MARCELLIN COLLEGE RANDWICK

YEAR 12

EXTENSION 2

HSC ASSESSMENT TASK 2

2016

STUDENT NAME: ____________ ____________ **MARK** /59

TEACHER: ____________

TIME ALLOWED: 90 minutes
WEIGHTING: 20%

Directions:

- Use a separate sheet for each question.
- Show all necessary working. Where more than one mark is allocated to a question, full marks may not be awarded for answers only.
- Marks may not be awarded for careless or badly arranged work.
- Calculators may be used

Section 1

5 marks

Attempt Questions 1 – 5

Allow about 10 minutes for this Section

The questions are of equal value

Answer each question on the multiple-answer sheet provided.

Question 1

Let $w = 2 - 3i$ and $z = 3 + 4i$. The value of $w\bar{z}$ is

A $-6-17i$ B $18-i$ C $-6+17i$ D $18+i$

Question 2

The remainder when $P(z) = z^3 + iz^2 + 3z - 4$ is divided by $(z - i)$ is

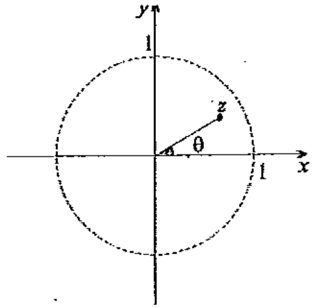
A $3i-4$ B $i-4$ C $5i-4$ D $-3i-4$

Question 3

The polynomial equation $x^3 + Ax^2 + Bx + C = 0$ has roots 3 and $1-2i$. It is known that *A*, *B* and *C* are real numbers. The value of *A* is

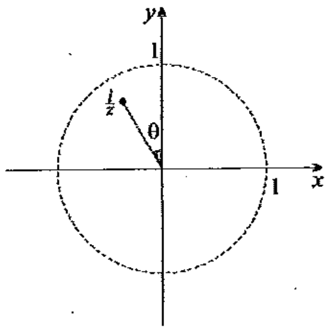
A -4 B -5 C -6 D -7

Question 4

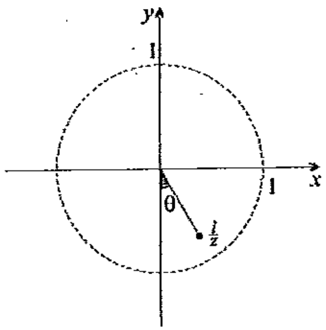


The Argand diagram above shows the complex number z. By considering the modulus and argument, which diagram below best represents the complex number $\frac{i}{z}$?

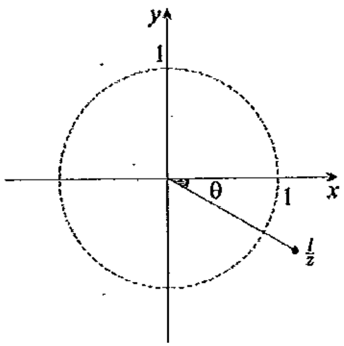
(A)



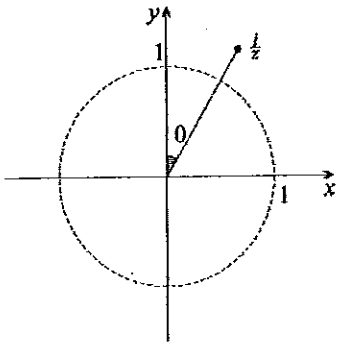
(B)



(C)

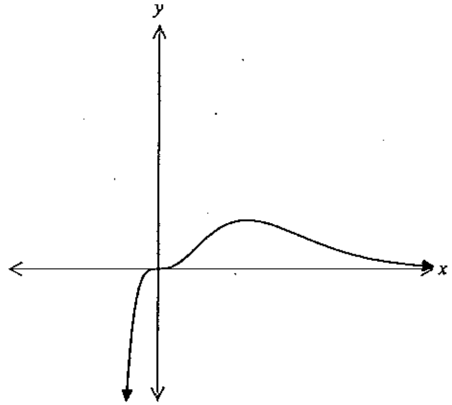


(D)



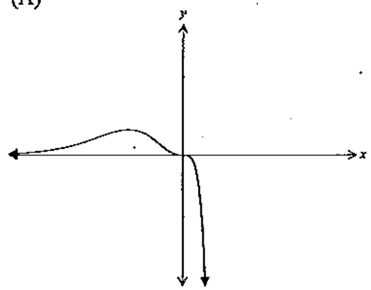
Question 5

The diagram shows the graph of $y = f(x)$.

