Student Number: ____________

2016 ASSESSMENT TASK 3

Mathematics

General Instructions

- Working time – 50 minutes
- Write using black pen
- Board-approved calculators may be used
- A reference sheet is provided at the back of this paper
- In Questions 4–5, show relevant mathematical reasoning and/or calculations

Total Marks – 33

Section I Page 2

3 marks

- Attempt Questions 1–3
- Allow about 5 minutes for this section

Section II Pages 3–4

30 marks

- Attempt Questions 4–5
- Allow about 45 minutes for this section

Section I

3 marks
Attempt Questions 1–3
Allow about 5 minutes for this section

Use the multiple-choice answer sheet for Questions 1–3.

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1 If $\log_x a = 0.77$ and $\log_x b = 3.08$, what is the value of $\log_x\left(\frac{a}{b}\right)$?

(A) -2.31 (B) 0.25 (C) 2.31 (D) 4.00

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2 Ariella correctly solved the equation $\sqrt{2}\cos x - 1 = 0$ for the domain $0 \le x \le 2\pi$. Which of the following is her solution?

(A) $\frac{\pi}{4}$ or $\frac{5\pi}{4}$ (B) $\frac{\pi}{4}$ or $\frac{7\pi}{4}$ (C) $\frac{\pi}{3}$ or $\frac{2\pi}{3}$ (D) $\frac{\pi}{3}$ or $\frac{5\pi}{3}$

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3 Which of the following is a solution to the equation $e^{2x} - 4e^x - 5 = 0$?

(A) 4 (B) 5 (C) $\ln 4$ (D) $\ln 5$

Section II

30 marks
Attempt Questions 4 – 5
Allow about 45 minutes for this section

Answer each question on the writing paper provided. Start each question on a new page. Extra writing paper is available.

In Questions 4 – 5, your responses should include relevant mathematical reasoning and/or calculations.

Question 4 (15 marks) Start a new page

(a) A quadrant with radius r has an arc length of 10 cm. Find the exact value of the radius. 1

(b) Differentiate the following with respect to x.

(i) $\ln 3x$ 1

(ii) $3xe^x$ 2

(c) Find the following indefinite integrals.

(i) $\int \frac{2x}{x^2+3}\,dx$ 1

(ii) $\int e^x(e^x+1)\,dx$ 2

(d) Find the area swept out by the 6 cm minute hand of a clock as it moves between 10:50 am and 11:45 am. Give your answer in terms of π (exact form). 2

(e) If $\log_a P = x$, $\log_a Q = y$ and $\log_a R = z$, write an expression in terms of x, y and z for each of the following.

(i) $\log_a \frac{PR}{Q}$ 1

(ii) $\log_a \frac{P^2\sqrt{Q}}{R^3}$ 1

Question 4 (continued)

(f) Draw a large, neat sketch of the graph of $y = 2 - 4\ln x$. On your sketch, indicate the coordinates of the point(s) where the curve crosses the coordinate axes. 2

(g) If $\sin x = -\frac{\sqrt{3}}{2}$ and $\pi < x < \frac{3\pi}{2}$, find the exact value of:

(i) $\tan x$ 1

(ii) $\sec^2 x$ 1

Question 5 (15 marks) Start a new page

(a) Using Simpson's rule with 5 function values, find an approximation for $\int_0^2 e^{x^2}\,dx$. Give your answer correct to 2 decimal places. **3**

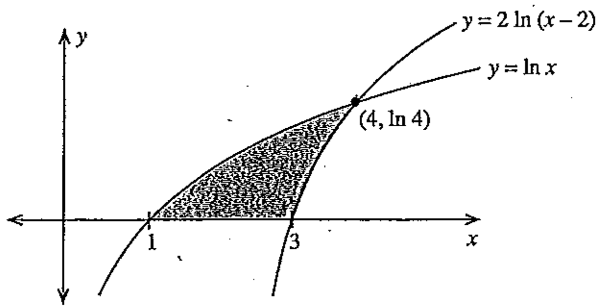
(b) (i) Show that $\frac{d}{dx}[(\ln x)^2] = \frac{2\ln x}{x}$. **1**

