SYDNEY BOYS HIGH
MOORE PARK, SURRY HILLS

2013
HIGHER SCHOOL CERTIFICATE
ASSESSMENT TASK #1

Mathematics Extension 2

General Instructions:

- Reading time – 5 minutes
- Working time – 1.5 hours
- Write using black or blue pen
  Black pen is preferred
- Board approved calculators may be used
- A table of standard integrals is provided on the back of the Multiple Choice answer sheet
- All necessary working should be shown in every question if full marks are to be awarded
- Marks may NOT be awarded for messy or badly arranged work
- Answer in simplest exact form unless otherwise stated

Total marks – 70 Marks
Section I Pages 2–3
7 marks

- Attempt Questions 1–7
- Answer on the Multiple Choice answer sheet provided.
- Allow about 10 minutes for this section

Section II Pages 4–6
63 marks

- Attempt Questions 8–10
- Allow about 1 hour 20 minutes for this section
- For Questions 8–10, start a new answer booklet per question

Examiner: Mr D. Hespe

This is an assessment task only and does not necessarily reflect the content or format of the Higher School Certificate.

Section I — 7 marks

Select the alternative A, B, C, or D that best answers the question.
Fill in the response oval on your multiple choice answer sheet.

Marks

1. The probability that a randomly chosen angle has a sine which is less than a half is **1**

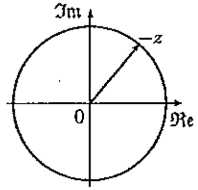
(A) $\frac{1}{2}$

(B) $\frac{2\pi}{3}$

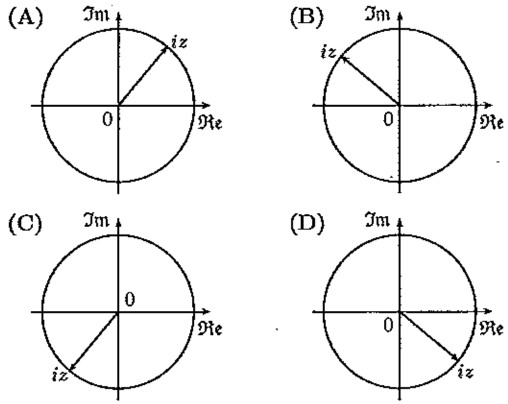
(C) $\frac{2}{3}$

(D) $\frac{4\pi}{3}$

2. **1**



This graph is a representation of $-z$ shown in an Argand diagram. Which of the following shows iz?



3. If $f(x) = e^{x+2}$ what is the inverse function $f^{-1}(x)$? **1**

(A) $f^{-1}(x) = e^{y-2}$

(B) $f^{-1}(x) = e^{y+2}$

(C) $f^{-1}(x) = \log_e x - 2$

(D) $f^{-1}(x) = \log_e x + 2$

4. What is the domain and range of $y = \cos^{-1}\left(\frac{2x}{5}\right)$? [1]

(A) Domain: $\left[-\frac{5}{2}, \frac{5}{2}\right]$; Range: $[0, \pi]$

(B) Domain: $[-1, 1]$; Range: $[0, \pi]$

(C) Domain: $\left[-\frac{5}{2}, \frac{5}{2}\right]$; Range: $[-\pi, \pi]$

(D) Domain: $[-1, 1]$; Range: $[-\pi, \pi]$

5. Form a polynomial $f(x)$ with real coefficients having the given degree and zeroes. [1]
Degree: 3, Zeroes: $1+i$ and -5.

(A) $f(x) = x^3 + x^2 - 8x + 10$

(B) $f(x) = x^3 - 5x^2 - 8x - 12$

(C) $f(x) = x^3 + 3x^2 - 8x + 10$

(D) $f(x) = x^3 + 3x^2 + 10x - 8$

6. Find the indefinite integral $\int 2t^2(1+t^3)^4 dt$ using the substitution $u = 1 + t^3$. [1]

(A) $\frac{1}{5}(1+t^3)^5 + c$

(B) $\frac{2}{5}(1+t^3)^5 + c$

(C) $\frac{2}{3}(1+t^3)^5 + c$

(D) $\frac{2}{15}(1+t^3)^5 + c$

7. If $(x-3)^2 + (y+2)^2 = 0$, then $x + y =$ [1]

(A) 1

(B) 2

(C) 3

(D) 5

Section II— 63 marks

Marks

Question 8 (21 marks) (use a separate answer booklet)

(a) You are dealt a hand of 5 cards from a standard deck of 52 (in 4 suits of 13 cards).