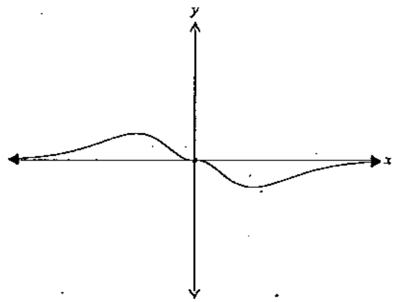


Which of the following is the graph of $y = f(-x)$?

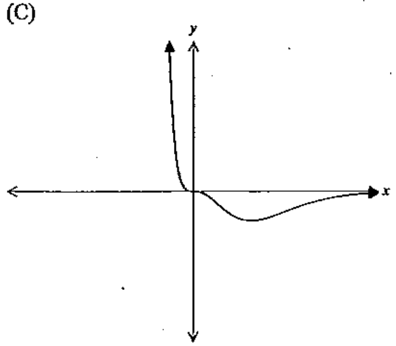
(A)



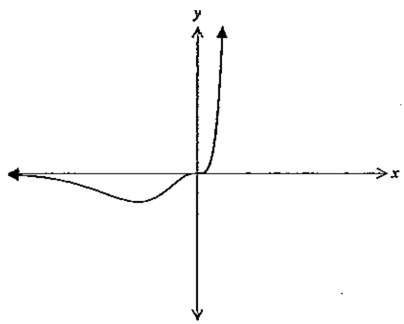
(B)



(C)



(D)



Section 2

55 marks

Attempt Questions 6 – 9

Answer each question on a SEPARATE page.

Question 6 (15 marks) **Marks**

(a) Let $z = \frac{3-2i}{2+i}$ and find $|z|$. **2**

(b) Find all pairs of real numbers x and y that satisfy $(x+iy)^2 = 77-36i$. **3**

(c) Let $\alpha = -1+i$ and $\beta = \sqrt{3}+i$.

(i) Find $\frac{\alpha}{\beta}$, in the form $x+iy$ where $x, y \in \mathbb{R}$. **2**

(ii) Write α in modulus-argument form. **2**

(iii) Given that β has the modulus-argument form $\beta = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$, **1**

find the modulus-argument form of $\frac{\alpha}{\beta}$.

(iv) Hence find the exact value of $\cos\left(\frac{7\pi}{12}\right)$. **1**

(d) Show that if $x=\alpha$ is a triple zero of $P(x)$, then $x=\alpha$ is also a double zero of $P'(x)$. **2**

You may assume that $P(x) = (x-\alpha)^3 Q(x)$ for some polynomial $Q(x)$ where $Q(\alpha) \neq 0$.

(e) Sketch the region on the Argand diagram where the inequalities $|z| \leq 4$ and **2**

$0 \leq \arg(z+2) \leq \frac{\pi}{3}$ both hold.

Question 7 (15 marks) Begin writing on a new page. **Marks**

(a) (i) Find the roots of $z^5 - 1 = 0$, in modulus-argument form. **2**

(ii) Hence show that, for $z \neq 1$, **3**

$$\frac{z^5-1}{z-1} = \left(z^2 - 2\cos\frac{2\pi}{5}z + 1\right)\left(z^2 - 2\cos\frac{4\pi}{5}z + 1\right).$$

(b) The polynomial $P(x) = x^4 - 4x^3 + 7x^2 - 6x - 4$ has zeros α, β, γ and δ.

(i) Show that the polynomial which has zeros $(\alpha-1), (\beta-1), (\gamma-1)$ and $(\delta-1)$ is **2**

$Q(x) = x^4 + x^2 - 6$.

(ii) Find the zeros of $Q(x)$ and hence find the zeros of $P(x)$. **2**

(c) Let $z = \cos\theta + i\sin\theta$..