(ii) Hence, or otherwise, evaluate the definite integral $\int_e^{e^2} \frac{\ln x}{x}\,dx$. **3**

(c) The area under the curve $y = \sqrt{e^{2x}+1}$ between $x = 0$ and $x = 1$ is rotated around the x-axis to create a solid of revolution. Find the volume of this solid. Give your answer in exact form. **3**

(d) (i) If $y = 2\ln(x-2)$, show that $x = e^{\frac{y}{2}} + 2$. **2**

(ii)



The graphs of the curves $y = \ln x$ and $y = 2\ln(x-2)$ are shown in the diagram above. The curves intersect at $(4, \ln 4)$. The area bound by the two curves and the x-axis is shaded in the diagram. **3**

Calculate the exact area of the shaded region.

End of paper

Reference Sheet

Factorisation

$a^2 - b^2 = (a+b)(a-b)$

$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$

$a^3 - b^3 = (a+b)(a^3 - ab + b^3)$

Angle Sum of a Polygon

$S = (n-2) \times 180°$

Equation of a Circle

$(x-h)^2 + (y-k)^2 = r^2$

Equation of a Circle

$\sin\theta = \frac{\text{opposite side}}{\text{hypotenuse}}$ $\quad \operatorname{cosec}\theta = \frac{1}{\sin\theta}$

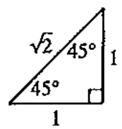
$\cos\theta = \frac{\text{adjacent side}}{\text{hypotenuse}}$ $\quad \sec\theta = \frac{1}{\cos\theta}$

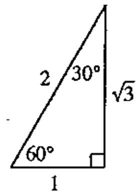
$\tan\theta = \frac{\text{opposite side}}{\text{adjacent side}}$ $\quad \tan\theta = \frac{\sin\theta}{\cos\theta}$

$\cot\theta = \frac{\cos\theta}{\sin\theta}$

$\sin^2\theta + \cos^2\theta = 1$

Exact Ratios





Sine Rule

$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

Cosine Rule

$c^2 = a^2 + b^2 - 2ab\cos C$

Area of a Triangle

Area $= \frac{1}{2}ab\sin C$

Distance Between Two Points

$d = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$

Perpendicular Distance of a Point from a Line

$d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2+b^2}}$

Slope (Gradient) of a Line

$m = \frac{y_2 - y_1}{x_2 - x_1}$

Point-Gradient Form of the Equation of a Line

$y - y_1 = m(x - x_1)$

nth term of an Arithmetic Series

$T_n = a + (n-1)d$

Sum to n terms of an Arithmetic Series

$S_n = \frac{n}{2}[2a + (n-1)d]$ or $S_n = \frac{n}{2}(a + l)$

nth term of an Geometric Series

$T_n = ar^{n-1}$

Sum to n terms of an Geometric Series

$S_n = \frac{a(r^n - 1)}{r-1}$ or $S_n = \frac{a(1-r^n)}{1-r}$

Limiting Sum of a Geometric Series

$S = \frac{a}{1-r}$

Compound Interest

$A_n = P\left(1 + \frac{r}{100}\right)^n$

Differentiation from First Principles

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Derivatives

If $y = x^n$, then $\frac{dy}{dx} = nx^{n-1}$

If $y = uv$, then $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

If $y = \frac{u}{v}$, then $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

If $y = F(u)$, then $\frac{dy}{dx} = F'(x)\frac{du}{dx}$

If $y = e^{f(x)}$, then $\frac{dy}{dx} = f'(x)e^{f(x)}$

If $y = \log_e f(x) = \ln f(x)$, then $\frac{dy}{dx} = \frac{f'(x)}{f(x)}$

If $y = \sin f(x)$, then $\frac{dy}{dx} = f'(x)\cos f(x)$

If $y = \cos f(x)$, then $\frac{dy}{dx} = -f'(x)\sin f(x)$

If $y = \tan f(x)$, then $\frac{dy}{dx} = f'(x)\sec^2 f(x)$

Solution of a Quadratic Equation

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Sum and Product of Roots of a Quadratic Equation

