15. POLYNOMIALS

1. A polynomial $P(x)$ is given by $P(x) = 9x^3 - 25x^2 + 10Kx - K^2$.

   (i) Find the remainder when $P(x)$ is divided by $x - 1$.

   (ii) Find the value(s) of K if $P(x)$ is divisible by $x - 1$.

   (iii) For these values of K, solve the equation $P(x) = 0$ for the real roots.

2. If α, β, γ are the roots of $x^3 - x^2 + 4x - 2 = 0$, find the values of:

   (a) $\alpha + \beta + \gamma$ (b) $\alpha\beta + \beta\gamma + \gamma\alpha$

   (c) $\alpha\beta\gamma$ (d) $\alpha^2 + \beta^2 + \gamma^2$

3. If $x^2 - 1$ is a factor of the polynomial $P(x) = x^3 + ax^2 + bx - 2$, find a and b. Solve the equation $P(x) = 0$ completely for the real roots.

4. Using one step of Newton's method, find a better approximation to the root of the equation $f(x) = 0$, where $f(x) = x^5 - 36.4$, and $x_1 = 2$ is the first approximation of the root. Give your answer to 2 decimal places.

5. The equation $f(x) = 0$, where $f(x) = \sin x + \frac{x}{2} - 1$, has a root near $x = 0.6$. Find a better approximation to the root, by Newton's method, giving your answer to 2 decimal places.

6. A monic polynomial $P(x)$ of degree 4 is known to have zeros 2 and −2.

   (a) Write down an expression for $P(x)$ to the extent specified so far.

   (b) Given further that $P(0) = 4$ and $P(1) = -3$, find $P(x)$.

   (c) Solve $P(x) = 0$ for real roots.

7. Find the relationship between p, q, r if the roots of the equation $x^3 + px^2 + qx + r = 0$ are in arithmetic progression. Solve the equation $4x^3 - 24x^2 + 23x + 18 = 0$, given that the roots are in arithmetic progression.

8. What relation must be satisfied by the coefficients of the equation $ax^3 + bx^2 + cx + d = 0$, if the sum of two of the roots is zero. Solve $ax^3 - x^2 - 18x + 9 = 0$, given that the sum of two of its roots is zero.

9. (a) State the remainder theorem

   (b) When a polynomial P(x) is divided by $x - a$, the remainder is a^2 and when divided by $x - b$, the remainder is b^2. Show that when $P(x)$ is divided by $(x-a)(x-b)$, the remainder is $(a+b)x - ab$.

10. A polynomial $P(x)$ has the following properties:

    (i) $P(x)$ is an even function

    (ii) $P(x)$ has a zero at $x = 2$

    (iii) $P(x)$ is monic

    (iv) $P(x)$ is of degree 4.

    (a) Write down a general expression for $P(x)$

    (b) If $P(1) = 2$, find the particular polynomial $P(x)$

    (c) Solve $P(x) = 0$ for the real roots.

SOLUTIONS 9→10

9. (a) When $P(x)$ is divided by $x - a$, the remainder is $P(a)$.

   (b) Let $P(x) = (x-a)(x-b)Q(x) + Lx + M$

   When $P(x)$ is divided by $x - a$ and $x - b$, the remainders are $P(a) = a^2$ and $P(b) = b^2$ respectively.

   $\therefore La + M = a^2$ ... (1)

   $Lb + M = b^2$ ... (2)

   Subtracting (1) − (2):

   $L(a-b) = a^2 - b^2$, $a \neq b$

   $\therefore L = a + b$

   Then from (1):

   $a^2 + ab + M = a^2$

   $M = -ab$

   Hence the remainder when $P(x)$ is divided by $(x-a)(x-b)$ is $Lx + M = (a+b)x - ab$.

10. (a) $P(x) = (x^2 - 4)(x^2 + b)$

    (b) $P(1) = 2$

    $\therefore 2 = -3(1+b)$

    $b = -\frac{5}{3}$

    $P(x) = (x^2 - 4)(x^2 - \frac{5}{3})$

    (c) $P(x) = 0$

    $(x^2 - 4)(x^2 - \frac{5}{3}) = 0$

    $\therefore x = \pm 2, \pm\sqrt{\frac{5}{3}}$ are the required roots.

15. POLYNOMIALS

1. $P(x)=9x^3-25x^2+10kx-k^2$

(i) The remainder when $P(x)$ is divided by $x-1$ is
$P(1)=9-25+10k-k^2$
$=-k^2+10k-16$

(ii) If $P(x)$ is divisible by $x-1$, then $P(1)=0$.
$\therefore -k^2+10k-16=0$
$(k-2)(k-8)=0$
$k=2$ or 8

(iii) For $k=2$,
$P(x)=9x^3-25x^2+20x-4$
The roots of $P(x)=0$ are given by
$(x-1)(9x^2-16x+4)=0$
$x=1,\ x=\dfrac{9\pm2\sqrt7}{9}$
For $k=8$,
$P(x)=8x^3-25x^2+80x-64$
$=(x-1)\cdot(9x^2-16x+64)$
The real root of $P(x)=0$ is $x=1$, the other two roots are not real.

