BOS Number: ____________

Student Name: ____________

Teacher: ____________

Strathfield Girls High School

Mid-Year Examination 2016

Mathematics Extension 1

General Instructions

- Write using black or blue pen
  Black pen is preferred
- Attempt all questions:
  - Attempt Questions 1-7 on the Multiple Choice answer sheet attached at the back of this exam.
  - Attempt Questions 8-10 in separate booklets.
- Board- approved calculators may be used
- A formula sheet is provided at the back of this paper.
- Show all necessary working in Questions 8-10
- Full marks may not be awarded for careless or badly arranged work.

READING TIME - 5 minutes
TIME ALLOWED: 90 minutes

SECTION I

- 7 Marks
- Allow about 10 minutes for this section.

SECTION II

- 43 Marks
- Allow about 1 hour 20 minutes for this section.

TOTAL MARKS – 50

Exam Requirements

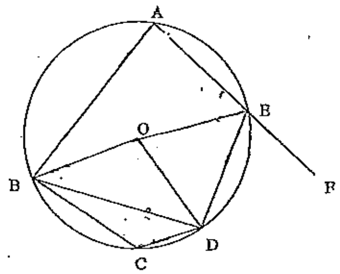
- Examination paper
- 3 answer booklets

This paper is not to be removed from the examination hall

Section 1 Multiple Choice

Answer questions 1 - 7 on the multiple choice answer sheet provided.

Question 1



Given: O is the centre of the circle
$\angle BCD = 130°$
$\angle ABO = 30°$

What is the size of $\angle DEF$

(A) 70°

(B) 110°

(C) 97.5°

(D) 82.5°

Question 2

Four different Science books and six different History books are to be place in a row on a bookshelf. How many different arrangements are possible if the subjects are kept together.

(A) ${}^{4}C_4 \times {}^{6}C_6 \times 2$

(B) ${}^{10}C_4 \times {}^{10}C_6 \times 2$

(C) ${}^{4}P_4 \times {}^{6}P_6 \times 2$

(D) ${}^{10}P_4 \times {}^{10}P_6 \times 2$

Question 3

$$\int 3^{2x} + e^2 + \frac{2}{x}\, dx =$$

(A) $3^{2x} + e^2 + ln2x + C$
(B) $\frac{3^{2x}}{2ln3} + e^2 + 2lnx + C$
(C) $\frac{3^{2x}}{3ln2} + e^2x + 2lnx + C$
(D) $\frac{3^{2x}}{2ln3} + e^2x + 2lnx + C$

Question 4

$$\int \frac{2}{x\ln x}\, dx =$$

(A) $2\ln x + C$
(B) $2\ln(\ln x) + C$
(C) $2ln\frac{1}{x} + C$
(D) $\frac{2}{x} + C$

Question 5

What is the natural domain for the graph $y = \ln\left(\frac{x}{x-1}\right)$?

(A) $0 \leq x < 1$
(B) $0 < x < 1$
(C) $x \leq 0 \; and \; x > 1$
(D) $x < 0 \; and \; x > 1$

Question 6

A curve has parametric equations $x = 2t - 1$ and $y = t^2$.
What is Cartesian equation of this curve?

(A) $y = {}^1/_4\sqrt{(x+1)}$
(B) $y = 4\sqrt{(x-1)}$
(C) $y = {}^1/_4(x+1)^2$
(D) $x^2 = 4y$

Question 7

The polynomial equation $f(x) = 2x^3 + \frac{x^2}{2} - 6x - 5$ has roots α, β, γ.
Find the exact value of $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$

(A) $-\frac{6}{5}$
(B) $\frac{1}{12}$
(C) $-\frac{1}{10}$
(D) $-\frac{5}{6}$

End of Section 1

Section II - Free Response

Answer questions 8 - 10 in the booklets provided

Question 8 (Start a new booklet) (14 marks)

(a) Ten keys are to be placed on a key ring. What is the number of ways of arranging the keys if

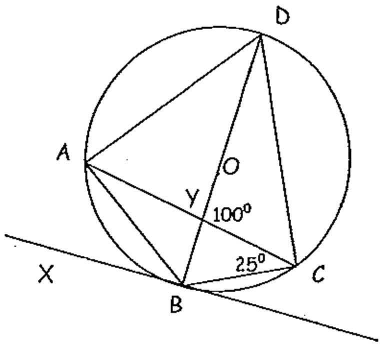
(i) the keys are different colours? 1

(ii) there are three green, two gold, 4 blue and 1 red key? 1

(b) Find $\int x\sqrt{5x^2-4}\ \ dx$ 2

(c) In the diagram, the points A, B, C and D are on the circumference of a circle, whose centre O lies on BD. The chord AC intersects the diameter BD at Y. The tangent at B passes through the point X.