(i) What is the chance that you have a flush (*i.e.* all cards from the same suit)? [1]

(ii) What is the chance of four of a kind (*i.e.* 4 cards of the same value)? [1]

(b) (i) What is $\int \sec^2\psi\, d\psi$? [1]

(ii) Evaluate $\int_0^{\pi} \sin^3\theta\, d\theta$. [3]

(iii) Find $\int \frac{dx}{\sqrt{x^2 - 4x + 20}}$ by using the substitution $y = x - 2$. [3]

(c) Mark on an Argand diagram the points representing the numbers $2 + 3i$ and $-3 + 4i$. [2]

(d) Find, correct to three significant figures, the modulus and argument of $\frac{1}{12+5i}$. [2]

(e) Simplify:

(i) $(5-3i)(2+i)$, [1]

(ii) $\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$, [1]

(iii) $\frac{1}{5-3i} - \frac{1}{5+3i}$. [1]

(f) The cubic equation $x^3 + ax - b$ has roots α, β, γ. Given that $\gamma = \alpha\beta$, express each of a and b in terms of γ only, and hence show that $(a+b)^2 = b$. [5]

Marks

Question 9 (21 marks) (use a separate answer booklet)

(a) The letters of the word EXCELLENT are arranged in a random order. Find the probability that:

(i) the same letter occurs at each end. [2]

(ii) X, C and N occur together in any order. [2]

(iii) the letters occur in alphabetical order. [1]

(b) Evaluate

(i) $\sin\left(\sin^{-1}\dfrac{1}{2}\right)$ [1]

(ii) $\sin\left(\cos^{-1}\dfrac{1}{2}\right)$ [1]

(iii) $\cos^{-1}\left(\cos\dfrac{4\pi}{3}\right)$ [1]

(iv) Sketch the function $f(x) = -2\pi\cos^{-1}\left(\dfrac{3x}{\pi}\right)$. [3]

(c) Use the substitution $x = \tan\theta$ to evaluate $\displaystyle\int_0^1 \frac{x^2\,dx}{(1+x^2)^2}$. [3]

(d) (i) Use limits to show why $\dfrac{dx}{dy} = \dfrac{1}{\frac{dy}{dx}}$. [1]

(ii) Prove that $\dfrac{d}{dx}(\tan^{-1}x) = \dfrac{1}{1+x^2}$. [3]

(iii) Hence find a primitive of $\dfrac{x^2}{1+x^2}$. [3]

Marks

Question 10 (21 marks) (use a separate answer booklet)

(a) A company has to place three orders for supplies among five distributors. Each order is randomly assigned, and a distributor may receive multiple orders. Find the probabilities of the following events.

(i) Each order goes to a different distributor. [1]

(ii) All orders go to the same distributor. [1]

(iii) Exactly two of the three go to one of the distributors. [1]

(b) (i) Use De Moivre's theorem to express $\cos 3\theta$ in terms of $\cos\theta$ and $\sin 3\theta$ in terms of $\sin\theta$. [2]

(ii) Use the result to solve the equation [3]

$$8x^3 - 6x + 1 = 0.$$

(c) When a rational integral function is divided by $(x-\alpha)^2$ the remainder is $R_2(x-\alpha) + R_1$, *i.e.* $f(x) = (x-\alpha)^2Q(x) + R_2(x-\alpha) + R_1$.

