Student Number:______________________

St Catherine's School
Waverley

Year 12 Mathematics **Extension 1**

Task 2

March 2017

Time allowed: 90 minutes plus 5 minutes reading time

Total marks: 52 marks

Weighting: 25%

There are 10 questions including 7 multiple choice questions.

INSTRUCTIONS

- In Section I, answer the 7 multiple choice questions on the attached multiple choice answer sheet.
- In Section II, start questions 8, 9 and 10 in three separate answer booklets. You may request more booklets if needed.
- Show all necessary working for questions 8, 9 and 10.
- Marks may be deducted for careless or badly arranged work.
- A reference sheet is provided.
- Write using blue or black pen. Black pen is preferred.

Multiple choice		/7 marks
Question 8		/15 marks
Question 9		15 marks
Question 10		/15 marks
Total		/52 marks

Section 1

7 marks

Attempt questions 1 – 7

Use the multiple-choice answer sheet for questions 1 – 7.

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1. What is a focal length of the parabola given with $x = 10t$, $y = -5t^2$?

(A) -5 (B) $-\frac{5}{2}$

(C) $\frac{5}{2}$ (D) 5

2. If $f(x) = \frac{1}{5}x + 3$, find the gradient of $y = f^{-1}(x)$.

(A) 5 (B) $\frac{1}{5}$

(C) -5 (D) $-\frac{1}{5}$

3. Which of the following is an asymptote to the curve $y = \frac{x^2-9}{x}$?

(A) $x = 3$ (B) $y = 0$

(C) $y = x$ (D) $x = 0$

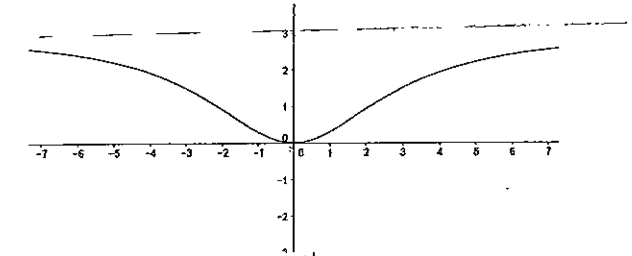
4. Find the exact value of $\tan\left(\sin^{-1}\left(-\frac{2}{3}\right)\right)$.

(A) $-\frac{2}{\sqrt{5}}$ (B) $-\frac{2}{5}$

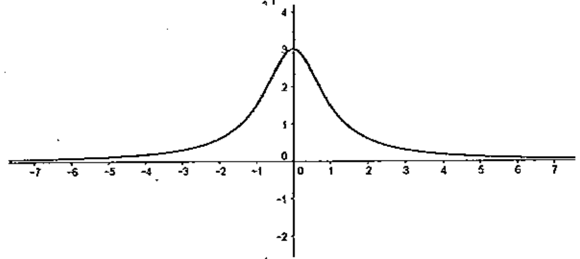
(C) $\frac{2}{5}$ (D) $\frac{2}{\sqrt{5}}$

5. Which of the following curves represents $y = \frac{3x^2}{x^2+9}$?

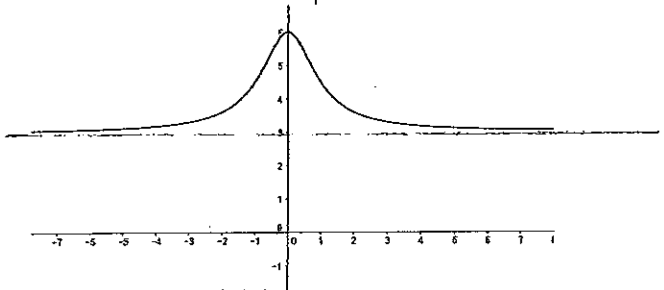
(A)



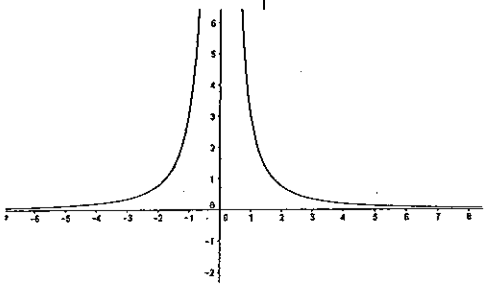
(B)



