SYDNEY GIRLS HIGH SCHOOL

2011
HSC ASSESSMENT TASK 2

Mathematics Extension 2

General Instructions
- Reading time – 5 minutes
- Writing time – 90 minutes
- Board-approved calculators may be used
- Diagrams are NOT to scale
- A table of standard integrals is provided at the back of this paper
- Topics – Complex Numbers

Total marks – 75
- Attempt Questions 1-3
- All questions are of equal value
- Start each question on a new page
- Write on one side of the paper only
- Write using black or blue pen
- All necessary working should be shown in every question

Name : _____

Teacher : _____

QUESTION 1 (25 Marks)

(a) If $\omega = 3 - 4i$ and $z = -1 + 3i$, express the following in the form $a + ib$ where a and b are real numbers.

    (i) $\omega + 2z$ 2

    (ii) $z - \bar{z}$ 2

    (iii) $\dfrac{\omega}{z}$ 2

    (iv) $\text{Im}\left[\omega^2 + iz\right]$ 3

(b) (i)   Find $\sqrt{6i - 8}$ and express each answer in the form $x + iy$.   4

    (ii)   Hence solve the equation $2z^2 - (3 + i)z + 2 = 0$, expressing z in the form $x + iy$.   2

(c) If ω is a complex cube root of unity (i.e. a root of $z^3 = 1$) :

    (i)   Prove that ω^2 is also a complex cube root of unity.   2

    (ii)   Prove that $1 + \omega + \omega^2 = 0$.   2

    (iii)   Evaluate $\left(3\omega^2 + 3\omega^4\right)^3$.   2

(d) Find all the complex numbers $z = a + ib$, where a and b are real, such that $|z|^2 + 5\bar{z} + 10i = 0$.   4

- End of Question 1 -

QUESTION 2 (continued)

(d)   (i)   Express $z = 1 + i$ in modulus/argument form.   1
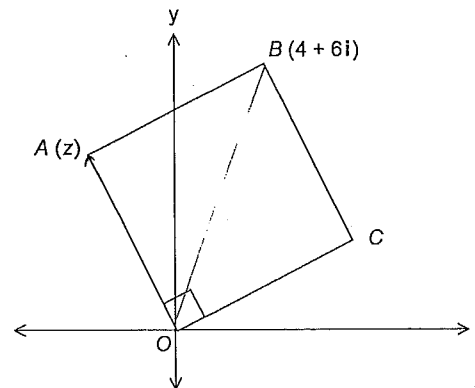
     (ii)   Hence show that $z^9 = 16z$.   1

     (iii)   Hence express $(1+i)^9 + (1-i)^9$ in the form $a + ib$ 2

         where a and b are real.

- End of Question 2 -

QUESTION 3 (continued)

(d) $OABC$ is a square. A represents the complex number z. B represents the complex number $4 + 6i$.

   (i)   Write down the complex number represented by C in terms of z.   1

   (ii)   Hence, or otherwise, find z in the form of $x + iy$.   2

(e)   Find the maximum value for $|z|$ where z satisfies the condition   3

$$\left| z - \frac{3}{z} \right| = 5 \quad .$$

- End of paper -

a.
$w = 3 - 4i \qquad z = -1 + 3i$

i.
$$w + 2z = 3 - 4i + 2(-1 + 3i)$$
$$= 3 - 4i - 2 + 6i$$
$$= 1 + 2i$$

ii.
$$z - \bar{z} = 2i \times \text{Im}(z)$$
$$= 2i \times 3$$

iii.
$$\frac{w}{z} = \frac{3 - 4i}{-1 + 3i} \times \frac{-1 - 3i}{-1 - 3i}$$
$$= \frac{-3 - 9i + 4i + 12i^2}{1 - 9i^2}$$
$$= \frac{-15 - 5i}{10}$$
$$= -\frac{3}{2} - \frac{1}{2}i$$

iv.
$$\text{Im}(w^2 + iz) = \text{Im}\left\{(3 - 4i)^2 + i(-1 + 3i)\right\}$$
$$= \text{Im}(9 - 24i + 16i^2 - i + 3i^2)$$
$$= \text{Im}(9 - 25i - 19)$$
$$= \text{Im}(-10 - 25i)$$
$$= -25$$

b.

