Student Number: ____________
~~Student Name~~: ____________
Teacher Name: ____________

Strathfield Girls High School

2016

Year 12 Mid-Year Examination

Mathematics

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
  Black pen is preferred
- Board- approved calculators may be used
- A reference section is provided
- Show all necessary working in Questions 7-11

Exam Requirements

- Examination paper
- 5 answer booklets
- Multiple Choice answer sheet (attached)

Section I

- 6 Marks
- Attempt Questions 1-6 on the Multiple Choice answer sheet attached at the back of this examination
- Allow about 10 minutes for this section

Section II

- 60 Marks
- Attempt Questions 7-11 in separate booklets
- Allow about 1 hours 50 minutes for this section

Total marks – 66

SECTION I MULTIPLE CHOICE

Indicate your answers for this section on the Multiple Choice answer sheet attached at the end of the examination paper. You must detach the answer sheet and place it with your solutions.

QUESTION 1

The domain of the function $f(x)=\sqrt{x^2+5x-6}$ is: for real x

A $-6\le x\le 1$ B $x\le -6\ ,\ x\ge 1$

C $-1\le x\le 6$ D $x\le -1\ ,\ x\ge 6$

QUESTION 2

Two dice with 8 faces each, numbered 1 to 8 are rolled together.
What is the probability that the sum of the two faces is 10 ?

A $\frac{7}{36}$ B $\frac{1}{8}$

C $\frac{7}{64}$ D $\frac{3}{36}$

QUESTION 3

For the sequence 7 , 18 , 29 , 40, . . . , which is the first term to exceed 1000 ?

A 1006 B 1007

C 1008 D 1009

QUESTION 4

Given $f(x)=2^x$, the derivative of the function is

A $f'(x)=2^x$ B $f'(x)=(\ln 2)e^x$

C $f'(x)=\frac{2^x}{\ln 2}$ D $f'(x)=(\ln 2)2^x$

QUESTION 5

The exact value of $\log_3\frac{1}{9}$ is

A -2 B 2 C -3 D 3

QUESTION 6

$$\int_1^{16}\frac{dx}{\sqrt{x}}=$$

A $\frac{1}{6}$ B 6 C $\frac{2}{3}$ D $\frac{3}{2}$

QUESTION 7 *(Start a new booklet)* [Marks]

(a) Evaluate $\int_{-3}^{2}(x^2-2x)dx$ [2]

(b) Differentiate $y = x\ln x - x$ hence evaluate $\int_{1}^{5}\ln x dx$ giving the answer in exact form [2]

(c) Use Simpson's rule with 5 function values to find an approximation for $\int_{1}^{5}\ln x\, dx$ correct to 2 decimal places. [2]

(d) The digits 5, 6, 7, 8 and 9 are written on 5 cards, which are then placed face-down on a desk. Two cards are then selected to create a two-digit number [3]

(i) list all the possible outcomes

(ii) find the probability that the number is even

(iii) find the probability that the number is greater than 70

(e) (i) Find the limiting sum of the series $24 + 8 + \frac{8}{3} + \ldots$ [1]

(ii) Find the first term of an infinite Geometric Series with $r = -\frac{2}{3}$ and a limiting sum of 15. [2]

QUESTION 8 *(Start a new booklet)* [Marks]

(a) Differentiate $y = \ln\frac{x^2-7}{2x+1}$ and write the result with a single denominator. [2]

(b) A function is defined as $f(x) = \frac{\ln x}{x}$

(i) state its domain [1]

(ii) find its derivative [1]

(iii) find any stationary points and determine their nature [2]

(iv) evaluate $f(10)$ hence sketch $y = f(x)$ on a number plane, showing all asymptotes and intercepts with the axes. Make your graph at least 1/3 page [2]

(c) Sketch the function $y = x^2 - 3x - 4$ showing any intercepts with the x-axis, hence find the area enclosed by the curve, the x-axis and the lines $x = 1$ and $x = 6$ [4]

QUESTION 9 *(Start a new booklet)*

(a) At what point/s on the curve $y = \sqrt{9-x^2}$ is/are the tangent/s parallel to the line $y = x + 5$ [4]

(b) Given that the probability of event 'A' occurring is $\frac{1}{3}$, the probability of event 'B' occurring is $\frac{1}{4}$ and the probability of 'A' and 'B' occurring is $\frac{1}{6}$, find the probability of [2]

(i) 'A' or 'B' occurring

(ii) neither 'A' nor 'B' occurring

(c) Evaluate the following sums

(i) $\sum_{n=10}^{30}(2n-4)$ [2]