(C)



(D)

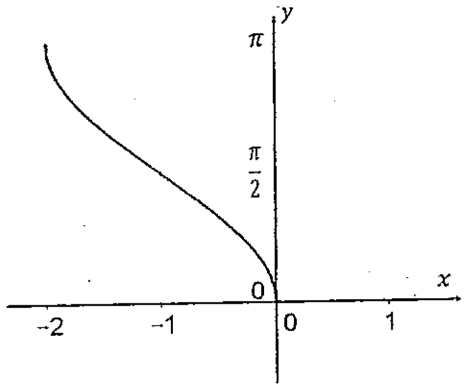


6. $\displaystyle\int_1^2 \frac{1}{\sqrt{4-x^2}}\,dx$

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{3}$

(C) $\frac{\pi}{2}$ (D) $\frac{\pi}{4}$

7.



Which one is the equation of the function above?

(A) $y = sin^{-1}(x+1)$ (B) $y = cos^{-1}(x+1)$

(C) $y = cos^{-1}(x) + 1$ (D) $y = sin^{-1}(x) + 1$

End of section I

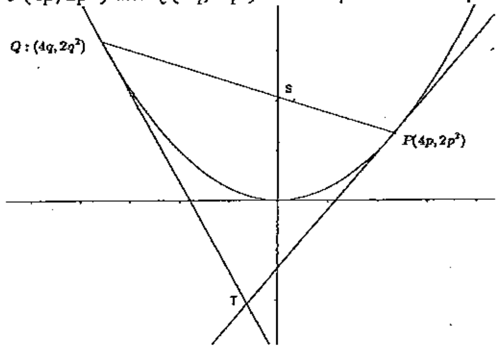
Section II

45 marks
Attempt questions 8 – 10
Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.
In Questions 8 – 10, your responses should include relevant mathematical reasoning and/or calculations.

Question 8 (15 marks) **Use a separate writing booklet** **Marks**

a) Differentiate $y = tan^{-1}\frac{1}{x+1}$ **2**

b) $P(4p, 2p^2)$ and $Q(4q, 2q^2)$ are two points on the parabola $x^2 = 8y$.



(i) Show that the equation of the tangent at point P is $px - y = 2p^2$ **2**

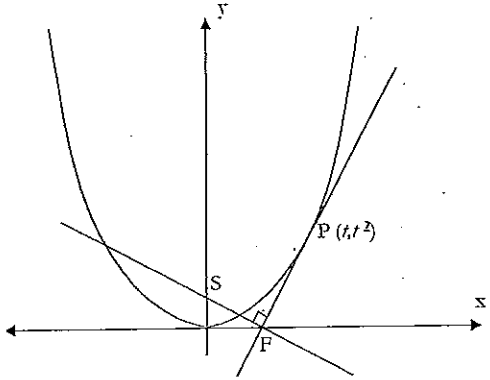
(ii) The tangents at P and Q intersect at point T. Write down the equation of the tangent at Q and hence show that the coordinates of point T are $(2(p+q), 2pq)$. **2**

(iii) P and Q move on the parabola such that PQ always passes through the focus. Show that $pq = -1$ **3**

(iv) Hence find the equation of the locus of T. **1**

Question 8 continues over the next page

c) Consider the parabola $y = x^2$.



It is given that the equation of the tangent at a point $P(t, t^2)$ is $y = 2tx - t^2$. Do not prove this.

A line SF is drawn perpendicular to the tangent at P from the focus S, meeting the tangent at F.

(i) Show that the equation of SF is $2x + 4ty = t$. **2**

(ii) Show that the coordinates of point F are $(\frac{t}{2}, 0)$. **1**

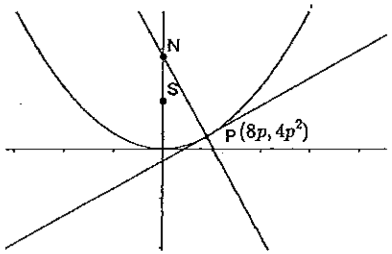
(iii) Find the Cartesian equation of the locus of M, the midpoint of SF. **2**

End of question 8

Question 9 (15 marks) **Use a separate writing booklet** **Marks**

a) Find $\int \frac{dx}{\sqrt{9-5x^2}}$ **2**

b)



(i) Show that the equation of the normal to the parabola $x^2 = 16y$ at the point $P(8p, 4p^2)$, is given by $x + py = 8p + 4p^3$. **2**

(ii) This normal meets the y-axis at the point N. S is the focus of the parabola. Find the distance SN in terms of p. **2**

c) Find the exact value of: **3**

$$\int_0^{\frac{1}{\sqrt{15}}} \frac{4}{1+5x^2}\,dx$$

Question 9 continues over the next page.