i.
$$\sqrt{6i - 8} = a + ib$$
$$6i - 8 = a^2 + 2abi - b^2$$
Equating real and imaginary parts:
$$a^2 - b^2 = -8 \rightarrow (1)$$
$$2ab = 6$$
$$b = \frac{3}{a} \rightarrow (2)$$
Sub (2) into (1):
$$a^2 - \left(\frac{3}{a}\right)^2 = -8$$
$$a^4 + 8a^2 - 9 = 0$$
$$(a^2 + 9)(a^2 - 1) = 0$$
a is real $\therefore a = \pm 1$
Sub $a = \pm 1$ into (2):
$$b = \pm 3$$
$$\therefore \sqrt{6i - 8} = \pm(1 + 3i)$$

ii.
$$2z^2 - (3 + i)z + 2 = 0$$
$$z = \frac{(3 + i) \pm \sqrt{(3 + i)^2 - 4 \times 2 \times 2}}{4}$$
$$= \frac{(3 + i) \pm \sqrt{9 + 6i + i^2 - 16}}{4}$$
$$= \frac{(3 + i) \pm \sqrt{6i - 8}}{4}$$
$$= \frac{(3 + i) \pm (1 + 3i)}{4}$$
$$z = \frac{4 + 4i}{4} \quad \text{or} \quad z = \frac{2 - 2i}{4}$$
$$= 1 + i \qquad\qquad = \frac{1}{2} - \frac{1}{2}i$$

c.

i.
$$z^3 = 1$$
$$z = cis\left(\frac{2k\pi}{3}\right) \qquad k = 0, 1, 2$$
$$= 1, \ cis\frac{2\pi}{3}, \ cis\frac{4\pi}{3}$$
let $w = cis\frac{2\pi}{3}$
$$w^2 = \left(cis\frac{2\pi}{3}\right)^2$$
$$= cis\frac{4\pi}{3} \quad \text{which is also a root.}$$

ii.
$$1 + w + w^2 = \frac{1(w^3 - 1)}{w - 1} \qquad (w^3 = 1)$$
$$= \frac{1(1 - 1)}{w - 1}$$
$$= 0$$

iii.
$$(3w^2 + 3w^4)^3 = \left[3w^2(1 + w^2)\right]^3$$
$$= \left[3w^2(-w)\right]^3$$
$$= (-3)^3$$
$$= -27$$

d.
$$|z|^2 + 5\bar{z} + 10i = 0$$
$$a^2 + b^2 + 5a - 5bi + 10i = 0$$
$$a^2 + b^2 + 5a + i(-5b + 10) = 0$$
Equating real and imaginary parts:
$$-5b + 10 = 0$$
$$b = 2$$
$$a^2 + b^2 + 5a = 0$$
$$a^2 + 4 + 5a = 0$$
$$(a + 4)(a + 1) = 0$$
$$a = -4, -1$$
$$\therefore z = -1 + 2i, \ -4 + 2i$$

2 (a) (i) $|z_1| = \sqrt{1^2 + (\sqrt{3})^2}$

$= 2$

$\tan\theta = \dfrac{\sqrt{3}}{1}$

$\theta = \dfrac{\pi}{3}$

$\therefore z_1 = 2\,\text{cis}\,\dfrac{\pi}{3}$ ✓

$|z_2| = \sqrt{1^2 + (-1)^2}$

$= \sqrt{2}$

$\tan\theta = \dfrac{-1}{1}$

$\theta = -\dfrac{\pi}{4}$

$\therefore z_2 = \sqrt{2}\,\text{cis}\left(-\dfrac{\pi}{4}\right)$ ✓

$\dfrac{z_1}{z_2} = \dfrac{2}{\sqrt{2}}\,\text{cis}\left(\dfrac{\pi}{3} - -\dfrac{\pi}{4}\right)$

$= \sqrt{2}\,\text{cis}\,\dfrac{7\pi}{12}$ ✓

(ii) $\left(\dfrac{z_1}{z_2}\right)^n = r\,\text{cis}\left(\dfrac{k\pi}{2}\right)$

k an integer

$(\sqrt{2})^n\,\text{cis}\,\dfrac{7n\pi}{12} = r\,\text{cis}\left(\dfrac{k\pi}{2}\right)$ ✓