(i) Prove that $R_1 = f(\alpha)$ and $R_2 = f'(\alpha)$, where $f'(x)$ is the differential coefficient of $f(x)$ with respect to x. [2]

(ii) Show that $x^n - nx + n - 1$ is exactly divisible by $(x-1)^2$ for any integral value of n greater than 1. [2]

(iii) Deduce that $2^{4n} - 15n - 1$ is exactly divisible by 225 for any integral value of n greater than 1. [2]

(d) (i) If a, b are the complex numbers represented by points A, B in the Argand diagram, what geometrical properties correspond to the modulus and argument of b/a? [2]

(ii) Show that, if the four points representing the complex numbers z_1, z_2, z_3, z_4 are concyclic, the fraction [5]

$$\frac{(z_1 - z_2)(z_3 - z_4)}{(z_3 - z_2)(z_1 - z_4)}$$

must be real.

End of Paper

Section I — 7 marks

Select the alternative A, B, C, or D that best answers the question.
Fill in the response oval on your multiple choice answer sheet.

Marks

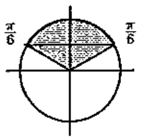
1. The probability that a randomly chosen angle has a sine which is less than a half is [1]

   (A) $\frac{1}{2}$

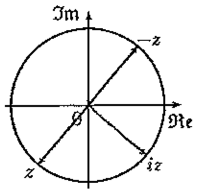
   (B) $\frac{2\pi}{3}$

   (C) $\frac{2}{3}$

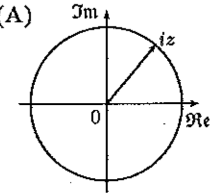
   (D) $\frac{4\pi}{3}$

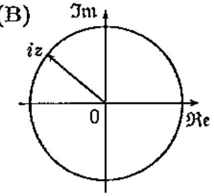
   Solution: 

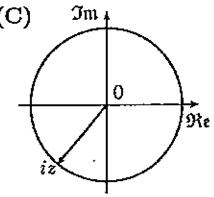
   $\frac{2\pi - \frac{2\pi}{3}}{2\pi} = \frac{2}{3}$, so (C).

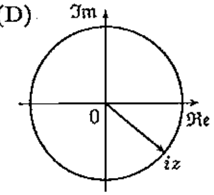
2. 

   This graph is a representation of $-z$ shown in an Argand diagram. Which of the following shows iz? [1]

   (A) 

   (B) 

   (C) 

   (D) 

   Solution: z is π from $-z$ and iz is rotated $\frac{\pi}{2}$ anticlockwise from z, so (D).

3. If $f(x) = e^{x+2}$ what is the inverse function $f^{-1}(x)$? [1]

   (A) $f^{-1}(x) = e^{y-2}$

   (B) $f^{-1}(x) = e^{y+2}$

   (C) $f^{-1}(x) = \log_e x - 2$

   (D) $f^{-1}(x) = \log_e x + 2$

   Solution: $\log_e x = f^{-1}(x) + 2,$
   $f^{-1}(x) = \log_e x - 2$, so (C).

4. What is the domain and range of $y = \cos^{-1}\left(\frac{2x}{5}\right)$? [1]

   (A) Domain: $\left[-\frac{5}{2}, \frac{5}{2}\right]$; Range: $[0, \pi]$

   (B) Domain: $[-1, 1]$; Range: $[0, \pi]$

   (C) Domain: $\left[-\frac{5}{2}, \frac{5}{2}\right]$; Range: $[-\pi, \pi]$

   (D) Domain: $[-1, 1]$; Range: $[-\pi, \pi]$

   Solution: $-1 \leqslant \frac{2x}{5} \leqslant 1,$
   $-\frac{5}{2} \leqslant x \leqslant \frac{5}{2}.$
   $0 \leqslant y \leqslant \pi$, so (A).

5. Form a polynomial $f(x)$ with real coefficients having the given degree and zeroes. [1]
   Degree: 3, Zeroes: $1 + i$ and -5.

   (A) $f(x) = x^3 + x^2 - 8x + 10$

   (B) $f(x) = x^3 - 5x^2 - 8x - 12$

   (C) $f(x) = x^3 + 3x^2 - 8x + 10$

   (D) $f(x) = x^3 + 3x^2 + 10x - 8$

   Solution: $S_1 = (1+i) + (1-i) - 5,$
   $= -3.$
   $S_2 = (1+i)(1-i) + (1+i)(-5) + (1-i)(-5),$
   $= -8.$
   $S_3 = (1+i)(1-i)(-5),$
   $= -10.$
   So the solution is $f(x) = x^3 + 3x^2 - 8x + 10$, *i.e.* (C).

