$ ✓

y-coord of R $= apq$

$= a \times -1$

$= -a$

$\therefore$ R lies on the directrix $y = -a$ ✓

c) $\cos 2\theta + 3\sin\theta - 2 = 0 \qquad 0 \le \theta \le 2\pi$

$1 - 2\sin^2\theta + 3\sin\theta - 2 = 0$ ✓

$2\sin^2\theta - 3\sin\theta + 1 = 0$

$(2\sin\theta - 1)(\sin\theta - 1) = 0$ ✓

$\therefore \sin\theta = \frac{1}{2}$ or $\sin\theta = 1$

$\therefore \theta = \frac{\pi}{2}$

*S	A*
T	C

Related ∠ $= \frac{\pi}{6}$

$\therefore \theta = \frac{\pi}{6}, \frac{5\pi}{6}$

$\therefore \theta = \frac{\pi}{6}, \frac{\pi}{2}$ or $\frac{5\pi}{6}$ ✓

d) i) $\frac{1}{2} \times OA^2 \times \theta = 4 \times \frac{1}{2} \times OC^2 \times \theta$ ✓

$OA^2 = 4r^2$

$\therefore OA = 2r \quad (\text{since } OA > 0)$

$AC = OA - OC$

$= 2r - r$

$= r$ ✓

ii) $48 = 2r + 2r + 2r\theta + r\theta$

$= 4r + 3r\theta$

$\therefore \theta = \frac{48 - 4r}{3r}$ as required ✓

iii) $A_{OCD} = \frac{1}{2}r^2\theta$

$= \frac{1}{2} \times r^2 \times \frac{48-4r}{3r}$ ✓

$= \frac{48r}{6} - \frac{4r^2}{6}$

$= 8r - \frac{2}{3}r^2$

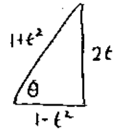
$\frac{dA}{dr} = 8 - \frac{4}{3}r$ ✓

$= 0$ when $r = 6$

$\frac{d^2A}{dr^2} = -\frac{4}{3}$

< 0 ∩ ∴ max occurs when $r = 6$ ✓ (must justify max)

QUESTION 13:

a) 

LHS $= \operatorname{cosec}\theta + \cot\theta$

$= \frac{1}{\sin\theta} + \frac{1}{\tan\theta}$

$= \frac{1+t^2}{2t} + \frac{1-t^2}{2t}$ ✓ (for either)

$= \frac{2}{2t}$

$= \frac{1}{t}$

$= \frac{1}{\tan\frac{\theta}{2}}$

$= \cot\frac{\theta}{2}$

$=$ RHS as required. ✓

b) i) $2\sin x + 4\cos x = R\sin(x+\alpha)$

$= R\sin x\cos\alpha + R\cos x\sin\alpha$

$R\cos\alpha = 2$ ①

$R\sin\alpha = 4$ ②

squaring + adding:

$R^2(\cos^2\alpha + \sin^2\alpha) = 2^2 + 4^2$

$= 20$

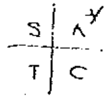
$\therefore R = 2\sqrt{5}$ $(R > 0)$ ✓

$\therefore \cos\alpha = \frac{2}{2\sqrt{5}}$

$= \frac{1}{\sqrt{5}}$

$\sin\alpha = \frac{4}{2\sqrt{5}}$

$= \frac{2}{\sqrt{5}}$



$\alpha = \cos^{-1}\left(\frac{1}{\sqrt{5}}\right)$

$= 63 \cdot 4349\ldots$

$= 63°26'$ (to nearest min)

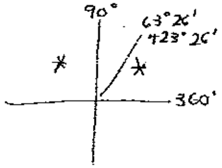
$\therefore 2\sin x + 4\cos x = 2\sqrt{5}\sin(x + 63°26')$ ✓

(to nearest minute)

ii) $2\sqrt{5}\sin(x + 63°26') = 3$

$0° \le x \le 360°$

$63°26' \le x + 63°26' \le 423°26'$



$\sin(x + 63°26') = \dfrac{3}{2\sqrt{5}}$

Related $\angle = \sin^{-1}\left(\dfrac{3}{2\sqrt{5}}\right)$

$= 42 \cdot 1304\ldots$

$x + 63°26' = 180° - 42\cdot1304\ldots°$ OR $360° + 42\cdot1304\ldots°$

$= 137\cdot8695\ldots°$ $\quad = 402\cdot1304\ldots°$

$\therefore x = 74°26'$ OR $338°42'$

c) Test first case:

When $n = 1$:

$LHS = \frac{1}{2}$

$RHS = 2 - \frac{1+2}{2}$

$= \frac{1}{2}$

$= LHS$ $\therefore$ true for $n = 1$

Assume true for $n = k$:

$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots + \frac{k}{2^k} = 2 - \frac{k+2}{2^k}$$

Prove true for $n = k+1$:

[Required to prove $\Sigma = 2 - \dfrac{k+3}{2^{k+1}}$]

$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots + \frac{k}{2^k} + \underbrace{\frac{k+1}{2^{k+1}}} = 2 - \frac{k+2}{2^k} + \underbrace{\frac{k+1}{2^{k+1}}}$$

$$= 2 - \frac{2(k+2)}{2(2^k)} + \frac{k+1}{2^{k+1}}$$

$$= 2 - \left(\frac{2k+4}{2^{k+1}} - \frac{k+1}{2^{k+1}}\right)$$

$$= 2 - \frac{k+3}{2^{k+1}} \text{ as required.}$$

The result now follows for all integers ≥ 1 by the Principle of Mathematical Induction.

d) i) $\tan 60° = \dfrac{h}{AB}$

$\therefore AB = \dfrac{h}{\tan 60°}$

$= \dfrac{h}{\sqrt{3}}$

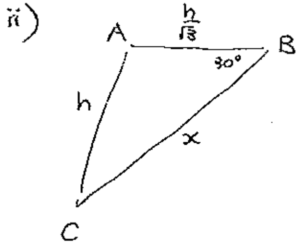
$\tan 45° = \dfrac{h}{AC}$

$\therefore AC = \dfrac{h}{\tan 45°}$

$= \dfrac{h}{1}$

$= h$

ii)



$$h^2 = x^2 + \left(\frac{h}{\sqrt{3}}\right)^2 - 2 \times x \times \frac{h}{\sqrt{3}} \times \cos 30°$$

$$= x^2 + \frac{h^2}{3} - \frac{2xh}{\sqrt{3}} \times \frac{\sqrt{3}}{2}$$

$$3h^2 = 3x^2 + h^2 - 3xh$$

$\therefore 2h^2 + 3xh - 3x^2 = 0$ as required.

iii) METHOD 1:

$$2h^2 + 3xh - 3x^2 = 0$$ *

$a = 2$
$b = 3x$
$c = -3x^2$

$$h = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-3x \pm \sqrt{(3x)^2 - 4 \times 2 \times -3x^2}}{2 \times 2} \quad ✓$$

$$= \frac{-3x \pm \sqrt{33x^2}}{4}$$

$$= \left(\frac{-3 \pm \sqrt{33}}{4}\right) x$$

BUT $h > 0$ $\therefore$ $h = \left(\frac{-3+\sqrt{33}}{4}\right)x$

$$\therefore \frac{h}{x} = \frac{-3+\sqrt{33}}{4} \quad ✓$$

(must justify positive answer only)

METHOD 2: *

divide throughout by x^2:

$$2\frac{h^2}{x^2} + \frac{3xh}{x^2} - \frac{3x^2}{x^2} = 0 \quad (\text{since } x^2 \neq 0)$$

$$2\left(\frac{h}{x}\right)^2 + 3\left(\frac{h}{x}\right) - 3 = 0 \quad ✓$$

$$\therefore \frac{h}{x} = \frac{-3 \pm \sqrt{3^2 - 4 \times 2 \times -3}}{2 \times 2}$$

$$= \frac{-3 \pm \sqrt{33}}{4}$$

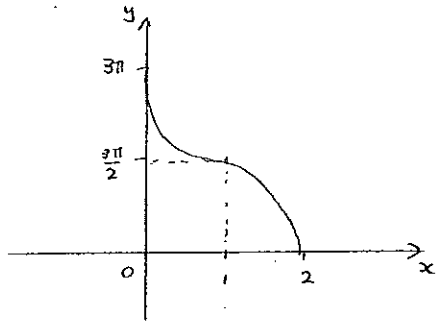
BUT $\frac{h}{x} > 0$ $\therefore \frac{h}{x} = \frac{-3+\sqrt{33}}{4}$ ✓

(must justify positive answer only)

QUESTION 14:

a) $y = 3\cos^{-1}(x-1)$

i)