$\alpha + \beta = -\frac{b}{a}$ $\qquad$ $\alpha\beta = \frac{c}{a}$

Equation of a Parabola

$(x - h)^2 = \pm 4a(y - k)$

Integrals

$$\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C$$

$$\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + C$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$$

$$\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) + C$$

$$\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + C$$

$$\int \sec^2(ax+b)\,dx = \frac{1}{a}\tan(ax+b) + C$$

Trapezoidal Rule (one application)

$$\int_a^b f(x)\,dx \approx \frac{b-a}{2}[f(a) + f(b)]$$

Simpson's Rule (one application)

$$\int_a^b f(x)\,dx \approx \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

Logarithms – Change of Base

$$\log_a x = \frac{\log_b x}{\log_b x}$$

Angle Measure

$180° = \pi$ radians

Length of an Arc

$l = r\theta$

Area of a Sector

Area $= \frac{1}{2}r^2\theta$

Angle Sum Identities

$\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$

$\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$

$\tan(\theta + \phi) = \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi}$

t Formulae

If $t = \tan\frac{\theta}{2}$, then

$\sin\theta = \frac{2t}{1+t^2}$

$\cos\theta = \frac{1-t^2}{1+t^2}$

$\tan\theta = \frac{2t}{1-t^2}$

General Solution of Trigonometric Equations

$\sin\theta = a, \quad \theta = n\pi + (-1)^n \sin^{-1} a$

$\cos\theta = a, \quad \theta = 2n\pi \pm \cos^{-1} a$

$\tan\theta = a, \quad \theta = n\pi + \tan^{-1} a$

Division of an Interval in a Given Ratio

$$\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right)$$

Parametric Representation of a Parabola

For $x^2 = 4ay$,
$x = 2at \quad y = at^2$

At $(2at, at^2)$,
tangent: $y = tx - at^2$
normal: $x + ty = at^3 + 2at$

At (x_1, y_1),
tangent: $xx_1 = 2a(y + y_1)$
normal: $y - y_1 = -\frac{2a}{x_1}(x - x_1)$

Chord of Contact from (x_0, y_0):
$xx_0 = 2a(y + y_0)$

Acceleration

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left[\tfrac{1}{2}v^2\right]$$

Simple Harmonic Motion

$x = b + a\cos(nt + \alpha)$

$\ddot{x} = -n^2(x - b)$

Further Integrals

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a} + C$$

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

Sum and Product of Roots of a Cubic Equation

$\alpha + \beta + \gamma = -\frac{b}{a}$

$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$

$\alpha\beta\gamma = -\frac{d}{a}$

Estimation of Roots of a Polynomial Equation

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

Binomial Theorem

$$(a+b)^n = \sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k} = \sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k$$

2 UNIT MATHEMATICS
2016 ASSESSMENT TASK 3

SUGGESTED MARKING SCHEME

MC: PC
Q4: SU
Q5: KM

SECTION I

1 $\log_n\left(\frac{a}{b}\right) = \log_n a - \log_n b$
$= 0.77 - 3.08$
$= -2.31$

1 A

2
$\sqrt{2}\cos x - 1 = 0$
$\sqrt{2}\cos x = 1$
$\cos x = \frac{1}{\sqrt{2}}$
$x = 45°$ or $315°$
$= \frac{\pi}{4}$ or $\frac{7\pi}{4}$

2 B

3 $e^{2x} - 4e^x - 5 = 0$
$(e^x)2 - 4(e^x) - 5 = 0$
Let $t = e^x$
$t^2 - 4t - 5 = 0$
$(t-5)(t+1) = 0$
$t = 5$ or -1
When $t = 5$,
$e^x = 5$
$\ln e^x = \ln 5$
$x = \ln 5$
When $t = -1$,
$e^x = -1$, which is not possible.
$\therefore x = \ln 5$

3 D

SECTION II

QUESTION 4

(a)
$l = r\theta$
$10 = r \times \frac{\pi}{2}$
$20 = \pi r$
$r = \frac{20}{\pi}$ cm ① CORRECT ANSWER

(b) (i) $y = \ln 3x$
$\frac{dy}{dx} = \frac{3}{3x}$
$= \frac{1}{x}$ ① CORRECT ANSWER

(ii) $y = 3xe^x$
$\frac{dy}{dx} = 3x.e^x + e^x.3$ ① USE OF PRODUCT RULE
$= 3xe^x + 3e^x$
$= 3e^x(x+1)$ ① CORRECT ANSWER

(c) (i) $\int \frac{2x}{x^2+3}\,dx = \ln(x^2+3) + C$ ① CORRECT ANSWER ACCEPT IF +C IS MISSING.

