KAMBALA

Student Number: ______ ____

HSC Task 1
March 2017

HSC Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 75 minutes
- Write using black pen
- Black pen is preferred
- Board-approved calculators may be used
- Answer questions 1 – 4 on the Multiple Choice Answer Sheet provided.
- Answer Questions 5 – 7 on the paper provided.
- **Start each question on a new page.**
- A reference sheet is provided
- Show all necessary working in Questions 5 – 7

Total marks – 40

Section I
4 marks
- Attempt Questions 1 – 4
- Allow about 7 minutes for this section

Section II
36 marks
- Attempt Questions 5 – 7
- Allow about 68 minutes for this section

Section I

4 Marks
Attempt Questions 1 – 4
Allow about 7 minutes for this section

Use the Multiple-Choice Answer Sheet for Questions 1 – 4.

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1 A parabola has the parametric equations $x=12t$ and $y=-6t^2$.
What are the coordinates of the focus?

(A) $(-6,0)$
(B) $(0,-6)$
(C) $(6,0)$
(D) $(0,6)$

2 The equation of the tangent to the parabola $x^2=4y$ at the point $(-1,\frac{1}{4})$ is:

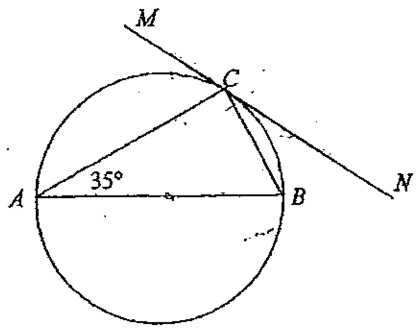
(A) $2x+4y+1=0$
(B) $2x+2y+1=0$
(C) $2x+2y-1=0$
(D) $2x-4y+1=0$

3 The equation of the normal to the parabola $x^2=4ay$ at the variable point $P\left(2ap,ap^2\right)$ is given by $x+py=2ap+ap^3$.

How many different values of p are there such that the normal passes through the focus of the parabola?

(A) 0
(B) 1
(C) 2
(D) 3

4 In the diagram below, AB is a diameter of the circle and MCN is the tangent to the circle at C. $\angle CAB = 35^\circ$. What is the size of $\angle MCA$?



(A) 35°

(B) 45°

(C) 55°

(D) 65°

Section II

36 Marks
Attempt Questions 5 – 7
Allow about 68 minutes for this section

Answer each question on the writing paper provided. Start each question on a new page.

In Questions 5 – 7, your responses should include relevant mathematical reasoning and/or calculations.

Question 5 (12 marks) *Start a new page.*

(a) Find the Cartesian equation of the curve defined by the parametric equations $x = \sin\theta$ and $y = \cos^2\theta - 3$. **2**

(b) The point $(-6t, 9t^2)$, where t is a variable, lies on a curve. Find the Cartesian equation of the curve. **2**

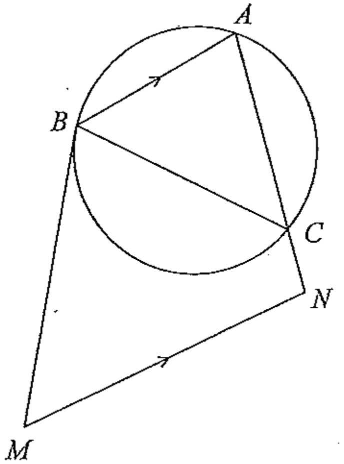
(c) (i) The chord joining P $(2ap, ap^2)$ and Q $(2aq, aq^2)$ on the parabola $x^2 = 4ay$ subtends a right angle at the vertex of the parabola. **2**

Show that $pq = -4$.

(ii) Show that the locus of the point M, the midpoint of PQ, is also a parabola and state its vertex. **3**

Question 5 continued

(d) ABC is a triangle inscribed in a circle. M is a point on the tangent to the circle at B and N is a point on AC produced so that MN is parallel to BA.

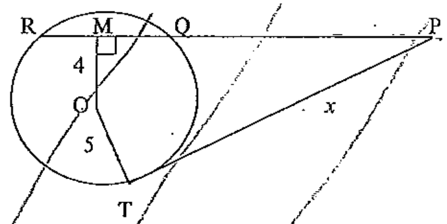


(i) State why $\angle MBC = \angle BAC$. 1

(ii) Prove that $MNCB$ is a cyclic quadrilateral. 2

Question 6 (10 marks) *Start a new page.*

(a) 2

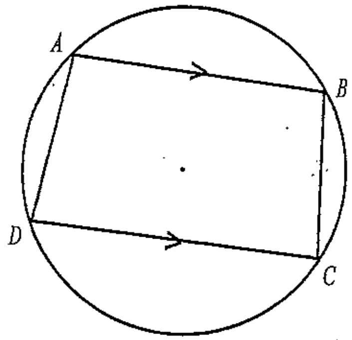


IGNORE THIS Q (NO POSSIBLE ANS)

PT is a tangent to the circle, centre O. OM is perpendicular to the secant RQ.