(ii) $\sum_{n=1}^{8}3\times 2^{n-1}$ [2]

(iii) $27 - 18 + 12 - 8 + \ldots$ [2]

QUESTION 10 *(Start a new booklet)* [Marks]

(a) A rectangular prism is designed so that:

* it has a square cross-section of x cm by x cm
* it has a height of h cm
* it has a volume of $8\,000$ cm^3
* the sum of its length, breadth and height is a minimum

(i) Show that the sum of its three dimensions is $S = 2x + \frac{8000}{x^2}$ [1]

(ii) Find the values of x and h so that S is a minimum [4]

(b) There are 4 red and 5 black playing cards. If two are chosen at random (without replacement) find the probability that at least one of them is red [3]

(c) Find the exact volume of the solid generated when the region enclosed by the curve $y = x^2 + 1$, the x-axis, the y-axis and the line $x = 3$ is rotated about the x-axis. [4]

QUESTION 11 *(Start a new booklet)* [Marks]

(a) Evaluate $\int_2^3 \frac{x^2}{x^3+1}dx$ leaving the answer in exact form [2]

(b) The derivative of a function is $f'(x) = 3x^2 - 2x + 1$ [2]
Find the equation of the function given that it passes through (1 , - 5)

(c) Find the exact area enclosed by the hyperbola $xy = 4$, the y-axis and the horizontal lines $y = 1$ and $y = 3$ [3]

(d) For a given function $y = f(x)$, $f''(x) = 6$

(ii) show that the function is a parabola and that it is concave up [2]

(ii) given that its vertex is at (2 , 7) find the equation of the function [3]

. . . o o o O o o o . . .

Multiple Choice: 1B , 2C , 3C , 4D , 5A , 6B Question 7 Solution	Marks and Feedback
a) evaluate $\int_{-3}^{2}(x^2-2x)dx$ $=\left[\frac{x^3}{3}-\frac{2x^2}{2}\right]_{-3}^{2}$ ① $=\left(\frac{2^3}{3}-2^2\right)-\left(\frac{(-3)^3}{3}-(-3)^2\right)$ $=\left(\frac{8}{3}-4\right)-\left(\frac{-27}{3}-9\right)$ $=\left(2\frac{2}{3}-4\right)+9+9$ $=16\frac{2}{3}$ units ① or 50/3	Generally very well done Careless arithmetic errors occurred when substituting the limits.
b) Differentiate $y=x\ln x-x$ hence evaluate $\int_{1}^{5}\ln x dx$ giving the answer in exact form differentiate $y=x\ln x-x$ $\frac{dy}{dx}=x\times\frac{1}{x}+1\times\ln x-1$ $=1+\ln x-1$ $=\ln x$ ①	Typically the differentiation resulted in an incorrect derivative of 1/x

Hence integrate

$\int_1^5 \ln x dx$ as $f(x) = x\ln x - x$ and $f'(x) = \ln x$

$= x\ln x - x\Big|_1^5$

$= (5\ln 5 - 5) - (1\ln 1 - 1)$

$= 5\ln 5 - 4$ ① in an exact form only

$= 4.04718...$

$= 4.05(2dp)$

Hence integrate ....This implies that you are using what you just differentiated and you have been given the answer virtually so in this case the integral is as shown . If not you could be awarded marks for for integrating non trivial derivatives.

c) Use Simpson's rule with 5 function values to find an approximation for $\int_1^5 \ln x\, dx$ correct to 2 decimal places.

x	1	2	3	4	5
fx)	ln1	ln2	ln3	ln4	ln5
wt	1	4	2	4	1

$\approx \frac{1}{3}\{ 1\ln 1 + 4\ln 2 + 2\ln 3 + 4\ln 4 + 1\ln 5 \}$

$\approx \frac{1}{3}\, 12.1244...$

≈ 4.04 (2dp)

OR

$\frac{b-a}{6}\{f(first) + 4f(even) + 2f(odd) + f(last)\}$

$\frac{1}{3}\{ 1\ln 1 + 4\ln 2 + 2\ln 3 + 4\ln 4 + 1\ln 5 \}$

But note b-a is the first and last values of one application eg 3-1

① to correctly use 2 applications of simpsons rule

① evaluate

This was well done in extension, however 2U were blindly following the formula on the sheet and not recognising that 2 applications are required.

Think 3 Function values 1 application, 5 function values 2 applications

Or 2 strips 1 application

4 strips 2 applications and remember (b-a)/6 is the width of the strip not the width of the two applications.!