6. Find the indefinite integral $\int 2t^2(1+t^3)^4 dt$ using the substitution $u = 1 + t^3$. [1]

   (A) $\frac{1}{5}(1+t^3)^5 + c$

   (B) $\frac{2}{5}(1+t^3)^5 + c$

   (C) $\frac{2}{3}(1+t^3)^5 + c$

   (D) $\frac{2}{15}(1+t^3)^5 + c$

   Solution: $du = 3t^2 dt,$
   $I = \frac{2}{3}\int 3t^2(1+t^3)^4 dt,$
   $= \frac{2}{3}\int u^4 du,$
   $= \frac{2}{3} \times \frac{u^5}{5} + c,$
   $= \frac{2(1+t^3)^5}{15}$, so (D).

7. If $(x-3)^2+(y+2)^2=0$, then $x+y=$ [1]

(A) 1

(B) 2

(C) 3

(D) 5

Solution:
$$\begin{aligned}(x-3)^2 \geqslant 0,\ (y+2)^2 &\geqslant 0,\\ (x-3)^2+(y+2)^2 &= 0,\\ \therefore x-3=0,\ y+2 &= 0,\\ x-3+y+2 &= 0,\\ x+y &= 1, \text{ so (A).}\end{aligned}$$

Section II— 63 marks

Marks

Question 8 (21 marks) (use a separate answer booklet)

(a) You are dealt a hand of 5 cards from a standard deck of 52 (in 4 suits of 13 cards).

(i) What is the chance that you have a flush (*i.e.* all cards from the same suit)? [1]

Solution: $\dfrac{4\times\binom{13}{5}}{\binom{52}{5}}=\dfrac{33}{16660}.$

(ii) What is the chance of four of a kind (*i.e.* 4 cards of the same value)? [1]

Solution: $\dfrac{13\times(52-4)}{\binom{52}{5}}=\dfrac{1}{4165}.$

(b) (i) What is $\displaystyle\int \sec^2\psi\,d\psi$? [1]

Solution: $\tan\psi+c$.

(ii) Evaluate $\displaystyle\int_0^{\pi}\sin^3\theta\,d\theta$. [3]

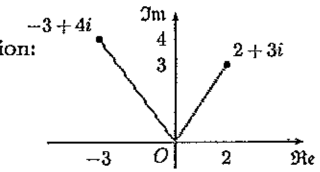
Solution:
$$\begin{aligned}\text{I} &= \int_0^{\pi}\sin^2\theta\sin\theta\,d\theta,\\ &= \int_0^{\pi}(1-\cos^2\theta)\sin\theta\,d\theta,\\ &= \int_0^{\pi}\sin\theta\,d\theta-\int_0^{\pi}\cos^2\theta\sin\theta\,d\theta,\\ &= \big[-\cos\theta\big]_0^{\pi}-\left[-\frac{\cos^3\theta}{3}\right]_0^{\pi},\\ &= 1-{}^-1-\left(\frac{1-{}^-1}{3}\right),\\ &= \frac{4}{3}.\end{aligned}$$

(iii) Find $\displaystyle\int\frac{dx}{\sqrt{x^2-4x+20}}$ by using the substitution $y=x-2$. [3]

Solution:
$$\begin{aligned} x^2-4x+4+16 &= (x-2)^2+4^2, \\ dy &= dx, \\ \text{I} &= \int \frac{dy}{\sqrt{y^2+4^2}}, \\ &= \ln(y+\sqrt{y^2+4^2})+c, \\ &= \ln(x-2+\sqrt{x^2-4x+20})+c. \end{aligned}$$

(c) Mark on an Argand diagram the points representing the numbers $2+3i$ and $-3+4i$. [2]

Solution:



(d) Find, correct to three significant figures, the modulus and argument of $\dfrac{1}{12+5i}$. [2]

Solution:
$$\begin{aligned} |12+5i| &= \sqrt{144+25}, \\ &= 13. \\ \arg(12+5i) &= \tan^{-1}\left(\frac{5}{12}\right), \\ &\approx 0.395 \text{ or } 22.6^\circ. \\ \text{So } \left|\frac{1}{12+5i}\right| &= \frac{1}{13}, \\ &\approx 0.0769. \\ \arg\left(\frac{1}{12+5i}\right) &\approx -0.395. \end{aligned}$$

(e) Simplify:

(i) $(5-3i)(2+i)$, [1]

Solution: $10+5i-6i+3=13-i$.