(ii) $\int e^x(e^x+1)\,dx = \int (e^x)^2 + e^x\,dx$ ① EXPANSION
$= \int e^{2x} + e^x\,dx$
$= \frac{1}{2}e^{2x} + e^x + C$ ① CORRECT ANSWER ACCEPT IF +C IS MISSING.

(d) Angle swept out $= \frac{55}{60} \times 360$
$= 330°$
$= \frac{11\pi}{6}$ ① ANGLE
$\therefore$ Area $= \frac{1}{2}r^2\theta$
$= \frac{1}{2} \times 6^2 \times \frac{11\pi}{6}$
$= 33\pi\text{ cm}^2$ ① CORRECT ANSWER

(e) (i)
$\log_a \frac{PR}{Q} = \log_a PR - \log_a Q$
$= (\log_a P + \log_a R) - \log_a Q$
$= (x+z) - y$
$= x - y + z$ ① CORRECT ANSWER.

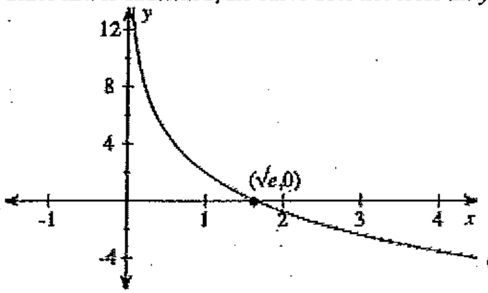
(ii)
$\log_a \frac{P^2\sqrt{Q}}{R^3} = \log_a(P^2\sqrt{Q}) - \log_a R^3$
$= \log_a(P^2Q^{\frac{1}{2}}) - \log_a R^3$
$= (\log_a P^2 + \log_a Q^{\frac{1}{2}}) - \log_a R^3$
$= (2\log_a P + \frac{1}{2}\log_a Q) - 3\log_a R$
$= (2x + \frac{1}{2}y) - 3z$
$= 2x + \frac{1}{2}y - 3z$ ① CORRECT ANSWER.

(f) When $y = 0$,

$$0 = 2 - 4\ln x$$
$$4\ln x = 2$$
$$\ln x = \tfrac{1}{2}$$
$$x = e^{\frac{1}{2}}$$
$$= \sqrt{e}$$

$\therefore$ The curve crosses the x axis at $(\sqrt{e}, 0)$. ① x INTERCEPT

Since $\ln x$ is undefined, the curve does not cross the y axis.



① CORRECT GRAPH SHAPE, DOESN'T CROSS Y-AXIS ETC.

(g) $\sin x = \dfrac{\sqrt{3}}{2}$

$\therefore$ The related acute angle is 60°.

(i) Since x is in the third quadrant,

$$\tan x = \tan 60°$$
$$= \sqrt{3}$$ ① CORRECT ANSWER

(ii) Since x is in the third quadrant,

$$\sec^2 x = (\sec 60°)^2$$
$$= \frac{1}{(\cos 60°)^2}$$
$$= \frac{1}{(\frac{1}{2})^2}$$
$$= \frac{1}{\frac{1}{4}}$$
$$= 4$$ ① CORRECT ANSWER.