It is given that $\angle CYB = 100°$ and $\angle BCY = 25°$.



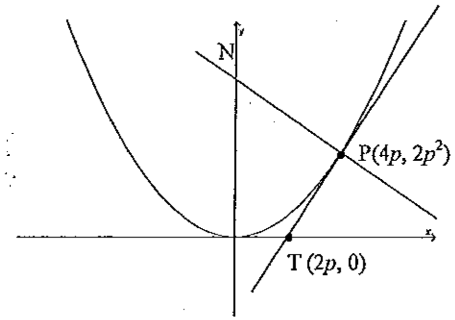
Copy or trace the diagram into your writing booklet.

Answer the following questions giving reasons

(i) What is the size of $\angle ACD$ and $\angle ABX$? 2

(ii) Find the size of $\angle CAD$. 3

(d) The diagram shows the graph of the parabola $x^2 = 8y$. The tangent to the parabola at $P(4p, 2p^2)$ meets the x axis at $T(2p, 0)$. The normal to the tangent at P meets the y-axis at N.



(i) Show that the equation of the normal at P is $x + py = 4p + 2p^3$ 2

(ii) Find the coordinates of N. 1

(iii) The point K divides NT externally in the ratio 2:1

Find the coordinates of K. 2

Question 9 (Start a new booklet) (14 marks)

(a) Use the substitution $u = 3x + 1$ to evaluate, 3

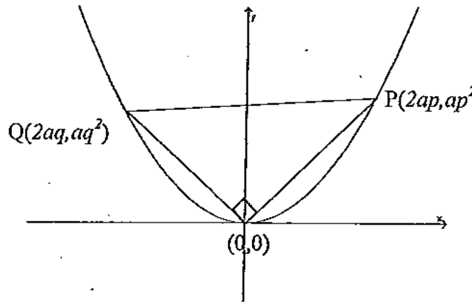
$$\int_0^1 \frac{3x-2}{\sqrt{3x+1}}dx$$

(b) Use mathematical induction to prove that, for all positive integers n, 4

$$\frac{4}{1\times2\times3}+\frac{4}{2\times3\times4}+\frac{4}{3\times4\times5}+\cdots\ldots+\frac{4}{n(n+1)(n+2)}=1-\frac{2}{(n+1)(n+2)}$$

(c) The points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the parabola $x^2 = 4ay$.

The chord PQ subtends a right angle at the vertex



Copy the diagram into your answer booklet

(i) Show that $pq = -4$ 2

(ii) You may assume these equations. Do not find them. 2

Normal at P: $x + py = 2ap + ap^3$

Normal at Q: $x + qy = 2aq + aq^3$

Given that the normal at P and Q meet at N,

Show that N has the coordinates:

$$\left[-apq(p+q),\ a(p^2+pq+q^2+2)\right].$$

(iii) Hence, show that the locus of N is a parabola.

Find its vertex and focal length. 3

Question 10 (Start a new booklet) (15 marks)

(a) The graphs of $x^2 = 4y$ and $y = \frac{1}{8}x^3$ intersect at $x = 2$. Find the acute angle between the two curves at $x = 2$. Answer to the nearest minute. 2

(b) A company wants to employ four apprentices consisting of 2 females and 2 males. 4 females (including Mary) and 5 males (including Jack) apply.