(ii) $\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$, [1]

Solution: $\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}=i$.

(iii) $\dfrac{1}{5-3i}-\dfrac{1}{5+3i}$. [1]

Solution: $\dfrac{5+3i-(5-3i)}{25+9}=\dfrac{3i}{17}$.

(f) The cubic equation x^3+ax-b has roots α, β, γ. Given that $\gamma=\alpha\beta$, express each of a and b in terms of γ only, and hence show that $(a+b)^2=b$. [5]

Solution:
$$\begin{aligned} 0 &= \alpha+\beta+\gamma, \\ \alpha+\beta &= -\gamma, \\ a &= \alpha\beta+\beta\gamma+\gamma\alpha, \\ &= \gamma+\gamma(\alpha+\beta), \\ &= \gamma-\gamma^2. \\ b &= \alpha\beta\gamma, \\ &= \gamma^2. \\ a+b &= \gamma, \\ (a+b)^2 &= \gamma^2, \\ &= b. \end{aligned}$$

Marks

Question 9 (21 marks) (use a separate answer booklet)

(a) The letters of the word EXCELLENT are arranged in a random order. Find the probability that:

(i) the same letter occurs at each end. [2]

> **Solution:** $\dfrac{\frac{7!}{2!}+\frac{7!}{3!}}{\frac{9!}{3!\times 2!}} = \dfrac{1}{9}.$

(ii) X, C and N occur together in any order. [2]

> **Solution:** $\dfrac{\frac{3!\times 7!}{3!\times 2!}}{\frac{9!}{3!\times 2!}} = \dfrac{1}{12}.$

(iii) the letters occur in alphabetical order. [1]

> **Solution:** $\dfrac{1}{\frac{9!}{3!\times 2!}} = \dfrac{1}{30240}$

(b) Evaluate

(i) $\sin\left(\sin^{-1}\frac{1}{2}\right)$ [1]

> **Solution:** $\frac{1}{2}$

(ii) $\sin\left(\cos^{-1}\frac{1}{2}\right)$ [1]

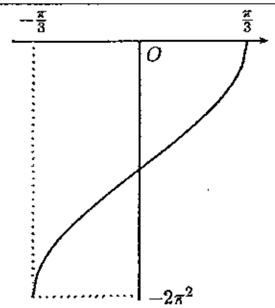
> **Solution:** $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

(iii) $\cos^{-1}\left(\cos\frac{4\pi}{3}\right)$ [1]

> **Solution:** $cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$

(iv) Sketch the function $f(x) = -2\pi\cos^{-1}\left(\frac{3x}{\pi}\right)$. [3]

Solution:



(c) Use the substitution $x = \tan\theta$ to evaluate $\displaystyle\int_0^1 \frac{x^2\,dx}{(1+x^2)^2}$. [3]

> **Solution:**
> $$\begin{aligned} dx &= \sec^2\theta d\theta, \\ x = 0 &\implies \theta = 0, \\ x = 1 &\implies \theta = \tfrac{\pi}{4}, \\ \text{I} &= \int_0^{\frac{\pi}{4}} \frac{\tan^2\theta\sec^2\theta d\theta}{(1+\tan^2\theta)^2}, \\ &= \int_0^{\frac{\pi}{4}} \frac{\sin^2\theta}{\cos^2\theta}\times\frac{d\theta}{\cos^{\theta}}\times\frac{1}{(\sec^2\theta)^2}, \\ &= \int_0^{\frac{\pi}{4}} \sin^2\theta d\theta, \\ &= \int_0^{\frac{\pi}{4}} \frac{1-\cos 2\theta}{2}\,d\theta, \\ &= \tfrac{1}{2}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{4}}, \\ &= \tfrac{1}{2}\left(\tfrac{\pi}{4} - \tfrac{1}{2} - (0-0)\right), \\ &= \frac{\pi-2}{8}. \end{aligned}$$