3

d) I

	5	6	7	8	9
5		56	57	58	59
6	65		67	68	69
7	75	76		78	79
8	85	86	87		89
9	95	96	97	98	

(1) correct table

II
P(even) = 8/20 = 2/5 (1) cfpa from table if used duplicates

III
P(n>70) = 12/20 =3/5 (1) cfpa from table if used duplicates

Generally well done

Although quite a few did not read carefully and created a table with duplicates I have very generously given CFPA.
Read the question carefully.

e) i) Find the limiting sum of the series $24 + 8 + \frac{8}{3} + \ldots$ a = 24 , r= 8/24 =1/3 $S_\infty = \frac{a}{1-r}$ $= \frac{24}{1-\frac{1}{3}}$ ① correct answer $= \frac{24}{\frac{2}{3}}$ $= 36$	Excellent. well done all!
(ii) Find the first term of an infinite Geometric Series with $r = -\frac{2}{3}$ and a limiting sum of 15. $S_\infty = 15 = \frac{a}{1-r}, r = -\frac{2}{3}$ $15 = \frac{a}{1--\frac{2}{3}}$ ① set up correct equation $= \frac{a}{\frac{5}{3}}$ $\frac{3a}{5} = 15$ ① solve answer $a = 25$	Generally well done A few careless errors with 1 - -2/3 Or failure to recognise that S_∞ = 15

5.

Question 8

Solution	Marks and Feedback
(a) $y = \ln\left(\frac{x^2-7}{2x+1}\right)$ $y = \ln(x^2-7) - \ln(2x+1)$ $y' = \dfrac{2x}{x^2-7} - \dfrac{2}{2x+1}$ $= \dfrac{2x}{x^2-7} - \dfrac{2}{2x+1}$ $= \dfrac{2x(2x+1) - 2(x^2-7)}{(x^2-7)(2x+1)}$ $= \dfrac{2x^2+2x+14}{(x^2-7)(2x+1)}$ or $\dfrac{2x^2+2x+14}{2x^3+x^3-14x-7}$	Total - 2 marks 1 mark - for differentiating. Students who got the question incorrect often used a mixture of differentiating $\ln f(x) = \frac{f'(x)}{f(x)}$ where $f'(x)$ would be found through quotient rule. Although this is a correct method of differentiating, it is not recommended and most students who pursued the method found $f'(x)$ but did not divide by $f(x)$. 1 mark - for simplifying correctly. Most students got this part correct after receiving the previous mark.
(b) i) Domain: $x > 0$ $\{x \text{ is real}, x > 0\}$ ii) $f'(x) = \frac{(x)\left(\frac{1}{x}\right) - (\ln x)(1)}{x^2}$ $= \frac{1-\ln x}{x^2}$	Total marks - 6 marks 1 mark - most students identified that $x > 0$ must be the domain since $\ln x$ is the numerator. 1 mark - for differentiating using quotient rule. Most students obtained the answer correctly.

iii) $f'(x) = 0$ at stationary points

$\therefore 1 - \ln x = 0$

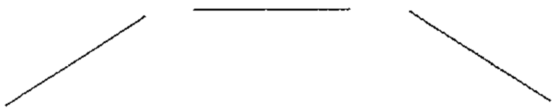
$\ln x = 1 \quad (\log_e e = 1)$

$\therefore x = e, y = \dfrac{1}{e}$

1 mark – for obtaining the correct values for the stationary point. CFPA was given to those who received a different point due to incorrect first derivative.

Table of values

x	2.6	e	2.8
y'	0.007	0	-0.004



$\therefore \left(e, \frac{1}{e}\right)$ or $(e, 0.37)$is a local maximum

1 mark – for determining the nature of the stationary point is a maximum. Some students decided to use quotient rule and unsuccessfully differentiated whilst most students who used the table of values got it correct.