(i) Show that $z^n + \frac{1}{z^n} = 2\cos n\theta$. **1**

(ii) By considering the expansion of $\left(z + \frac{1}{z}\right)^3$, show that **2**

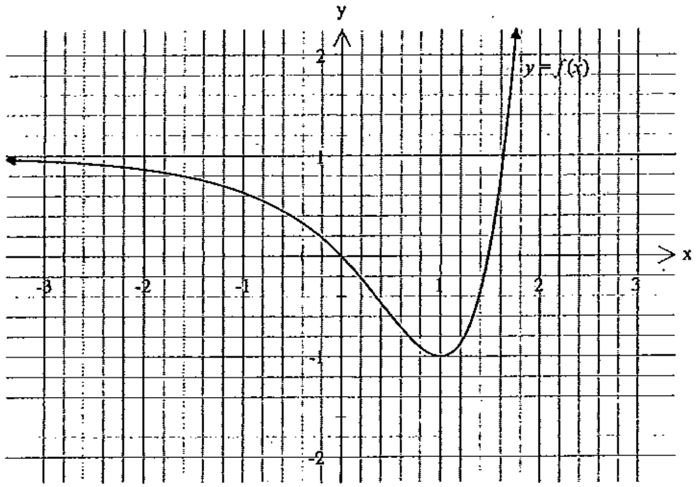
$8\cos^3\theta = 2\cos 3\theta + 6\cos\theta$.

(iii) Hence solve $8x^3 - 6x - 1 = 0$. **2**

(iv) Deduce that $\cos\frac{\pi}{9}\cos\frac{5\pi}{9}\cos\frac{2\pi}{9} = -\frac{1}{8}$. **1**

Question 8 (12 marks) Begin writing on a new page. **Marks**

(a) The graph of $y = f(x)$ is shown below.



Sketch graphs of the following on the separate answer sheet provided.

(i) $y = f(|x|)$ **1**

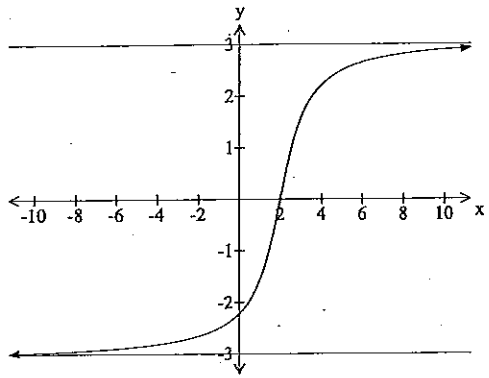
(ii) $y = \dfrac{1}{f(x)}$ **3**

(iii) $y = [f(x)]^2$ **2**

(b) The equation of a curve is $x^2 + 3xy + 4y^2 = 58$. Find the equation of the normal to the curve at the point (2, 3). **3**

Question 8 continued **Marks**

(c) The diagram shows the graph of the function $y = f(x)$.



Draw separate one-third page sketches of the graphs of the following:

(i) $y = f(x+2)$ **1**

(ii) $y = \sqrt{f(x)}$ **2**

Question 9 (12 marks) Begin writing on a new page. **Marks**

(a) Evaluate $\displaystyle\int_0^1 \frac{2x+1}{x^2+1}\,dx$. **3**

(b) (i) Find A and B such that $\dfrac{4}{4-x^2} \equiv \dfrac{A}{2-x} + \dfrac{B}{2+x}$. **2**

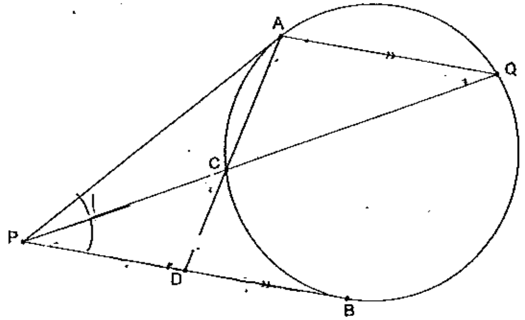
(ii) Hence find $\displaystyle\int \frac{4}{4-x^2}\,dx$ **2**

(c) In the diagram below, PA and PB are tangents to the circle. The chord AQ is parallel to the tangent PB. PCQ is a secant to the circle and chord AC produced meets PB at D.