(d) (i) Use limits to show why $\dfrac{dx}{dy} = \dfrac{1}{\frac{dy}{dx}}$. [1]

> **Solution:** We consider a small change in x, δx and a corresponding small change in y, δy.
> Now $\dfrac{dx}{dy} = \lim\limits_{\delta y\to 0}\dfrac{\delta x}{\delta y}$,
> $= \lim\limits_{\delta y\to 0}\dfrac{1}{\frac{\delta y}{\delta x}}$
> If y is a continuous function of x then $\delta y \to 0$ as $\delta x \to 0$;

whence $\dfrac{dx}{dy} = \lim_{\delta x \to 0} \dfrac{1}{\frac{\delta y}{\delta x}}$,

$= \dfrac{1}{\frac{dy}{dx}}$.

(ii) Prove that $\dfrac{d}{dx}\left(\tan^{-1} x\right) = \dfrac{1}{1+x^2}$. [3]

Solution: Let $y = \tan^{-1} x,\ x \in \mathbb{R},\ -\frac{\pi}{2} < y < \frac{\pi}{2}$,

$$\begin{aligned} x &= \tan y, \\ \frac{dx}{dy} &= \sec^2 y, \\ &= 1 + \tan^2 y, \\ &= 1 + x^2. \\ \therefore \frac{dy}{dx} &= \frac{1}{1+x^2},\ x \in \mathbb{R},\ \frac{dy}{dx} > 0. \end{aligned}$$

(iii) Hence find a primitive of $\dfrac{x^2}{1+x^2}$. [3]

Solution: $$\begin{aligned} \int \frac{x^2 dx}{1+x^2} &= \int \left(1 - \frac{1}{1+x^2}\right) dx, \\ &= x - \tan^{-1} x + c. \end{aligned}$$

Marks

Question 10 (21 marks) (use a separate answer booklet)

(a) A company has to place three orders for supplies among five distributors. Each order is randomly assigned, and a distributor may receive multiple orders. Find the probabilities of the following events.

(i) Each order goes to a different distributor. [1]

Solution: $$\begin{aligned} \text{Total of all arrangements} &= 5 \times 5 \times 5, \\ &= 125. \\ \text{Ways of distributing} &= 5 \times 4 \times 3, \\ &= 60. \\ \therefore \text{P(each different)} &= \frac{60}{125}, \\ &= \frac{12}{25}. \end{aligned}$$

(ii) All orders go to the same distributor. [1]

Solution: There are only 5 ways this can occur,

$$\begin{aligned} \text{so}\ \ \text{P(all to same)} &= \frac{5}{125}, \\ &= \frac{1}{25}. \end{aligned}$$

(iii) Exactly two of the three go to one of the distributors. [1]

Solution: Method 1—

This is the only other possibility

$$\begin{aligned} \text{so}\ \ \text{P(2 of 3 to 1)} &= 1 - \left(\tfrac{12}{25} + \tfrac{1}{25}\right), \\ &= \tfrac{12}{25}. \end{aligned}$$

Method 2—

$$\begin{aligned} \tbinom{3}{2} \times 5 \times 4 &= 60. \\ \text{So P(2 of 3 to 1)} &= \tfrac{60}{125}, \\ &= \tfrac{12}{25}. \end{aligned}$$

(b) (i) Use De Moivre's theorem to express $\cos 3\theta$ in terms of $\cos\theta$ and $\sin 3\theta$ in terms of $\sin\theta$. [2]