✓ shape
✓ intercepts

ii) METHOD 1: *

$$A = \int_0^{3\pi} \cos\frac{y}{3} + 1 \, dy \quad ✓$$

$$= \left[3\sin\frac{y}{3} + y\right]_0^{3\pi}$$

$$= 3\sin\frac{3\pi}{3} + 3\pi - (3\sin 0 + 0)$$

$$= 3\pi \text{ unit}^2 \quad ✓$$

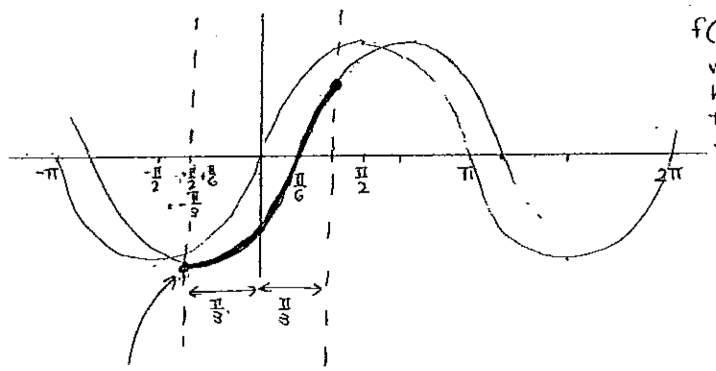
METHOD 2: *

From symmetry about $\left(1, \frac{3\pi}{2}\right)$: ✓

$$A = \frac{3\pi}{2} \times 2$$

$$= 3\pi \text{ unit}^2 \quad ✓$$

b) $f(x) = \sin\left(x - \frac{\pi}{6}\right)$



$f(x) = \sin\left(x - \frac{\pi}{6}\right)$ will fail the horizontal line test outside of $-\frac{\pi}{3} \leq x \leq \frac{2\pi}{3}$ ✓

beyond this point, the inverse will no longer be a function

$-\frac{\pi}{2} + \frac{\pi}{6} = -\frac{\pi}{3}$

$\therefore$ max possible value of $a = \frac{\pi}{3}$. ✓

c) i) $\frac{dx}{dt} = 2a$

$\frac{dy}{dt} = -2at$

$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$

$= -2at \times \frac{1}{2a}$

$= -t$

$\therefore m_\perp = \frac{1}{t}$ ✓

some working required.

$y - (-at^2) = \frac{1}{t}(x - 2at)$

$ty + at^3 = x - 2at$

$\therefore x - ty = 2at - at^3$ as required. ✓

ii) $x - ty = 2at + at^3$ ①

$x = -at$ ②

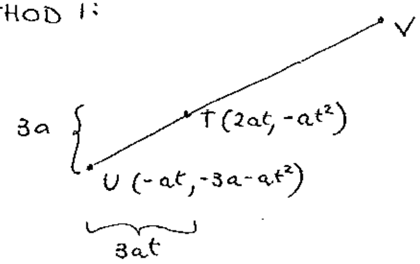
Sub ② into ①:

$-at - ty = 2at + at^3$ ✓

$\therefore y = -3a - at^2$

$\therefore U(-at, -3a - at^2)$ ✓

iii) METHOD 1: *



✓ (showing Δx + Δy)

$V(2at + 2 \times 3at, -at^2 + 2 \times 3a)$

$\rightarrow V(8at, 6a - at^2)$ ✓

METHOD 2: *

$x = \frac{3 \times 2at + (-2) \times (-at)}{-2 + 3}$

$= \frac{6at + 2at}{1}$

$= 8at$ ✓

$y = \frac{3 \times (-at^2) + (-2) \times (-3a - at^2)}{3 + (-2)}$

$= -3at^2 + 6a + 2at^2$

$= 6a - at^2$

$\therefore V(8at, 6a - at^2)$. ✓

iv) $x = 8at \longrightarrow t = \dfrac{x}{8a}$ ①

$y = 6a - at^2$ ②

Sub ① into ②:

$$y = 6a - a\left(\frac{x}{8a}\right)^2$$

$$= 6a - \frac{x^2}{64a}$$ ✓

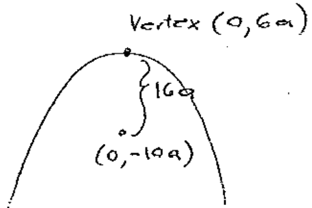
$$x^2 = (6a - y) \times 64a$$

$$= -64a(y - 6a)$$ ✓

$$4A = 64a$$

$$A = \frac{64a}{4}$$

$$= 16a$$



∴ The focus is $(0, -10a)$ ✓