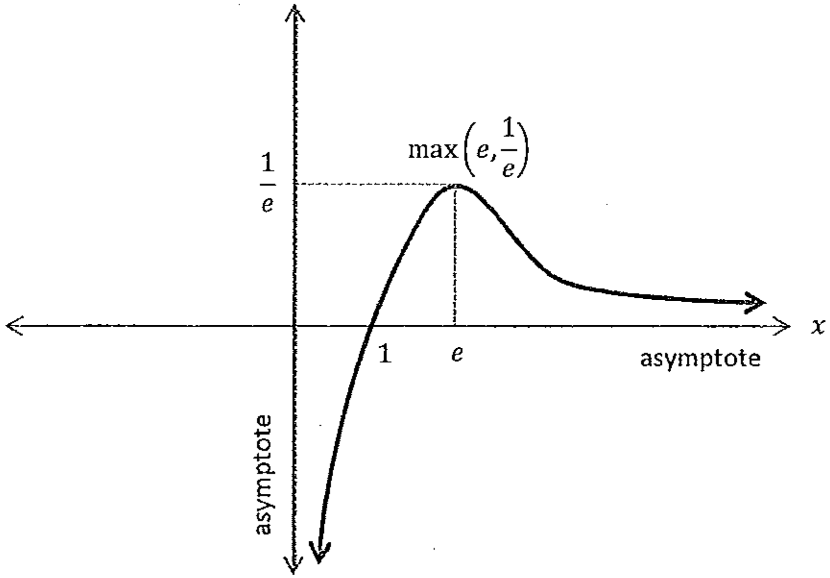
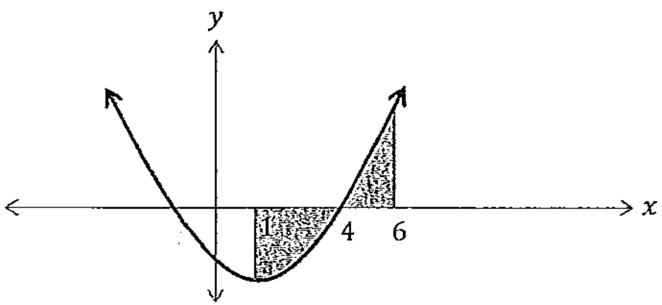
iv) $f(10) = \frac{\ln 10}{10} \approx 0.23\ (2\ d.p.)$

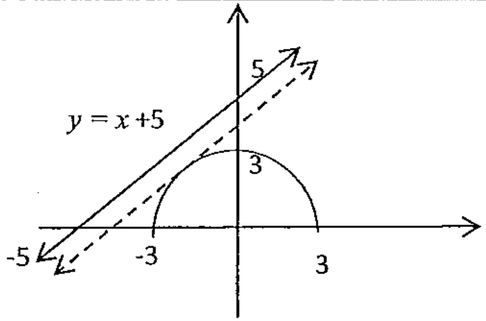
Also when $y = 0$,

$\dfrac{\ln x}{x} = 0$

$x = 1$

7

 	1 mark - for correct shape. 1 mark - for both asymptotes or curve approaching correct asymptotes. Students did poorly for this as many did not try and use limits to identify where the curve approaches as $x \to 0$ and $x \to \infty$. They did not use the fact that f(10) = 0.23 indicated that the curve was approaching 0 as x was larger. Furthermore, many students did not label their asymptotes using dotted lines but did draw the shape so that it looked like it approached one.
(c) $y = x^2 - 3x - 4$ $y = (x-4)(x+1)$ cuts the x axis at $x = 4,\ x = -1$  	1 mark - for showing x intercepts correctly. Most students who attempted this question got this correct.

Question 9 Solutions	Marks and Comment
a)  $y=\sqrt{9-x^2}=(9-x^2)^{1/2}$ $\frac{dy}{dx}=\frac{1}{2}(9-x^2)^{1/2}\times(-2x)$ $\frac{dy}{dx}=\frac{-x}{\sqrt{9-x^2}}$ Now the line $y = x + 5$ has gradient m1=1, ∴the gradient of tangent parallel to this line is also m2=1 (1 mark) $\therefore \frac{-x}{\sqrt{9-x^2}}=1$ $\sqrt{9-x^2}=-x$ $(\sqrt{9-x^2})^2=(-x)^2$ $9-x^2=x^2$ $9=2x^2$ $x^2={}^9/_2$ $x=\pm\sqrt{{}^9/_2}=\pm\frac{3}{\sqrt{2}}$ (1 mark) But there is only one point where m=1, since the equation in question is a positive semicircle $y=\sqrt{9-x^2}$ $\therefore x=-\frac{3}{\sqrt{2}}$ Only point is $(-\frac{3}{\sqrt{2}}\ ,\frac{3}{\sqrt{2}})$ (1 mark, eliminating for correct answer)	**1 mark for the gradient of tangent m=1** **1 mark for correct differentiation** **1 mark for $x=\pm\frac{3}{\sqrt{2}}$** **1 mark for final solution $(-\frac{3}{\sqrt{2}}\ ,\frac{3}{\sqrt{2}})$** Many students experienced great difficulty answering this question as they did not have a good understanding of the concepts and strategies in finding the tangent on a point of a curve. Differentiation was necessary, but many students tried to find the point of intersection between the line and semicircle mentioned in the question, which was not the question asked. (no marks given for this working) It's good to draw a diagram as it was very easy to draw. Very few students used this good strategy to see what the question involved. The tangent can only pass through one point, so elimination of one of the answers was necessary, as there cannot be two tangents parallel the line on the semi circle.
b) i) P(A or B)= P (A U B) = P(A) +P(B)- P(A B) $=\frac{1}{3}+\frac{1}{4}-\frac{1}{6}$ $=\frac{5}{12}$	**i) 1 mark correct answer with correct working $\frac{5}{12}$** **ii) 1 mark correct answer with correct working $\frac{7}{12}$ (CFPA)**