(i) Show that ΔCDP is similar to ΔPDA. **2**

(ii) Hence show that $PD^2 = AD \times CD$. **1**

(iii) Hence, or otherwise, prove that AD bisects PB. **2**



MARCELLIN COLLEGE RANDWICK

4 UNIT HSC ASSESSMENT 2, 2016.

SAMPLE SOLUTIONS

Section 1

1. $w = 2-3i$ $\quad \bar{z} = 3-4i$

$z = 3+4i$

$= (2-3i)(3-4i)$

$\therefore = 6 - 12 - 8i - 9i$

$= -6 - 17i$

$= A$

2. Remainder theorem.

$i^3 + i^3 + 3i - 4$

$= -i - i + 3i - 4$

$= i - 4$

$= B$

3. $x^3 + Ax^2 + Bx + C$.

Roots are 3, $1-2i$

other root must be $1+2i$, because all coefficients are real.

ie roots are 3, $1-2i$, $1+2i$

$-b/a$ = Sum of roots

ie $\dfrac{-A}{1} = 3 + (1-2i) + (1+2i)$

$-A = 5$, $A = -5$ $= B$

4. let $z = x+iy$ ie $|z| < 1$.

where $x, y < 1$.

$\dfrac{i}{z} = \dfrac{i}{x+iy} \times \dfrac{x-iy}{x-iy}$

$= \dfrac{y+ix}{x^2+y^2}$

$= D$.

ie A is incorrect because it is iz

B is incorrect because it is z/i

C is incorrect because both components are negative.

ie $\frac{1}{|z|}$

ie. D is correct

5. $y = f(x)$

Flipped about y axis $= y = f(-x)$

$= A$.

6. a) $z = \frac{3-2i}{2+i}$

$|z| = \frac{|3-2i|}{|2+i|}$

$= \frac{\sqrt{3^2+2^2}}{\sqrt{2^2+1^2}} = \frac{\sqrt{13}}{\sqrt{5}} = \sqrt{\frac{13}{5}}$

b). $(x+iy)^2 = 77-36i$

Expand and equate coefficients

$x^2 - y^2 + 2ixy = 77-36i$

So $x^2 - y^2 = 77$ (1)

$2ixy = -36i \Rightarrow xy = -18$ (2)

$x = \frac{-18}{y}$.

Sub back into (1)

$\left(\frac{-18}{y}\right)^2 - y^2 = 77.$

$\frac{324}{y^2} - y^2 = 77.$

$324 - y^4 = 77y^2$

let $u = y^2$

$324 - u^2 = 77u.$

$u^2 + 77u - 324 = 0.$

Quadratic in u.

$a = 1$

$b = 77$

$c = -324.$

$\frac{-77 \pm \sqrt{5929 - 4(-324)}}{2}$

$= \frac{-77 \pm 85}{2}$

$= \frac{8}{2}$ or $\frac{-162}{2}$.

but y cannot be $\sqrt{\frac{-162}{2}}$ since it must be real

So $y = \sqrt{\frac{8}{2}} = \sqrt{4} = \pm 2$.

if $y = 2$

$x^2 - 4 = 77$

$x^2 = 81$

$x = \pm 9.$

ie Solns are.

$9 + 2i$

$-9 + 2i$

Test.

$(9+2i)^2$

$= 81 - 4 + 36i$

Incorrect

So must be

$-9+2i$ $(x=-9,\ y=2)$

OR.

$9-2i$ $(x=9,\ y=-2)$

ie $\pm(9-2i)$

c) $\alpha = -1+i$

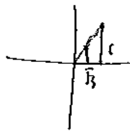
$\beta = \sqrt{3}+i$

$\frac{\alpha}{\beta}$ = Best to do mod arg form   in cartesian form.

$\alpha \rightarrow$

$= \sqrt{2}\angle 135°$

β :



ii) $\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

$= 2\angle 30°$

$= \frac{\sqrt{2}\angle 135}{2\angle 30} = \frac{\alpha}{\beta}$

$= \frac{\sqrt{2}}{2}\angle(135-30)$

iii) $= \frac{1}{\sqrt{2}}\angle 105 = \frac{1}{\sqrt{2}}\left(\cos\frac{7\pi}{12} - i\sin\frac{7\pi}{12}\right)$