Find the value of x to 1 decimal place.

(b) $ABCD$ is a cyclic quadrilateral with $AB \parallel DC$.

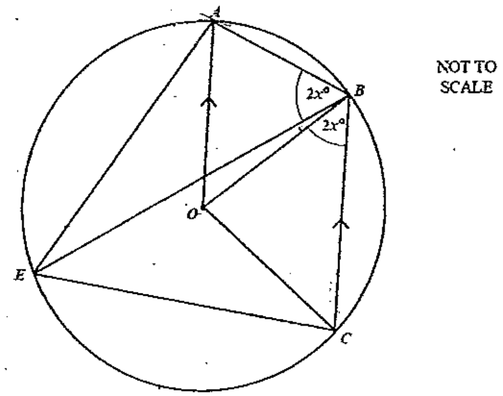


(i) Prove that $\angle ADC = \angle BCD$. 2

(ii) Hence, use congruent triangles to prove that any trapezium inscribed in a circle must be isosceles (i.e. has its non-parallel sides equal in length). 2

Question 6 continued

(c) In the diagram, $ABCE$ is a cyclic quadrilateral such that AO is parallel to BC. O is the centre of the circle and $\angle ABE = \angle OBC = 2x°$.



(i) Prove that $\angle AEB = x°$. **2**

(ii) Prove that $\angle BCE = 3x°$. **2**

Question 7 (12 marks) *Start a new page.*

(a) Let P($2ap$, $2ap^2$) and Q($2aq$, $2aq^2$) be points on the parabola $y = \dfrac{x^2}{2a}$.

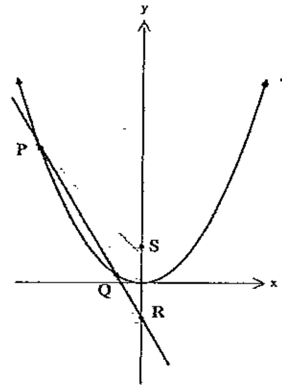
(i) Find the equation of the chord PQ. **2**

(ii) If PQ is a focal chord, find the relationship between p and q. **2**

(iii) Show that the locus of the midpoint of PQ is a parabola. **2**

(b) A straight line through R(0, $-a$) cuts the parabola $x^2 = 4ay$ at points P($2ap$, ap^2) and Q($2aq$, aq^2) which have parameters p and q respectively.

Point S(0, a) is the focus of the parabola.



(i) Show that the equation RP is given by $2py = x(p^2+1) - 2ap$. **2**

(ii) Prove that for the line RP to pass through Q, $pq = 1$. **2**

(iii) Hence, prove that $\dfrac{1}{SP} + \dfrac{1}{SQ} = \dfrac{1}{a}$. **2**

END OF ASSESSMENT TASK

Student Number: ________

Mathematics Extension 1

HSC Task 1
March 2017

Section I

Multiple-Choice Answer Sheet
Circle the correct response.

1.	A	B	C	D
2.	A	B	C	D
3.	A	B	C	D
4.	A	B	C	D

4

Question 5 $\frac{11}{12}$

a) $x = \sin\theta$, $y = \cos^2\theta - 3$

$\sin\theta = x$
$\sin^2\theta = x^2$ — ①

$y = \cos^2\theta - 3$
$\cos^2\theta = y + 3$

Using $\sin^2\theta + \cos^2\theta = 1$:
$x^2 + y + 3 = 1$
$x^2 + y + 2 = 0$
$y = -x^2 - 2$

$\frac{2}{2}$

b) $x = -6t$ $y = 9t^2$ — ②
$t = \frac{x}{-6}$ — ①

sub ① into ② :
$y = 9\left(\frac{x^2}{36}\right)$
$y = \frac{x^2}{4}$
$4y = x^2$ $\therefore x^2 = 4y$

$\frac{2}{2}$

c) i) $\therefore$ vertex at $(0,0)$

$\therefore M_{OP} \times M_{OQ} = -1$ (since it subtends right (at vertex)