Solution: $$\begin{aligned} \cos 3\theta + i \sin 3\theta &= (\cos\theta + i\sin\theta)^3, \\ &= \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta. \\ \cos 3\theta &= \cos^3\theta - 3\cos\theta\sin^2\theta \text{ (real part)}, \\ &= \cos^3\theta - 3\cos\theta(1 - \cos^2\theta), \\ &= 4\cos^3\theta - 3\cos\theta. \\ \sin 3\theta &= 3\cos^2\theta\sin\theta - \sin^3\theta \text{ (imaginary part)}, \\ &= 3(1 - \sin^2\theta)\sin\theta - \sin^3\theta, \\ &= 3\sin\theta - 4\sin^3\theta. \end{aligned}$$

(ii) Use the result to solve the equation [3]

$$8x^3 - 6x + 1 = 0.$$

Solution: If we put $x = \cos\theta$, then

$$\begin{aligned} 8\cos^3\theta - 6\cos\theta + 1 &= 0, \\ 4\cos^3\theta - 3\cos\theta &= -\tfrac{1}{2}, \\ i.e.\ \cos 3\theta &= -\tfrac{1}{2}, \\ 3\theta &= \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}, \frac{10\pi}{3} \ldots \\ \theta &= \frac{2\pi}{9}, \frac{4\pi}{9}, \frac{8\pi}{9}, \frac{10\pi}{9} \ldots \\ x &= \cos\frac{2\pi}{9}, \cos\frac{4\pi}{9}, \cos\frac{8\pi}{9}, \cos\frac{10\pi}{9} \ldots \end{aligned}$$

As it is a cubic, only 3 values of x are possible.

Now, $\cos\frac{8\pi}{9} = -\cos\frac{\pi}{9}$, $\cos\frac{10\pi}{9} = -\cos\frac{\pi}{9}$, etc.

Hence $x = -\cos\frac{\pi}{9}, \cos\frac{2\pi}{9}, \cos\frac{4\pi}{9}$ only.

(c) When a rational integral function is divided by $(x-\alpha)^2$ the remainder is $R_2(x-\alpha) + R_1$, *i.e.* $f(x) = (x-\alpha)^2 Q(x) + R_2(x-\alpha) + R_1$.

(i) Prove that $R_1 = f(\alpha)$ and $R_2 = f'(\alpha)$, where $f'(x)$ is the differential coefficient of $f(x)$ with respect to x. [2]

Solution:

$$\begin{aligned} f(x) &= (x-\alpha)^2 Q(x) + R_2(x-\alpha) + R_1 \ldots\ldots\ldots\ldots \boxed{1} \\ f(\alpha) &= (\alpha-\alpha)^2 Q(\alpha) + R_2(\alpha-\alpha) + R_1, \\ &= 0 + 0 + R_1. \\ i.e.\ R_1 &= f(\alpha). \end{aligned}$$

From $\boxed{1}$, differentiating w.r.t.x,

$$\begin{aligned} f'(x) &= 2(x-\alpha)Q(x) + (x-\alpha)^2 Q'(x) + R_2 + (x-\alpha)R'_2, \\ \therefore f'(\alpha) &= 2(\alpha-\alpha)Q(\alpha) + (\alpha-\alpha)^2 Q'(\alpha) + R_2 + (\alpha-\alpha)R'_2, \\ &= 0 + 0 + R_2 + 0, \\ i.e.\ R_2 &= f'(\alpha). \end{aligned}$$

(ii) Show that $x^n - nx + n - 1$ is exactly divisible by $(x-1)^2$ for any integral value of n greater than 1. [2]

Solution: Let

$$\begin{aligned} f(x) &= x^n - nx + n - 1, \\ f(1) &= 1^n - n + n - 1, \\ &= 0 = R_1. \\ f'(x) &= nx^{n-1} - n, \\ f'(1) &= n.1^{n-1} - n, \\ &= n - n = 0 = R_2. \\ \therefore f(x) &= (x-1)^2 Q(x) + 0 + 0, \end{aligned}$$

i.e. $x^n - nx + n - 1$ is divisible by $(x-1)^2$.

n must be an integer since $x^n - nx + n - 1$ must be a rational function before the above procedure can be used.