Question 9 continued..

d) Consider the curve

$$y = \frac{x+1}{x^2-16}$$

(i) State the equation(s) of the vertical asymptote. **1**

(ii) Find the equation of any horizontal asymptote(s). **1**

(iii) Show that there are no stationary points. **2**

(iv) Sketch the curve showing the above features and include the x and y intercepts. **2**

End of question 9

Question 10 (15 marks) **Use a separate writing booklet** **Marks**

a) By using the expansion of $tan(A+B)$ prove that **3**

$tan^{-1}(\frac{1}{4}) + tan^{-1}(\frac{3}{5}) = \frac{\pi}{4}$.

This needs to be set out clearly. Showing calculations alone will not be acceptable.

b) $f(x) = 3cos^{-1}\sqrt{x}$

(i) Find the domain and range of $f(x)$ **2**

(ii) Sketch the function clearly indicating all important features **2**

(iii) Find $f^{-1}x$ **1**

c) Consider the function $f(x) = (x-2)^2 + 1$, for the domain $x \geq 2$

(i) Find the inverse function $f^{-1}(x)$ **2**

(ii) On the same coordinate axes sketch the functions $= f(x)$, $y = f^{-1}(x)$ and the line $y = x$, clearly indicating the end point and any points of intersection. **3**

(iii) Find the x coordinates of any point(s) of intersection of $y = f(x)$ and $y = f^{-1}(x)$ **2**

END OF PAPER

Solution Test. 2 ear. 1. 2017

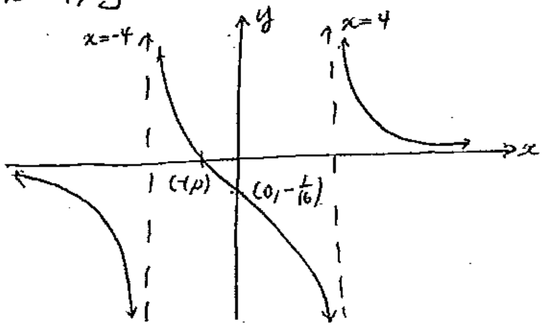
Qn	Solutions	Marks	Comments: Criteria
	Solution		
1)	$x = 10t$; $y = -5t^2$ $y = -5\frac{x^2}{100} = -\frac{x^2}{20}$; $x^2 = -20y$. focal length is 5 $\boxed{D}$		
2)	$y = \frac{x}{5} + 3$ $x = \frac{y}{5} + 3$ $y = 5x - 15$ $\boxed{A}$ $\boxed{S}$		
3)	$y = \frac{x^2-9}{x} = x - \frac{9}{x}$ $y = x$ $\boxed{C}$		
4)	let $\sin^{-1}\frac{2}{3} = \alpha$; $\sin\alpha = \frac{2}{3}$  $\sin^{-1}(-\frac{2}{3}) = -\sin^{-1}\frac{2}{3}$. $= -\alpha$ $\tan(\sin^{-1}(-\frac{2}{3})) = \tan(-\alpha)$ $= -\tan\alpha$ $= -\frac{2}{\sqrt{5}}$. $\boxed{A}$		
5)	$y = \frac{3x^2}{x^2+9} = 3 + \frac{-27}{x^2+9}$. $x^2+9 \overline{) 3x^2}$ quotient 3, $3x^2+27$, -27 $y = 3$ is an asymptote ; $x = 0$; $y = ?$ $\boxed{A}$		