iv) ie $\frac{1}{\sqrt{2}}\cos\frac{7\pi}{12} = \frac{(1-\sqrt{3})}{4}.$

$\cos\frac{7\pi}{12} = \frac{\sqrt{2}(1-\sqrt{3})}{4}$

$\frac{-1+i}{\sqrt{3}+i} \times \frac{\sqrt{3}-i}{\sqrt{3}-i}$

$= -\sqrt{3} + i + \sqrt{3}i + 1$

$\frac{1-\sqrt{3} + i(1+\sqrt{3})}{3+1}.$

$\frac{1-\sqrt{3}}{4} + i\,\frac{(1+\sqrt{3})}{4}$

i) $x = \frac{(1-\sqrt{3})}{4}$. $y = \frac{(1+\sqrt{3})}{4}$

d). $x = \alpha$. triple zero of $P(x)$.

ie $P(x) = (x-\alpha)^3 Q(x).$

$P'(x) \Rightarrow$ quotient rule

$= 3(x-\alpha)^2 Q(x) + Q'(x)(x-\alpha)^3$

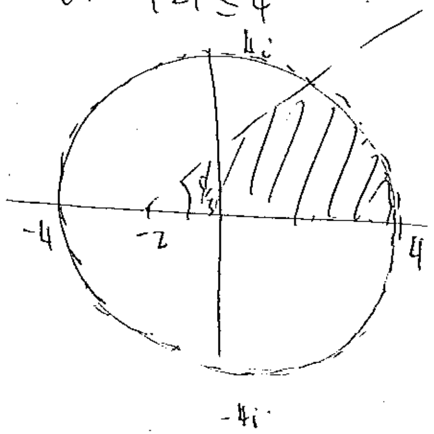
$= 0 + 0 = 0$ when $x = \alpha$

$\rightarrow = (x-\alpha)^2\left(3Q(x) + (x-\alpha)Q'(x)\right)$

So α is a double root.

e) $|z| \leq 4$

$0 \leq \text{Arg}(z+2) \leq \frac{\pi}{3}$



ii) $\frac{z^5-1}{z-1} = \left(z^2 - 2\cos\frac{2\pi}{5}z+1\right)\left(z^2 - 2\cos\frac{4\pi}{5}z+1\right)$

7. a) $z^5 - 1 = 0$

let $z = x+iy$.

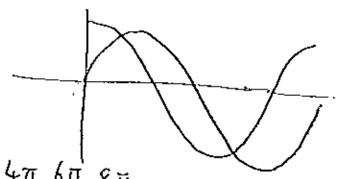
z in mod arg form

is $\underbrace{1\left(\cos\theta + i\sin\theta\right)}_{\text{standard form}}$.

so $z^5 = \cos 5\theta + i\sin 5\theta$.

$\cos 5\theta = 1$.

OR. $\sin 5\theta = 0$



ie at $\theta = 0, 2\pi, 4\pi, 6\pi, 8\pi$.

$z = \cos\left(\frac{2k\pi}{5}\right) + i\sin\left(\frac{2k\pi}{5}\right)$

for $k \in \mathbb{Z}$; $0 \leq k \leq 4$.

b. i) If α is a root

then

$\alpha^4 - 4\alpha^3 + 7\alpha^2 - 6\alpha - 4 = 0$

let $\alpha - 1 = y$, $\alpha = y+1$

$(y+1)^4 - 4(y+1)^3 + 7(y+1)^2 - 6(y+1) - 4 = 0$.

$0 = y^4 + 4y^3 + 6y^2 + 4y + 1 - 4y^3 - 12y^2 - 12y - 4 + 7y^2 + 14y + 7 - 6y - 6 - 4$

ie $0 = y^4 + y^2 - 6$

ie. $Q(x) = x^4 + x^2 - 6$.

let $u = x^2$.

$u^2 + u - 6 = 0$

$(u+3)(u-2) = 0$

ie $x^2 = -3$, $x^2 = 2$.

ie $x = \pm\sqrt{2}, \pm\sqrt{3}i$ } roots of $Q(x)$

So $\alpha = \sqrt{2}+1$

$\beta = 1 - \sqrt{2}$

$\gamma = 1 + \sqrt{3}i$

$\delta = 1 - \sqrt{3}i$

in no particular order.

$z = \cos\theta + i\sin\theta$.