2. $x^3-x^2+4x-2=0$

(a) $\alpha+\beta+\gamma=1$

(b) $\alpha\beta+\beta\gamma+\gamma\alpha=4$

(c) $\alpha\beta\gamma=2$

(d) $\alpha^2+\beta^2+\gamma^2$
$=(\alpha+\beta+\gamma)-2(\alpha\beta+\beta\gamma+\gamma\alpha)$
$=1-2\times4=-7$

3. $x^2-1=(x-1)(x+1)$
x^2-1 is a factor of $P(x)$, hence $(x-1)$ and $(x+1)$ are both the factors of $P(x)=x^3+ax^2+bx-2$
$\therefore\ P(1)=0$ and $P(-1)=0$
$\therefore\ a+b=1$ and $a-b=3$
Solving these:
$a=2,\ b=-1$
Then
$P(x)=(x-1)(x+1)(x+2)=0$
has the roots $-1,\ 1,\ -2$.

4. $f(x)=x^5-36.4$
$f'(x)=5x^4$
$x_1=2, f(2)=-4.4, f'(2)=80$
Using Newton's method
$$x_2=x_1-\frac{f(x_1)}{f'(x_1)}$$
$x_2=2+\dfrac{4.4}{80}=2.06$, to 2 decimal places.

5. $f(x)=\sin x+\dfrac{x}{2}-1$
$f'(x)=\cos x+\frac12$
$x_1=0.6$
$f(0.6)=-0.135\,36$
$f'(0.6)=1.325\,34$
Using Newton's method
$$x_2=x_1-\frac{f(x_1)}{f'(x_1)}$$
$=0.6+\dfrac{0.135\,36}{1.325\,34}$
$=0.70$
(2 decimal places)

6. (a) $P(x)=(x+2)(x-2)\cdot(x^2+ax+b)$

(b) $P(0)=4,\ P(1)=-3$
$\therefore\ -4b=4,\ b=-1$
and $-3(1+a+b)=-3$
$\therefore\ a=1$
$\therefore$
$P(x)=(x^2-4)(x^2+x-1)$
$=x^4+x^3-5x^2-4x+4$

(c) $P(x)=0$
$(x^2-4)(x^2+x-1)=0$
The real roots are:
$x=\pm2,\ \dfrac{-1\pm\sqrt5}{2}$

7. Since the roots are in A.P, let these be:
$\alpha=b-d,\ \beta=b,\ \gamma=b+d$
$x^3+px^2+qx+r=0$ ... (1)
$\alpha+\beta+\gamma=-p$
$\therefore\ b-d+b+b+d=-p$
$b=\dfrac{-p}{3}$
$\beta=b=-\dfrac{p}{3}$ satisfies (1).
$\therefore\ -\dfrac{p^3}{27}+\dfrac{p^2}{9}-\dfrac{pq}{3}+r=0$
This simplifies to
$2p^3-9pq+28r=0$,
which is the required relation between $p,\ q,\ r$.

To solve
$4x^3-24x^2+23x+18=0$,
divide both sides by 4, then
$x^3-6x^2+\frac{23}{4}x+\frac92=0$.
$\therefore\ p=-6$, i.e. $\beta=-\dfrac{p}{3}=2$
Divide $4x^3-24x^2+23x+18$ by $x-2$, then:
$(x-2)(4x^2-16x-9)=0$
$(x-2)(2x+1)(2x-9)=0$
The roots are:
$-\frac12,\ 2,\ \frac92$

8. $ax^3+bx^2+cx+d=0$ ... (1)
Let the roots be $\alpha,\ -\alpha,\ \beta$.
Then $\alpha+(-\alpha)+\beta=\dfrac{-b}{a}$
$\therefore\ \beta=\dfrac{-b}{a}$
β satisfies (1), so
$-\dfrac{b^3}{a^3}+\dfrac{b^3}{a^2}-\dfrac{bc}{a}+d=0$
This simplifies to
$ad-bc=0$ ... (2)

This is the required relation, so that the sum of the two roots is zero.
To solve
$ax^3-x^2-18x+9=0$,
whose roots are $\alpha,\ -\alpha,\ \beta$.
Using the relation (2):
$a\times9-(-1)\times(-18)=0$
$\therefore\ a=2$
Then
$2x^3-x^2-18x+9=0$ ... (1)
$\alpha-\alpha+\beta=\frac12$ gives $\beta=\frac12$
Factorising the L.H.S. of (1):
$x^2(2x-1)-2(2x-1)=0$
$(2x-1)(x^2-2)=0$
The roots are:
$x=\frac12,\ \pm\sqrt2$