In how many ways can the company select these four people if jack or Mary are selected as apprentices. Leave your answer as an expression. 2

(c) Using the principles of mathematical induction, prove that 3

$$4\ln[(n+2)!] > n+2, \quad \text{for} \quad n \geq 4$$

(d) Consider the function $h(x) = \dfrac{e^x}{4+e^x}$

(i) Show that $h(x)$ is increasing for all values of x 2

(ii) Find any point/s of inflexion, given 2

$$h''(x) = \frac{4e^x(4-e^x)}{(4+e^x)^3}$$

(iii) Describe the behaviour of $h(x)$ for very large positive and negative values of x, i.e. as $x \to \pm\infty$ 2

(iv) Sketch the curve $y = h(x)$, showing important features 2

STRATHFIELD GIRLS

2016 EXT1 MID YEAR

SAMPLE SOLUTIONS

SECTION 1 - MULTIPLE CHOICE

1. A
2. C
3. D
4. B
5. D
6. C
7. A

8. a) Key ring is circular

So i) $(10-1)! = 9!$

ii) $\dfrac{9!}{3! \times 2! \times 4!}$

b) $\int x\sqrt{5x^2-4}\, dx$

let $u = 5x^2$

$\dfrac{du}{dx} = 10x$

$\dfrac{du}{10x} = dx$

$\int x\sqrt{u-4}\left(\dfrac{du}{10x}\right)$

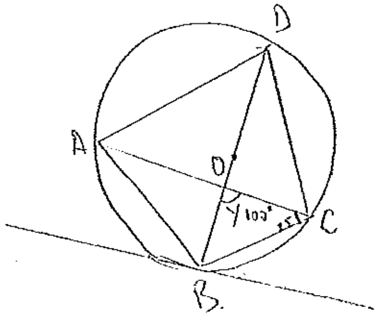
$= \dfrac{1}{10}\int \sqrt{u-4}\, du.$

$= \dfrac{1}{10}\int (u-4)^{\frac{1}{2}}\, du.$

$= \dfrac{1}{10}\left[(u-4)^{\frac{3}{2}} \times \dfrac{2}{3} \right]$

$= \dfrac{2(5x^2-4)^{\frac{3}{2}}}{30} + C$

$= \dfrac{(5x^2-4)^{\frac{3}{2}}}{15} + C.$



$\angle BCD = 90°$ ($\angle$ in semicircle is a right angle)

SINCE BD IS DIAMETER.

So $\angle ACD = 90 - 25 = 65°$ #

$\angle BCY = 25°$ (given)

$\therefore \angle ADB = 25°$ (Same Segment theorem)

$\therefore \angle ABX = 25°$ (Alternate segment theorem) #

ii) $\angle AYD = 100°$ (Vertically opp.)

we know $\angle ADB = 25°$ (see i)

So $\angle DAC = \angle CAD = 180 - 25 - 100$

$= 55°$ #

ii) Gradient of tangent

$$= \frac{2p^2-0}{4p-2p} = \frac{2p^2}{2p} = p.$$

So Gradient of Normal $= -\frac{1}{p}$ [negative Reciprocal]

Point - Gradient Formula.

$$y - 2p^2 = -\frac{1}{p}(x-4p).$$

$$yp - 2p^3 = -x + 4p$$

$$x + py = 4p + 2p^3.$$ // As required.

iii) Intercept of normal on Y axis

ie when $x=0$.

ie. $py = 4p+2p^3$

$$y = \frac{4p+2p^3}{p} = 4+2p^2 = y$$

ie $= (0, 4+2p^2)$ //

iii) ie the ~~ext~~ internal line division Ratio $-2:1$

$x_1 = 0$
$y_1 = 4+2p^2.$
$x_2 = 2p$
$y_2 = 0$

$$= \frac{0-2(2p)}{-1}, \frac{4+2p^2}{-1}$$

$$= \left(4p, -(4+2p^2)\right) = K.$$

9 a)

$$\int_0^1 \frac{3x-2}{\sqrt{3x+1}}\,dx$$

let $u = 3x+1$ $\quad \frac{du}{dx} = 3$ $\quad \frac{du}{3} = dx$

$u - 1 = 3x$

$$\int_1^4 \frac{3x-2}{\sqrt{u}} \left[\frac{du}{3}\right]$$

$$= \int_1^4 \frac{(u-1)-2}{\sqrt{u}}\left(\frac{du}{3}\right)$$

$$= \frac{1}{3}\int_1^4 \frac{u-3}{\sqrt{u}}\,du$$

$$= \frac{1}{3}\int_1^4 u^{-\frac{1}{2}}(u-3)\,du$$

$$= \frac{1}{3}\int_1^4 u^{\frac{1}{2}} - 3u^{-\frac{1}{2}}.$$

$$= \frac{1}{3}\left[\frac{2u^{\frac{3}{2}}}{3} - 6u^{\frac{1}{2}}\right]_1^4$$

$$= \frac{1}{3}\left[\left(\frac{16}{3}-12\right) - \left(\frac{2}{3}-6\right)\right]$$