$z^n = \cos n\theta + i\sin n\theta$

$\frac{1}{z^n} = z^{-n} = \cos(-n\theta) + i\sin(-n\theta)$

cos is even fn so $\cos(-n\theta) = \cos n\theta$

Sin is odd fn so $\sin(-n\theta) = -\sin(n\theta)$.

$= \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$

$\therefore 2\cos n\theta$ (as required).

ii) $\left(z + \frac{1}{z}\right)^3$: Expanding out.

$(2\cos\theta)^3 = 8\cos^3\theta$

$\left(z^2 + \frac{1}{z^2} + 2\right)\left(z + \frac{1}{z}\right)$.

$= z^3 + \frac{1}{z} + 2z + z + \frac{1}{z^3} + \frac{2}{z}$

$= z^3 + \frac{1}{z^3} + 2\left(z + \frac{1}{z}\right) + \left(z + \frac{1}{z}\right)$

$= z^3 + \frac{1}{z^3} + 3\left(z + \frac{1}{z}\right)$

$= 2\cos 3\theta + 3(2\cos\theta)$

$= 2\cos 3\theta + 6\cos\theta = 8\cos^3\theta$

iii) $8x^3 - 6x - 1 = 0$.

let $x = \cos\theta$

$8\cos^3\theta = 6\cos\theta + 1$.

ie $6\cos\theta + 1 = 2\cos 3\theta + 6\cos\theta$

ie $2\cos 3\theta = 1$

$\cos 3\theta = \frac{1}{2}$.

$\theta = \frac{\cos^{-1}\left(\frac{1}{2}\right)}{3}$

$= \frac{\pi}{9}$.

$x = \cos\left(\frac{\pi}{9}\right)$

$x = \cos\left(\frac{5\pi}{9}\right)$

$x = \cos\left(\frac{7\pi}{9}\right)$

iv) $\cos\frac{2\pi}{9} = -\cos\frac{7\pi}{9}$. (1)

product of roots.

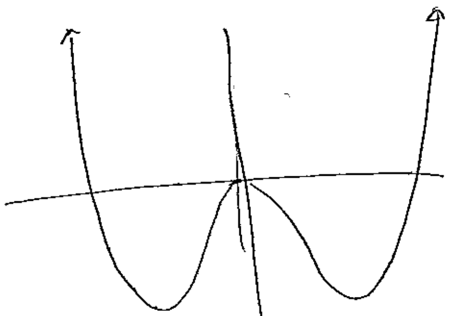
$\alpha\beta\gamma = -\frac{d}{a} = \frac{1}{8}$

but from (1)

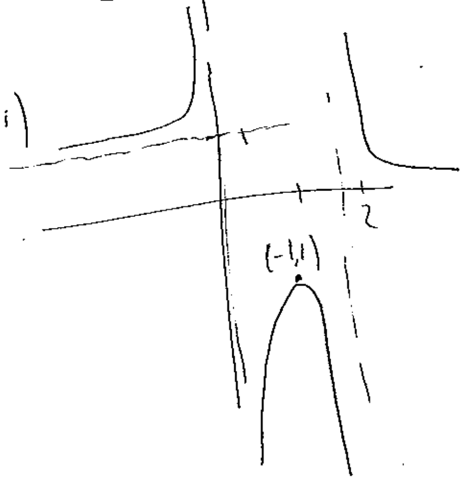
ie $\cos\frac{\pi}{9}\cos\frac{5\pi}{9}\cos\frac{2\pi}{9} = -\frac{1}{8}$

8. a).

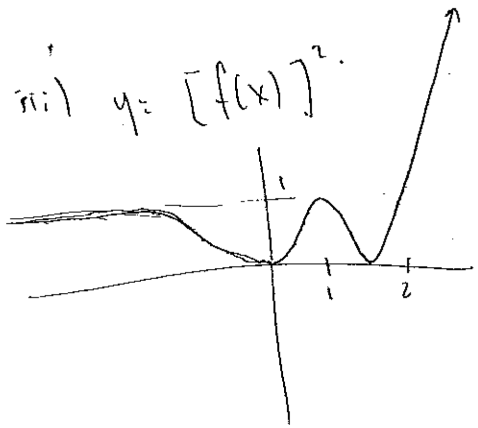
i) $y = f(|x|)$.



ii)



iii) $y = [f(x)]^2$.



b) Implicit Differentiation.