$\dfrac{7n\pi}{12} = \dfrac{k\pi}{2}$

$7n = 6k$

$\therefore n = 6$ ✓

(iii) $r = (\sqrt{2})^6 \sin\left(\dfrac{7\times 6\times\pi}{12}\right)$

$= 8\sin\dfrac{7\pi}{2}$

$= -8$ ✓

(b) (i) $\sqrt{x^2 + (y-3)^2} = \sqrt{(x+6)^2 + y^2}$ ✓

$x^2 + y^2 - 6y + 9 = x^2 + 12x + 36 + y^2$

$0 = 12x + 6y + 27$

$4x + 2y + 9 = 0$ ✓

(ii) $2\sqrt{x^2+y^2} = x+iy + x-iy + 4$

$4(x^2+y^2) = (2x+4)^2$ ✓

$4x^2 + 4y^2 = 4x^2 + 16x + 16$

$4y^2 = 16x + 16$ ✓

$y^2 = 4x + 4$

(iii) $\dfrac{z-i}{z+1} = \dfrac{x+iy-i}{x+iy+1}$

$= \dfrac{x+i(y-1)}{x+1+iy} \times \dfrac{x+1-iy}{x+1-iy}$ ✓

$= \dfrac{x(x+1) - ixy + i(y-1)(x+1) + y(y-1)}{(x+1)^2 + y^2}$ ✓

$\text{Re}\left(\dfrac{z-i}{z+1}\right) = 0$

$\dfrac{x(x+1) + y(y-1)}{(x+1)^2 + y^2} = 0$

$x(x+1) + y(y-1) = 0$ ✓

$x^2 + x + \left(\tfrac{1}{2}\right)^2 + y^2 - y + \left(-\tfrac{1}{2}\right)^2 = \dfrac{1}{2}$

$\left(x+\tfrac{1}{2}\right)^2 + \left(y-\tfrac{1}{2}\right)^2 = \dfrac{1}{2}$
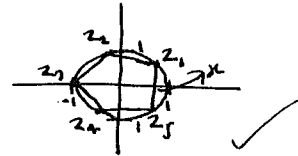
(c) (i) $r^5\,\text{cis}\,5\theta = -1$ ✓

$r^5 = 1 \qquad 5\theta = \pi, 3\pi, 5\pi, 7\pi, 9\pi$

$r = 1 \qquad \theta = \dfrac{\pi}{5}, \dfrac{3\pi}{5}, \pi, \dfrac{7\pi}{5}, \dfrac{9\pi}{5}$

$z_1 = \text{cis}\,\dfrac{\pi}{5},\; z_2 = \text{cis}\,\dfrac{3\pi}{5},\; z_3 = \text{cis}\,\pi$

$z_4 = \text{cis}\,-\dfrac{3\pi}{5},\; z_5 = \text{cis}\,-\dfrac{\pi}{5}$ ✓

(ii)

(iii) $x^2 = 1^2 + 1^2 - 2\times 1\times 1\times\cos\dfrac{2\pi}{5}$ ✓

$x = \sqrt{2 - 2\cos\dfrac{2\pi}{5}}$ ✓

$P = 5\left(2 - 2\cos\dfrac{2\pi}{5}\right)^{\frac{1}{2}}$

$= 5.87778\ldots$

$= 5.88\text{ m}$ ✓

(iv) $z^5 + 1 = (z+1)\left(z - \text{cis}\,\dfrac{\pi}{5}\right)\left(z - \text{cis}\,-\dfrac{\pi}{5}\right)$

$\left(z - \text{cis}\,\dfrac{3\pi}{5}\right)\left(z - \text{cis}\,-\dfrac{3\pi}{5}\right)$ ✓

$= (z+1)\left(z - \cos\dfrac{\pi}{5} - i\sin\dfrac{\pi}{5}\right)\left(z - \cos\dfrac{\pi}{5} + i\sin\dfrac{\pi}{5}\right)$

$\left(z - \cos\dfrac{3\pi}{5} - i\sin\dfrac{3\pi}{5}\right)\left(z - \cos\dfrac{3\pi}{5} + i\sin\dfrac{3\pi}{5}\right)$

$= (z+1)\left(z^2 - 2z\cos\dfrac{\pi}{5} + \cos^2\dfrac{\pi}{5} + \sin^2\dfrac{\pi}{5}\right)$

$\left(z^2 - 2z\cos\dfrac{3\pi}{5} + \cos^2\dfrac{3\pi}{5} + \sin^2\dfrac{3\pi}{5}\right)$ ✓

$= (z+1)\left(z^2 - 2\cos\dfrac{\pi}{5}\,z + 1\right)\left(z^2 - 2\cos\dfrac{3\pi}{5}\,z + 1\right)$ ✓

(d) (i) $|z| = \sqrt{1^2 + 1^2}$

$= \sqrt{2}$

$\tan\theta = \dfrac{1}{1}$

$\theta = 45°$

$z = \sqrt{2}\,\text{cis}\,\dfrac{\pi}{4}$ ✓

(ii) $z^9 = (\sqrt{2})^9\,\text{cis}\,\dfrac{9\pi}{4}$

$= 16\sqrt{2}\,\text{cis}\,\dfrac{\pi}{4}$ ✓

$= 16z$ ✓

G.C.O.