$+\frac{16}{3}$

$$= \frac{1}{3}\left[-\frac{4}{3}\right] = -\frac{4}{9}$$ //

5). Mathematical Induction

$$\frac{4}{1\times2\times3}+\frac{4}{2\times3\times4}+\frac{4}{3\times4\times5}+\dots+\frac{4}{n(n+1)(n+2)}=1-\frac{2}{(n+1)(n+2)}$$

CASE I. Prove for $n=1$

$$\frac{4}{1(2)(3)}=1-\frac{2}{(2)(3)}$$

$$\frac{4}{6}=1-\frac{2}{6}=\frac{4}{6} \implies \therefore \text{True for } n=1$$

Assume $n=k$. where $k\in\mathbb{Z}, k>0$

$$\left.\frac{4}{k(k+1)(k+2)}=1-\frac{2}{(k+1)(k+2)}\right]$$ Assumed Statement to be true.

Prove True for $n=k+1$.

$$\frac{4}{(k+1)(k+2)(k+3)}=1-\frac{2}{(k+2)(k+3)}$$

RTP LHS $=1-\frac{2}{(k+1)(k+2)}+\frac{4}{(k+1)(k+2)(k+3)}$

$$1+\frac{4-2(k+3)}{(k+1)(k+2)(k+3)}$$

$$=1+\frac{4-2k-6}{(k+1)(k+2)(k+3)}$$

$$=1+\frac{-2k-2}{(k+1)(k+2)(k+3)}$$

$$=1+\left(\frac{-2}{(k+2)(k+3)}\right)$$

$$=1-\frac{2}{(k+2)(k+3)}$$

$=$ RHS

$\therefore$ True for $n=k+1$

$\therefore$ By the principle of mathematical Induction True for $n=k$ and $n=k+1$.

c) let O denote the point $(0,0)$

$$M_{PO}=\frac{ap^2}{2ap}=\frac{p}{2}$$

$$M_{QO}=\frac{aq^2}{2aq}=\frac{q}{2}$$

but $PO\perp QO$ ie

$$\frac{p}{2}\times\frac{q}{2}=-1$$

$$\frac{pq}{4}=-1$$

$$pq=-4 \#$$

ii) Normal at P: $x+py=2ap+ap^3$ ①

at Q: $x+qy=2aq+aq^3$ ②

Solve for y.

$$2ap+ap^3-py=2aq+aq^3-qy.$$

$$y(q-p)=2a(q-p)+\underbrace{a(q^3-p^3)}_{=a(q-p)(q^2+pq+p^2)}$$

$$y=2a+a(q^2+pq+p^2)$$

$$=a(p^2+pq+q^2+2)$$

Sub into either equation.

①

$$x=2ap+ap^3-p\left(a(p^2+pq+q^2+2)\right)$$

$$=-apq(p+q)$$

ie N has co-ordinates

$$\left(-apq(p+q),\ a(p^2+pq+q^2+2)\right)$$

iii) $x=-apq(p+q)$

$$y=a(p^2+pq+q^2+2)$$

we want to eliminate the parameters, ie.

$$y=a\left((p+q)^2-pq+2\right)$$

but $pq=-4$

ie $y=a(p+q)^2+6$

$$x=4a(p+q)$$

$$p+q=\frac{x}{4a}$$

ie $y=a\left[\left(\frac{x}{4a}\right)^2+6\right]$

$$y=\frac{ax^2}{16a^2}+6a$$

$$=\frac{x^2}{16a}+6a \#$$

which is a quadratic form.

(Vertex & focal length - Follow Formulas.

10

$x^2 = 4y \quad y = \frac{1}{8}x^3$

① ②

Intersect at $x = 2$.

at $x = 2$.

Gradients of each curve.

① $y = \frac{x^2}{4} \quad y' = \frac{2x}{4} = \frac{x}{2}$.

at $x=2, \quad m_1 = 1$

② $y = \frac{1}{8}x^3 \quad y' = \frac{3}{8}x^2$

at $x = 2, \; y' = \frac{3}{8}(4) = \frac{12}{8} = \frac{3}{2} = m_2$.