9.

ii) P (neither A or B) $= 1 - \frac{5}{12}$ $= \frac{7}{12}$	Many students were confused about how to do b)i). It is worth doing some extra practice of questions with 'or'. The second part was easy as it needed a simple subtraction.
C) i) $\sum_{n=10}^{30}(2n-4) = 16 + 18 + 20 + \cdots + 56$ AP with a=16, d=2, last term=56, n=30-10+1=21 $Sum = S_{21} = \frac{n}{2}(a+l)$ or $Sum = \frac{n}{2}(2a+(n-1)d)$ $S_{21} = \frac{21}{2}(16+56)$ $S_{21} = 756$	**1 mark for n=21** **1 mark for correct substitution in sum of arithmetic series** Many students answered this question correctly by realising it was an arithmetic series. Try and use the simplest methods eg finding first and last term. Don't forget to check your formula sheet and become familiar with them.
ii) $\sum_{n=1}^{8} 3\times 2^{n-1} = 3\times 2^0 + 3\times 2^1 + 3\times 2^2 + \ldots\ldots + 3\times 2^7$ $\sum_{n=1}^{8} 3\times 2^{n-1} = 3 + 6 + 12 + \ldots\ldots + 384$ Geometric Series with a=3, n=8, r= $\frac{6}{3} = \frac{12}{6} = 2$, $S_8 = \frac{a(r^n-1)}{r-1}$ $r > 1$ $S_8 = \frac{3(2^8-1)}{2-1}$ $S_8 = 765$	**1 mark r=2** **1 mark for correct substitution in sum of geometric series** Many students used the arithmetic series, and could not get full marks for this question. Some students just subbed from n=1 to n=8, and added all the numbers. This is a good idea for small sets of numbers like this question. Remember you do have a formula sheet which can help you use correct formulas.

iii) $27 - 18 + 12 - 8 + \ldots$ Geometric Series with a=27, $r = \frac{-18}{27} = \frac{12}{-18} = \frac{-2}{3}$ Since $\lvert r \rvert = \left\lvert \frac{-2}{3} \right\rvert = \frac{2}{3} < 1$, a limiting sum exists. (or -1<r<1) $S_\infty = \frac{a}{1-r}$ $= \frac{27}{1 - \frac{-2}{3}}$ $= \frac{27}{5/3}$ $= \frac{81}{5} = 16\,{}^1\!/_5$ OR $S_\infty = \frac{a(1-r^\infty)}{1-r}$ $r^\infty \to 0 \ when\, n \to \infty \ and \ -1 < \mathrm{r} < 1$ $S_\infty = \frac{a}{1-r}$ $= \frac{81}{5} = 16\,{}^1\!/_5$	**1 mark stating condition for limiting sum which exists when** $\lvert r \rvert = \left\lvert \frac{-2}{3} \right\rvert = \frac{2}{3} < 1$ or -1<r<1 **1 mark for correct substitution in limiting sum of geometric series(sum to infinity)** Most students neglected the condition for limiting sum. The other alternative was to use the sum of a geometric series where $r^\infty \to 0 \ when\, n \to \infty \ and \ -1 < \mathrm{r} < 1$. Even though some students used r^∞ , they did not clearly state a clear answer after this step.