QUESTION 5

(a) Using Simpson's rule: ① FUNCTION VALUE

x	$f(x)$	W	P
0.0	1.0000	1	1.0000
0.5	1.2840	4	5.1361
1.0	2.7183	2	5.4366
1.5	9.4877	4	37.9509
2.0	54.5982	1	54.5982
			104.1218

① SUM – CFPA

$$\therefore \int_0^2 e^{x^2}\,dx = \tfrac{1}{3} \times h \times \text{Sum}$$
$$= \tfrac{1}{3} \times 0.5 \times 104.1218$$
$$= 17.35$$ ① CORRECT VALUE CFPA

(b) (i) $\dfrac{d}{dx}[(\ln x)^2] = 2 \cdot \dfrac{1}{x} \cdot \ln x$ ① CORRECT PROCESS MUST BE SHOWN

$$= \frac{2\ln x}{x}$$

(ii) $\displaystyle\int_e^{e^2} \frac{\ln x}{x}\,dx = \frac{1}{2}\int_e^{e^2} \frac{2\ln x}{x}\,dx$ ① CORRECT USE OF RESULT FROM (i).

$$= \tfrac{1}{2}\left[(\ln x)^2\right]_e^{e^2}$$
$$= \tfrac{1}{2}[(\ln e^2)^2] - \tfrac{1}{2}[(\ln e)^2]$$
$$= \tfrac{1}{2}[(2\ln e)^2] - \tfrac{1}{2}[(\ln e)^2]$$ ① SUBSTITUTION AND PROCESS
$$= \tfrac{1}{2}[2^2] - \tfrac{1}{2}[1^2]$$
$$= \tfrac{1}{2}[4] - \tfrac{1}{2}[1]$$
$$= 2 - \tfrac{1}{2}$$
$$= 1\tfrac{1}{2}$$ ① CORRECT ANSWER OR CNE

(c) $y = \sqrt{e^{2x} + 1}$

$y^2 = e^{2x} + 1$ ①

$$\therefore \text{Volume} = \pi\int_0^1 y^2\,dx$$
$$= \pi\int_0^1 e^{2x} + 1\,dx$$
$$= \pi\left[\tfrac{1}{2}e^{2x} + x\right]_0^1$$ ① INTEGRATION.
$$= \pi[\tfrac{1}{2}e^2 + 1] - \pi[\tfrac{1}{2}e^0 + 0]$$
$$= \pi[\tfrac{1}{2}e^2 + 1] - \pi[\tfrac{1}{2}]$$
$$= \pi[\tfrac{1}{2}e^2 + \tfrac{1}{2}]$$
$$= \tfrac{\pi}{2}(e^2 + 1) \text{ cubic units}$$ ① CORRECT ANSWER.

(d) (i)

$$y = 2\ln(x-2)$$
$$\ln(x-2) = \frac{y}{2}$$ ①
$$x - 2 = e^{\frac{y}{2}}$$ ①
$$x = e^{\frac{y}{2}} + 2$$

(ii) From (i), when $y = 2\ln(x-2)$, we have $x = e^{\frac{y}{2}} + 2$.
Also, when $y = \ln x$, we have $x = e^{y}$.

$$
\begin{aligned}
\therefore \text{Area} &= \int_0^{\ln 4} x\,dy - \int_0^{\ln 4} x\,dy \\
&= \int_0^{\ln 4} e^{\frac{y}{2}} + 2\,dy - \int_0^{\ln 4} e^{y}\,dy && \text{(1) CORRECT EXPRESSION FOR AREA.} \\
&= \left[2e^{\frac{y}{2}} + 2y\right]_0^{\ln 4} - \left[e^{y}\right]_0^{\ln 4} && \text{(1) INTEGRATION. (CFPA)} \\
&= \left[[2e^{\frac{\ln 4}{2}} + 2\ln 4] - [2e^{\frac{0}{2}} + 2(0)]\right] - \left[[e^{\ln 4}] - [e^{0}]\right] \\
&= \left[[2e^{\frac{1}{2}\ln 4} + 2\ln 4] - [2e^{0} + 0]\right] - [[4] - [1]] \\
&= \left[[2e^{\ln 2} + 2\ln 4] - [2(1)]\right] - [3] \\
&= \left[[2(2) + 2\ln 4] - [2]\right] - 3 \\
&= 4 + 2\ln 4 - 2 - 3 \\
&= 2\ln 4 - 1 \text{ square units} && \text{(1) CORRECT ANSWER (CFPA)}
\end{aligned}
$$