$\tan\theta = \left| \frac{1 - \frac{3}{2}}{1 + (\frac{3}{2})} \right|$

$\theta = \tan^{-1}\left(\frac{1}{5}\right) \approx 11.31^\circ$ OR $11^\circ 19'$

b). ${}^3C_1 \times {}^4C_1$

c) $4\ln[(n+2)!] > n+2$

For $n \geqslant 4$

Prove true for lowest integer $n = 4$

$4\ln[(6)!] > 4+2$.

$4\ln(720) > 6$

True for $n = 4$

Assume $n = k$, where $k \geqslant 4 \quad k \in \mathbb{R}$.

~~$4\ln[(k+2)!] > k+2$~~

Prove true for $n = k+1$

$4\ln[(k+3)!] > (k+1)+2$.

$4\ln[(k+3)(k+2)!] > k+3$.

$= 4\left[\ln(k+3) + \ln[(k+2)!]\right] > k+3$

$\underbrace{4\ln(k+3) + \underbrace{4\ln[(k+2)!]}}_{①} > \underbrace{k+3}_{2}$

If we can prove that ① is bigger than something that is greater than $k+3$, then we know for sure it is 100% greater than $k+3$.

Sub into for $k+3$.

$4\ln(k+3) + 4\ln[(k+2)!] >$

$4\ln[(k+2)!] + 1$.

$4\ln(k+3) > 1$.

Since $\ln(k+3) > 1$

For $k \geqslant 4$

then this expression must hold true.

$\therefore$ by the principles of mathematical induction, this statement holds true for $n = k$ and $n = k+1$ and this is a true statement.

d) $h(x) = \dfrac{e^x}{4+e^x}$

$h'(x) \Rightarrow$ use quotient rule.

$u = e^x \quad u' = e^x$

$v = 4+e^x \quad v' = e^x$

$$\frac{e^x(4+e^x) - (e^{2x})}{(4+e^x)^2}$$

$h'(x) = \dfrac{4e^x}{(4+e^x)^2} \Rightarrow$ which is always positive for all x

domain of $h'(x) \Rightarrow x \in \mathbb{R}$

Range of $h'(x) \Rightarrow > 0$.

ie increasing for all x.

ii) $h''(x) = \dfrac{4e^x(4-e^x)}{(4+e^x)^3} = 0$

$4e^x(4-e^x) = 0$

$16e^x - 4e^{2x} = 0$

Quadratic in e^x

let $u = e^x$

$4u^2 - 16u = 0$

$u^2 - 4u = 0$

$u(u-4) = 0$

$u = 0$ OR $u = 4$.

ie $\underbrace{e^x = 0}_{\text{Impossible.}}$ or $e^x = 4$.

$e^x = 4 \Rightarrow$ log both sides

$x = \ln 4, \; h(x) = \dfrac{e^{\ln 4}}{4+e^{\ln 4}}$

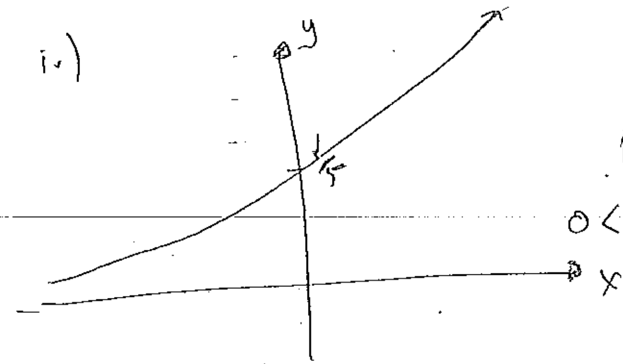
$= \dfrac{4}{4+4} = \dfrac{1}{2}$

iii) as $x \to \infty$

ie $\lim\limits_{x\to\infty} \dfrac{e^x}{4+e^x} = \dfrac{1}{\frac{4}{e^x}+1} \Rightarrow$ Approaches 1

$\lim\limits_{x\to-\infty} \dfrac{e^x}{4+e^x} = \dfrac{1}{\infty+1} \Rightarrow$ Approaches 0.

iv)



Bounded by the interval

$0 < y < 1$