$M_{OP} = \frac{ap^2 - 0}{2ap - 0} = \frac{ap^2}{2ap}$
$= \frac{p}{2}$

$M_{OQ} = \frac{aq^2 - 0}{2aq - 0} = \frac{aq^2}{2aq}$

$= \frac{q}{2}$

$M_{OP} \times M_{OQ} = -1$

$\therefore \frac{p}{2} \times \frac{q}{2} = -1$

$\frac{pq}{4} = -1$

$\therefore pq = -4$

2/2

ii) $M = \left(\frac{2ap + 2aq}{2}, \frac{ap^2 + aq^2}{2}\right)$

$= a(p+q), \frac{a(p^2+q^2)}{2}$

$x = a(p+q) \qquad y = \frac{a}{2}(p^2+q^2)$ — ②

$p+q = \frac{x}{a}$

squaring both sides:

$(p+q)^2 = \frac{x^2}{a^2}$

$p^2 + q^2 + 2pq = \frac{x^2}{a^2}$

$p^2 + q^2 = \frac{x^2}{a^2} - 2pq$ — ①

sub ① into ②:

$y = \frac{a}{2}\left[\frac{x^2}{a^2} - 2pq\right]$

Sub in $pq = -4$:

$y = \frac{a}{2}\left[\frac{x^2}{a^2} - 2(-4)\right]$

$2y = \frac{x^2}{a} + 8a$

$2ay = x^2 + 8a^2$

$x^2 = 2ay - 8a^2$

$x^2 = 2a(y - 4a)$

$\therefore$ vertex at $(0, 4a)$

d) i) $\angle MBC = \angle BAC$ ($\angle$ between chord and tangent is equal to $\angle$ in alternate segment)

ii) Let $\angle MBC$ be α.

$\angle MBC = \angle BAC = \alpha$

~~$\angle BAM$~~ ~~$\angle BAN + \angle CNM = 180°$ (cointerior $\angle$s on parallel lines are~~ ~~...)~~

$\angle BAC$ ~~$\angle BAN$~~ $+ \angle CNM = 180°$ (cointerior $\angle$s on parallel lines are ~~suppl~~)

~~$\angle CNM$~~ $\angle CNM = 180 - \alpha$

$\angle CNM + \angle MBC = \alpha + 180 - \alpha$

$= 180°$

2/2

$\therefore$ MNCB is cyclic quadrilateral (opposite interior $\angle$s are supple...)

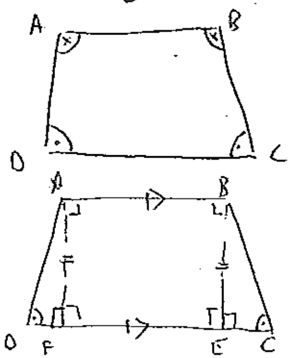
6

b) i) $\angle ADC + \angle DAB = 180°$ (cointerior $\angle$s on parallel lines are supplementary)
$\angle ADC = 180 - \angle DAB$

$\angle DAB + \angle BCD = 180°$ (opposite interior $\angle$s of cyclic quadrilateral are supplementary)
$\angle BCD = 180 - \angle DAB$

$\therefore \angle ADC = \angle BCD = 180 - \angle DAB$ 2

ii) we have



Construct point E such that
$BE \perp DC$
Construct point F such that
$AF \perp DC$
$\angle FAB = 90° = \angle EBA$ (Alt. $\angle$)
$\angle AFC = \angle BED = \angle 90°$ (Angle sum of str. line)
$\therefore ABEF$ is a rectangle
$BE = AF$ (opp. sides of rectangle are equal)
$\angle BCD = \angle ADC$ (proven in i)
$\therefore \triangle BEC \cong \triangle AFD$ (AAS)
$\therefore BC = AD$ (corresponding sides of congruent $\triangle$'s)

These are not insi[illegible] $\triangle ADB$ & $\triangle BCB$

1/2

6c) $\angle AOB = 2x$. (Alt. Angles)
$\therefore \angle AEB = x°$ (Angle at centre is twice angle at circumference subtended by same arc.)

ii) In $\triangle AEB$
$2x + x + \angle EAB = 180°$ ($\angle$ sum of $\triangle$)
$\angle EAB = 180 - 2x - x$
$= 180 - 3x$

$\angle BCE + \angle EAB = 180°$ (opp interior $\angle$s of cyclic quad are suppl. $\angle$s)
$\angle BCE = 180 - (180 - 3x)$
$\angle BCE = 3x$

2/2

Question 7

a i) $y = \frac{x^2}{2a}$ $\quad x^2 = 2ay$ $\quad \cancel{x^2 = 4 \times 4 a}$ $\quad x^2 = 4 \times \frac{a}{2} y$

$$m_{PQ} = \frac{2ap^2 - 2aq^2}{2ap - 2aq} = \frac{\cancel{2a}(p+q)\cancel{(p-q)}}{\cancel{2a}\cancel{(p-q)}}$$

$$m_{PQ} = p+q$$

chord PQ passes through $(2ap, 2ap^2)$:

$\cancel{y - y_1 = m(x - x_1)}$

$\cancel{y - 2ap^2 = (p+q)(x-2ap)}$

$$y - y_1 = m(x - x_1)$$

$$y - 2ap^2 = (p+q)(x - 2ap)$$

$$y - \cancel{2ap^2} = px + \cancel{qx} - \cancel{2ap^2} + qx - 2apq$$

2/2

$$y = px + qx - 2apq$$

$$y = (p+q)x - 2apq$$

ii) If PQ is a focal chord, focus $(0, \frac{1}{2}a)$ satisfies the equation:

$$\frac{a}{2} = (p+q)(0) - 2apq$$

$$\frac{a}{2} = -2apq$$

$$a = -4apq$$

2/2

$$1 = -4pq$$

$$pq = -\frac{1}{4}$$

iii) $M = \left(\frac{2ap + 2aq}{2}, \frac{2ap^2 + 2aq^2}{2}\right)$

$$= a(p+q), \quad a(p^2+q^2)$$

$x = a(p+q)$ $\quad y = a(p^2+q^2)$ — ②

$$p+q = \frac{x}{a}$$

squaring both sides:

$$(p+q)^2 = \frac{x^2}{a^2}$$

$$p^2 + q^2 + 2pq = \frac{x^2}{a^2}$$

$$p^2 + q^2 = \frac{x^2}{a^2} - 2pq \quad \text{— ①}$$

sub ① into ②:

$$y = a\left[\frac{x^2}{a^2} - 2pq\right]$$

$$y = a\left[\frac{x^2}{a^2} - 2\left(\frac{-1}{4}\right)\right]$$

$$y = a\left[\frac{x^2}{a^2} + \frac{1}{2}\right]$$

$$y = \frac{x^2}{a} + \frac{a}{2}$$

2/2

$$ay = x^2 + \frac{a^2}{2}$$

$$x^2 = ay - \frac{a^2}{2}$$

$$x^2 = a\left(y - \frac{a}{2}\right)$$

∴ The locus of the midpoint is a parabola

7. PART B.

$R(0,-a)$ $Q(2aq, aq^2)$
$P(2ap, ap^2)$ $S(0,a)$

i) Equation of the RP

$$m_{RP} = \frac{ap^2+a}{2ap} = \frac{a(p^2+1)}{a(2p)} = \frac{p^2+1}{2p}$$

Point gradient Formula $(y-y_1) = m(x-x_1) \rightarrow (0,-a)$ as ref.

$$(y+a) = \frac{p^2+1}{2p}(x)$$

$$(y+a)(2p) = (p^2+1)(x)$$

$$2py + 2ap = x(p^2+1)$$

$$\underline{2py = x(p^2+1) - 2ap} \quad (1)$$

ii) For the condition of Q to pass through, sub in $Q(2aq, aq^2)$ into (1)

$$2p(aq^2) = 2aq(p^2+1) - 2ap$$

$$2apq^2 = 2aqp^2 + 2aq - 2ap$$

$$pq^2 = qp^2 + q - p$$

$$pq^2 - qp^2 = (q-p)$$

$$pq(q-p) = (q-p)$$

$$\therefore pq = 1.$$

iii) $SP = \sqrt{(ap^2-a)^2 + (2ap)^2}$

$$= \sqrt{a^2(p^2-1)^2 + a^2(2p)^2}$$

$$= \sqrt{a^2[(p^2-1)^2 + (2p)^2]}$$

$$= \sqrt{a^2[p^4+1-2p^2+4p^2]}$$

$$= \sqrt{a^2[p^4+1+2p^2]}$$

$$= \sqrt{a^2[p^2+1]^2}$$

$$= a(p^2+1)$$

Similarly can be done for Q

ie $SQ = a(q^2+1)$

$$\frac{1}{SP} + \frac{1}{SQ} = \frac{a(q^2+1+p^2+1)}{a^2(q^2+1)(p^2+1)}$$

but we know $pq = 1 \Rightarrow q = \frac{1}{p}$.

$$\frac{\left(\frac{1}{p^2}+1+p^2+1\right)}{a\left(\frac{1}{p^2}+1\right)(p^2+1)}$$

$$\frac{p^2+\frac{1}{p^2}+2}{a\left(p^2+\frac{1}{p^2}+2\right)} = \frac{1}{a}.$$

AS REQUIRED