Question 10 12 marks

Solution	Marks and Feedback
(a)(i) $V = lbh$ $8000 = xxh$ $h = \frac{8000}{x^2}$ $S = x + x + h$ $S = 2x + \frac{8000}{x^2}$	1 mark for expressing h in terms of x (as a fraction) but since its value is given, you must show this as a product i.e. xxh=8000
(a)(ii) $\frac{dS}{dx} = 2 - 2(8000x^{-3})$ $= 2 - \frac{16000}{x^3}$	1 mark for the derivative (do not use the quotient rule)
$\frac{dS}{dx} = 0 \quad at\ st.pts.$ $2 = \frac{16000}{x^3}$ $x^3 = 8000$ $x = 20$	1 mark for the equation
$h = \frac{8000}{20^2}$ $h = 20$	1 mark for the value of h
$\frac{d^2S}{dx^2} = 0 + 6 \times 8000x^{-4}$ $= \frac{48000}{20^4} > 0 \quad at\ x = 20$ $\therefore concave\ up \quad \therefore$ minimum	1 mark for checking concavity This can also be done using a table of values for $\frac{dS}{dx}$

(b) P(at least one is Red)=1 - P(both are Black) $=1-\frac{5}{9}\times\frac{4}{8}$ $=1-\frac{5}{18}$ $=\frac{13}{18}$	1 mark for the expression / formula / tree (This can also be done by adding the probabilities for RR, RB and BR which takes longer and needs a probability tree) 1 mark for the calculation 1 mark for the answer
(c) $y=x^2+1$ $y^2=(x^2+1)^2$ $y^2=x^4+2x^2+1$ $V=\pi\int_0^3(x^4+2x^2+1)dx$ $=\pi\left[\frac{x^5}{5}+\frac{2x^3}{3}+x\right]_0^3$ $=\pi\left(\frac{243}{5}+18+3\right)$ $=\frac{348\pi}{5}$ *cubic units*	1 mark for squaring correctly (many students missed the middle term) 1 mark for substituting into the correct integral 1 mark for integrating correctly 1 mark for the substitution Do no convert to a decimal as the exact (fraction) value is preferred (69.6π looks silly)

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Question 11

	Answer	Marking Criteria/Comments
11.a	$\int_2^3 \frac{x^2\ dx}{x^3+1} = \frac{1}{3}\left[\ln(x^3+1)\right]_2^3$ $= \frac{1}{3}\left[\ln 28 - \ln 9\right]$ $= \frac{1}{3}\left[\ln\frac{28}{9}\right]$	**1 Mark** Few students couldn't figure out what the given integrand was. Remember the general form of log function is $\int_a^b \frac{f\prime(x)}{f(x)}$ and if the constant is not there manipulate it **1 Mark** (2)
b	$f'(x) = 3x^2-2x+1$ $f(x) = x^3-x^2+x+C$ When $x=1$, $y=-5$ $-5 = 1-1+1+C$ $\therefore C = -6$ $\therefore f(x) = x^3-x^2+x-6$	**1 Mark** **1 Mark** **Most of the students got this question correct.** (2)

	Answer	Marking Criteria/Comments
c	xy = 4 $x = \frac{4}{y}$	1 Mark for making x as the subject
	$A = \int_1^3 \frac{4}{y}\, dy$ $= 4\left[\ln y\right]_1^3$ $= 4(\ln 3 - \ln 1)$ $= 4 \ln 3$ or $\ln 3^4 = \ln 81$ $(\ln 1 = 0)$	The Question was to find the **area bounded the y axis** and between the **lines y = 1 and y = 3** **Few students didn't read the question well, so forgot to make 'x' as the subject** 1 Mark 1 Mark (3)
d. (i)	Given $f''(x) = 6$ $f'(x) = 6x + C_1$ $\therefore\ f(x) = 3x^2 + C_1 x + C_2$, → **which is the equation of a parabola, since its highest power of x is 2 and it is concave up since the coefficient of x^2 is 3 (>0)** →	The Question was to <u>show that the function</u> is concave up, given so from $f''(x)$ **you have to integrate twice to find f(x).** 1 Mark **Few even forgot the method to get back to f(x) and few even forgot to use different constant of integration.** 1 Mark (2)

	Answer	Marking Criteria/Comments
(ii)	We need to find C_1 and C_2, using the given conditions. **To find C_1** $f'(x) = 0$ when $x=2$ $6 \times 2 + C_1 = 0$ $C_1 = -12$ $\therefore f(x) = 3x^2 - 12x + C_2$ **To find C_2; given that the vertex is (2 , 7)** $f(2) = 7$ $3 \times (2^2) - 12 \times 2 + C_2 = 7$ $C_2 = 19$ $\therefore f(x) = 3x^2 - 12x + 19$	Poorly done by the students, as they forgot to integrate twice in order to find values of the two constants of integration, by using the given conditions. 1 Mark 1 Mark 1 Mark (3)