Qn	Solutions	Marks	Comments: Criteria
6)	$\int_1^2 \frac{1}{\sqrt{4-x^2}} = \left(\sin^{-1}\frac{x}{2}\right)_1^2$		
	$= \sin^{-1}1 - \sin^{-1}\frac{1}{2}$		
	$= \frac{\pi}{2} - \frac{\pi}{6} = \frac{\pi}{3}$ $\boxed{B}$		
7)	$\cos^{-1}(x+1)$ $\boxed{B}$		
	Q.8		
a)	$y = \tan^{-1}\frac{1}{x+1}$		
	$y' = \frac{1}{1+\frac{1}{(x+1)^2}} \times \frac{d}{dx}\frac{1}{x+1}$	(1)	
	$= \frac{(x+1)^2}{(x+1)^2+1}\left(\frac{-1}{(x+1)^2}\right)$	(1)	
	$= \frac{-1}{(x+1)^2+1}$.		
b) (i)	$x^2 = 8y$		
	$y = \frac{x^2}{8}$		
	$y' = \frac{x}{4}$		
	$y'_p = \frac{4p}{4} = p$	(1)	
	Eqn. of the tangent is		
	$y - 2p^2 = p(x-4p)$		
	$y - 2p^2 = px - 4p^2$		
	$px - y = 2p^2$	(1)	

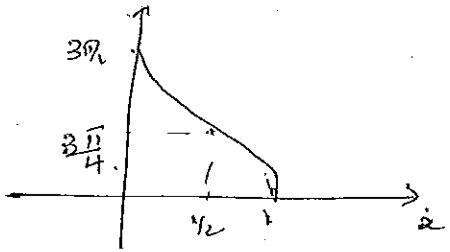
Qn	Solutions	Marks	Comments: Criteria
(ii)	Tgt. at Q:		
	$qx - y = 2q^2$ ①		
	$px - y = 2p^2$ ②		
	① − ②		
	$(q-p)x = 2(q^2-p^2)$		
	$x = 2(q+p)$ $q \neq p$		
	$y = q(2(q+p)) - 2q^2$		
	$= 2pq$		
	$\therefore T: (2(p+q), 2pq)$	(2)	
(iii)	focus: $(0,2)$		Alternate Eq. of chord PQ
	$m_{PS} = m_{QS}$		$m_{PQ} = \frac{2p^2-2q^2}{4p-4q}$
	$\frac{2q^2-2}{4q} = \frac{2p^2-2}{4p}$ (2)		$= \frac{2(p-q)(p+q)}{4(p-q)}$
	$\frac{q^2-1}{q} = \frac{p^2-1}{p}$		$= \frac{p+q}{2}$ (1)
	$pq^2 - p = p^2q - q$		$y - 2p^2 = \frac{p+q}{2}(x-4p)$
	$pq^2 - p^2q = p - q$		$2y - 4p^2 = (p+q)x - 4p(p+q)$
	$pq(q-p) = p-q$		sub $(0,2)$
	$\therefore pq = -1$. (1)		$4 - 4p^2 = (p+q)\times 0 - 4p(p+q)$ (1)
(iv)	For T:		$4 - 4p^2 = -4p(p+q)$
	$x = 2(p+q)$		$4 - \cancel{4p^2} = \cancel{-4p^2} - 4pq$
	$y = 2pq$		$4 = -4pq$
	also $pq = -1$		$\boxed{\therefore pq = -1}$ (1)
	$\therefore$ Locus of T is $y = -2$ (1)		