$x^2 + 3xy + 4y^2 = 58$

$2x + 3x\frac{dy}{dx} + 3y + 8y\frac{dy}{dx} = 0$

$\frac{dy}{dx}(3x + 8y) = -2x - 3y$

$\frac{dy}{dx} = \frac{-(2x+3y)}{(3x+8y)} = m_{\text{Tangent}}$.

So $= \frac{-(2(2) + 3(3))}{(3(2) + 8(3))} = \frac{-13}{30}$

$m_{\text{Normal}} = \frac{30}{13}$.

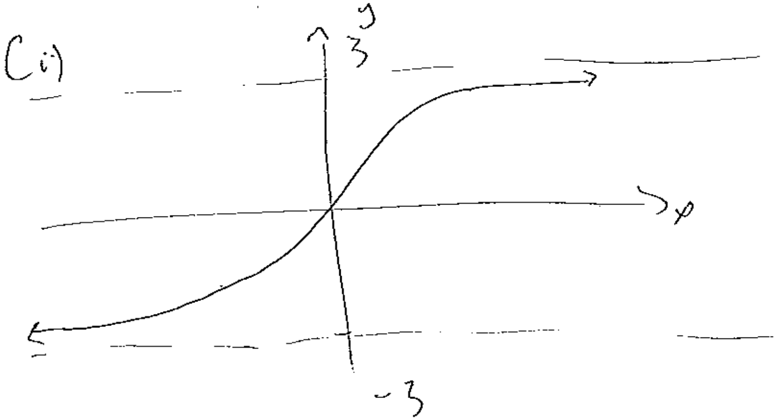
$(y-3) = \frac{30}{13}(x-2)$

$13(y-3) = 30(x-2)$.

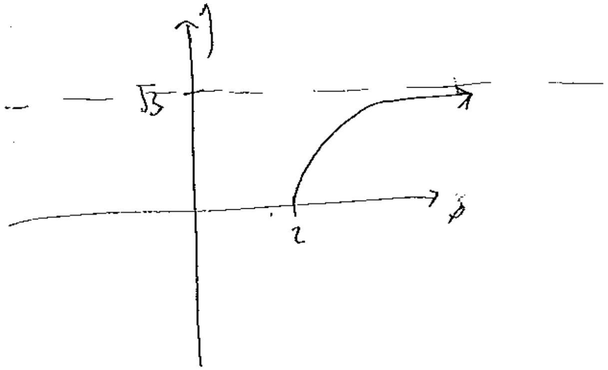
$13y - 39 = 30x - 60$

$30x - 13y - 21 = 0$.

C i)



ii)



9. $\int_0^1 \frac{2x+1}{x^2+1}\,dx$

$= \int_0^1 \frac{2x}{x^2+1} + \frac{1}{x^2+1}\,dx$

$= \left[\ln(x^2+1)\right]_0^1 + \tan^{-1}x\Big|_0^1$

$= \ln 2 - \ln(1) + \pi/4$

$= \ln 2 + \pi/4$

b) $\frac{4}{4-x^2} = \frac{A}{2-x} + \frac{B}{2+x}$

$A(2+x) + B(2-x) = 4$

equate coefficents

$2A + Ax + 2B - Bx = 4$

$2(A+B) = 4$

$A - B = 0 \rightarrow A = B$

$A = 1,\ B = 1$

$= \frac{1}{2-x} + \frac{1}{2+x}$

(i) $\int \frac{4}{4-x^2}\,dx = \int \frac{1}{2-x} + \frac{1}{2+x}\,dx$

$= -\ln(2-x) + \ln(2+x) + c$

c) i) in $\triangle COP$ and $\triangle PDA$

$\angle PDA$ is common.

$\angle DPC = \angle CQA$ (Alternate angles)

$\angle CQA = \angle DAP$ (Alternate segment theorem)

$\therefore \triangle COP \,|||\, \triangle PDA$ (equiangular)

ii) $\frac{PD}{AD} = \frac{CD}{PD}$ (corresponding sides of similar triangles)

$\therefore PD^2 = AD \times CD$.

iii). $PA = PB$ (tangents from same external point are equal)

$\therefore$ AD bisects PB (tangent secant theorem or $PD^2 = AD \times CD$)