(iii) Deduce that $2^{4n} - 15n - 1$ is exactly divisible by 225 for any integral value of n greater than 1. [2]

Solution:

$$x^n - nx + n - 1 = x^n - (x-1)n - 1,$$

Taking the $x = 16$ we have

$$\begin{aligned} x^n - (x-1)n - 1 &= 16^n - 15n - 1, \\ &= 2^{4n} - 15n - 1. \end{aligned}$$

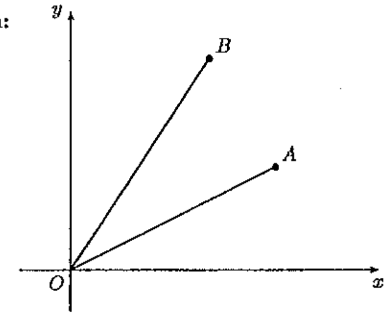
However $x^n - (x-1)n - 1$ is divisible by $(x-1)^2$, $n > 1$.

$\therefore 2^{4n} - 15n - 1$ is divisible by $(15)^2$ for any integral value of $n > 1$.

i.e. $2^{4n} - 15n - 1$ is divisible by 225 for any integral value of n greater than 1.

(d) (i) If a, b are the complex numbers represented by points A, B in the Argand diagram, what geometrical properties correspond to the modulus and argument of b/a? [2]

Solution:



The lengths of the vectors $\vec{OA}$, $\vec{OB}$ are equal respectively to $|a|$ and $|b|$ and the angles xOA, xOB are equal to $\arg a$ and $\arg b$.

Since $\left|\dfrac{b}{a}\right| = \dfrac{|b|}{|a|} = \dfrac{OB}{OA}$,

the modulus of b/a corresponds to the ratio OB/OA.

Again, since $\arg\left(\dfrac{b}{a}\right) = \arg b - \arg a = \angle xOB - \angle xOA$,

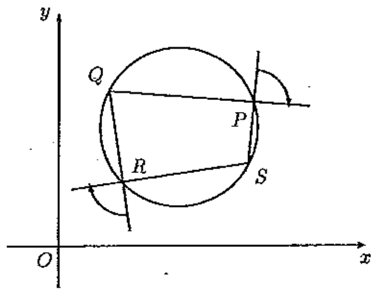
$= \angle AOB$.

(ii) Show that, if the four points representing the complex numbers z_1, z_2, z_3, z_4 are concyclic, the fraction [5]

$$\frac{(z_1 - z_2)(z_3 - z_4)}{(z_3 - z_2)(z_1 - z_4)}$$

must be real.

Solution: Let the representative points of the four complex numbers z_1, z_2, z_3, z_4 be P, Q, R, S.



Clearly, the sum of the marked angles is π (opposite $\angle$s of a cyclic quad. and vert. opp. $\angle$s). Since vectors $\vec{SP}$, $\vec{QP}$ represent repectively the complex numbers $z_4 - z_1$ and $z_2 - z_1$, the angle of turn from $\vec{SP}$ to $\vec{QP}$ is

$$\arg\left(\frac{z_2 - z_1}{z_4 - z_1}\right).$$

Similarly the angle of turn from $\vec{QR}$ to $\vec{SR}$ is $\arg\left(\dfrac{z_4 - z_3}{z_2 - z_3}\right)$. Hence

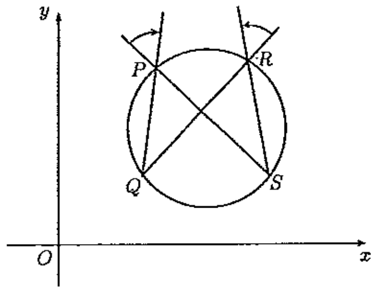
$$\arg\left(\frac{z_2 - z_1}{z_4 - z_1}\right) + \arg\left(\frac{z_4 - z_3}{z_2 - z_3}\right) = \pi.$$

But the sum of the arguments of two complex numbers is equal to the argument of their product, so that

$$\arg\left(\frac{z_2 - z_1}{z_4 - z_1} \times \frac{z_4 - z_3}{z_2 - z_3}\right) = \pi.$$

If the argument of a complex number is π, the number is a negative real number as required.

NOTE: even if we label the points around the circle in some other, non-cyclic order:



Then the sum of the arguments is zero and we have a positive real number.