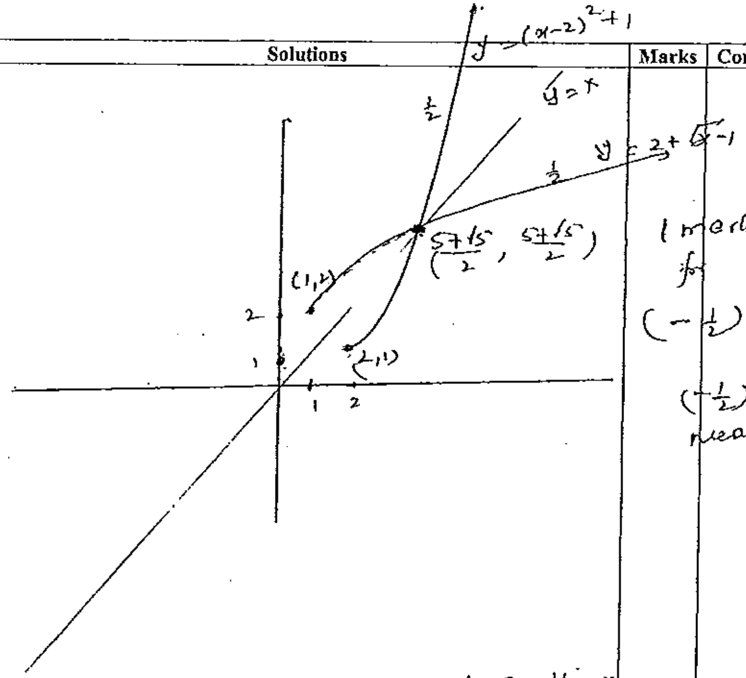
Qn	Solutions	Marks	Comments: Criteria
c) (i)	tgr. at P is $y = 2tx - t^2$ grad. of the tangent is $2t$. grad. of $SF = -\frac{1}{2t}$.	(1)	
	Eqn. of SF: $S: (0, \frac{1}{4})$ $y - \frac{1}{4} = -\frac{1}{2t}(x-0)$ $4ty - t = -2x$. $2x + 4ty = t$.	(1)	
(ii)	$2x + 4ty = t$ meet $y = 2tx - t^2$ $2x + 4t(2tx - t^2) = t$ $2x + 8t^2x - 4t^3 = t$ $x(2 + 8t^2) = t + 4t^3$ $x = \frac{t(1+4t^2)}{2(1+4t^2)} = \frac{t}{2}$ $y = 2t \cdot \frac{t}{2} - t^2 = 0$.	(1)	0 marks if co-ordinates of F were found by assuming it lies on the x-axis
(iii)	Mid. pt. of SF: $x = \frac{0 + \frac{t}{2}}{2} = \frac{t}{4}$ $y = \frac{\frac{1}{4} + 0}{2} = \frac{1}{8}$. Locus is $y = \frac{1}{8}$.	(2)	$(-\frac{1}{2})$ if the locus of M is not explicitly stated.

Qn	Solutions	Marks	Comment: Criteria
	Q9		
a)	$\int \frac{dx}{\sqrt{9-5x^2}} = \int \frac{dx}{\sqrt{5(\frac{9}{5}-x^2)}} = \frac{1}{\sqrt{5}} \int \frac{dx}{\sqrt{(\frac{3}{\sqrt{5}})^2 - x^2}}$	(1)	
	$= \frac{1}{\sqrt{5}} \sin^{-1} \frac{x}{3/\sqrt{5}} + c$	(1)	
	$= \frac{1}{\sqrt{5}} \sin^{-1} \frac{\sqrt{5}x}{3} + c$		
b)	$x^2 = 16y$ $y = \frac{1}{16}x^2$		
(i)	$y' = \frac{x}{8}$ y' at $P = \frac{8p}{p} = p$		
	gradient of the normal at P is $-\frac{1}{p}$	(1)	
	equation of the normal at P is $y - 4p^2 = -\frac{1}{p}(x - 8p)$	(1)	
	$yp - 4p^3 = -x + 8p$ $x + yp = 4p^3 + 8p$		
(ii)	N: $x = 0$, $y = 4p^2 + 8$ $(p \neq 0)$		
	$S: (0, N)$	(1)	
	$SN = 4p^2 + 8 - 4 = 4p^2 + 4$	(1)	
(c)	$\int_0^{\frac{1}{\sqrt{5}}} \frac{4}{1+5x^2}\,dx = 4\int_0^{\frac{1}{\sqrt{5}}} \frac{1}{5(\frac{1}{5}+x^2)}\,dx$	(1)	
	$= \frac{4}{5} \times \frac{1}{1/\sqrt{5}} \left[\tan^{-1} \frac{x}{1/\sqrt{5}}\right]_0^{\frac{1}{\sqrt{5}}}$	(1)	$\frac{4\sqrt{5}}{5} \tan^{-1} \sqrt{5}x$
	$= \frac{4\sqrt{5}}{5} \left(\tan^{-1} \frac{1}{\sqrt{5}} \times \sqrt{5}\right)$		

Qn	Solutions	Marks	Comment: Criteria
	$= \frac{4\sqrt{5}}{5} \tan^{-1}\frac{1}{\sqrt{3}}$		
	$= \frac{4\sqrt{5}}{5} \times \frac{\pi}{6}$	(1)	
	$= \frac{2\sqrt{5}\pi}{15}$		
(d) (i)	$x = 4$ or $x = -4$	(1)	
(ii)	$\lim_{x\to\infty} \frac{x+1}{x^2-16} = \lim_{x\to\infty} \frac{\frac{1}{x}+\frac{1}{x^2}}{1-\frac{16}{x^2}}$		
	$= 0$	(1)	
	$\therefore y = 0$		
(iii)	$y' = \frac{(x^2-16)(1) - (x+1)(2x)}{(x^2-16)^2}$	(1)	
	at stationary points : $y' = 0$		
	$x^2 - 16 - 2x^2 - 2x = 0$		
	$x^2 + 2x + 16 = 0$		
	$(x+1)^2 + 15 \neq 0$	(1)	or equivalent proof
	$\therefore$ there are no stationary points		
(iv)	$x = 0,\ y = -\frac{1}{16}$		
	$x = -1,\ y = 0$		
		(1) (1)	correct shape correct intercepts and asymptotes

Qn	Solutions	Marks	Comments: Criteria
10 (a)	Consider $\tan(\tan^{-1}\frac{1}{4} + \tan^{-1}\frac{3}{5})$		
	Let $\alpha = \tan^{-1}\frac{1}{4}$; $\beta = \tan^{-1}\frac{3}{5}$		
	$\tan\alpha = \frac{1}{4}$ $\tan\beta = \frac{3}{5}$	1M	
	α and β are acute		
	$\tan(\alpha+\beta)$		
	$= \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$		
	$= \frac{\frac{1}{4}+\frac{3}{5}}{1-\frac{1}{4}\times\frac{3}{5}}$ 1M		
	$= \frac{17/20}{17/20} = 1$		
	$\therefore \alpha+\beta = \frac{\pi}{4}$ 1M		
	i.e $\tan^{-1}\frac{1}{4} + \tan^{-1}\frac{3}{5} = \frac{\pi}{4}$		
b)	$f(x) = 3\cos^{-1}\sqrt{x}$		
	$\sqrt{x}$ is meaningful when $x > 0$		
	Domain of $\cos^{-1}x$ is $-1 \le x \le 1$		
	Combining both the requirements, we have $0 \le \sqrt{x} \le 1$		
	Domain $0 \le x \le 1$		
	Range $0 \le \cos^{-1}\sqrt{x} \le \frac{\pi}{2}$		
	$0 \le 3\cos^{-1}\sqrt{x} \le \frac{3\pi}{2}$		

Qn	Solutions	Marks	Comments: Criteria
		2M	label. $(1,0)$; $(0, 3\pi/2)$ general shape.
	$y = 3\cos^{-1}\sqrt{x}$		
	$x = 3\cos^{-1}\sqrt{y}$	1	
	$\cos^{-1}\sqrt{y} = \frac{x}{3}$		
	$\sqrt{y} = \cos\frac{x}{3}$		
	$y = \cos^2\frac{x}{3}$	1	
c)	$f(x) = (x-2)^2+1$		
	$y = (x-2)^2+1 \quad x \geq 2.$		
	Inverse $x = (y-2)^2+1$	1	
	$(y-2)^2 = x-1$		
	$y-2 = \pm\sqrt{x-1}$	1	no reason $(-\frac{1}{2})$
	$y = 2\pm\sqrt{x-1}$		
	$y = 2+\sqrt{x-1}$ since $y \geq 2$	1	

Qn	Solutions	Marks	Comments: Criteria
ii)			1 mark each for each curve $(-\frac{1}{2})$ no end point $(-\frac{1}{2})$ if not meeting on $y=x$
iii)	$y = f(x)$, $y = f^{-1}(x)$ meet on $y = x$		
	$f(x) = f^{-1}(x)$		
	$\equiv \quad f(x) = x$		
	$(x-2)^2+1 = x$		
	$x^2-4x+5 = x$		
	$x^2-5x+5 = 0$		
	$x = \frac{5\pm\sqrt{5}}{2}$		
	The inv. fn. exists for $x \geq 2$		
	$\therefore x = \frac{5+\sqrt{5}}{2}$ $\left(y = \frac{5+\sqrt{5}}{2}\right.$, not necessary$)$		