(iii) $z^9 + \overline{z}^9$

$= 16z + \overline{z^9}$ ✓

$= 16z + \overline{16z}$
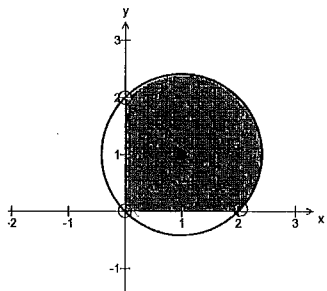
$= 16(1+i) + 16(1-i)$

$= 32$ ✓

$-2-$

QUESTION 3:

a) (i)
$$z^n + \frac{1}{z^n} = z^n + z^{-n}$$
$$= (\cos\theta + i\sin\theta)^n + (\cos\theta + i\sin\theta)^{-n}$$
$$= \cos n\theta + i\sin n\theta + \cos(-n\theta) + i\sin(-n\theta)$$
$$= \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$$
$$= 2\cos n\theta$$

(ii)
$$\left(z + \frac{1}{z}\right)^4 = z^4 + 4z^3 \cdot \frac{1}{z} + 6z^2 \cdot \frac{1}{z^2} + 4z \cdot \frac{1}{z^3} + \frac{1}{z^4}$$
$$(2\cos\theta)^4 = \left(z^4 + \frac{1}{z^4}\right) + 4\left(z^2 + \frac{1}{z^2}\right) + 6$$
$$16\cos^4\theta = 2\cos 4\theta + 8\cos 2\theta + 6$$
$$\cos^4\theta = \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}$$
$$= \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$$

(iii)
$$\cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$$
$$8\cos^4\theta = \cos 4\theta + 4\cos 2\theta + 3$$
$$8\cos^4\theta - 3 = \cos 4\theta + 4\cos 2\theta$$
$$\int(8\cos^4\theta - 3)d\theta = \int(\cos 4\theta + 4\cos 2\theta)d\theta$$
$$= \frac{1}{4}\sin 4\theta + 2\sin 2\theta + C$$

b) (i)
$$|z - 1 - i| \le \sqrt{2}$$
$$|z - \{1 + i\}| \le \sqrt{2}$$
Circle centre $(1,1)$ and radius $\sqrt{2}$ units

(ii) Required area = Area of semi-circle + Area of triangle
$$\text{Area} = \frac{1}{2} \times \pi \times (\sqrt{2})^2 + \frac{1}{2} \times 2 \times 2$$
$$= (\pi + 2)\text{u}^2$$

c) (i)
$$(\cos\theta + i\sin\theta)^4 = \cos^4\theta + 4\cos^3\theta(i\sin\theta) + 6\cos^2\theta(i\sin\theta)^2 + 4\cos\theta(i\sin\theta)^3 + (i\sin\theta)^4$$
$$\cos 4\theta + i\sin 4\theta = \cos^4\theta + i4\cos^3\theta\sin\theta - 6\cos^2\theta\sin^2\theta - i4\cos\theta\sin^3\theta + \sin^4\theta$$
$$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$$
$$\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$$

(ii)
$$\tan 4\theta = \frac{\sin 4\theta}{\cos 4\theta}$$
$$= \frac{4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta}{\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta}$$
$$= \frac{\dfrac{4\cos^3\theta\sin\theta}{\cos^4\theta} - \dfrac{4\cos\theta\sin^3\theta}{\cos^4\theta}}{\dfrac{\cos^4\theta}{\cos^4\theta} - \dfrac{6\cos^2\theta\sin^2\theta}{\cos^4\theta} + \dfrac{\sin^4\theta}{\cos^4\theta}}$$
$$= \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$$
$$= \frac{4t - 4t^3}{1 - 6t^2 + t^4} \quad (\text{using } t = \tan\theta)$$

(iii)
$$\tan 4\theta = 1$$
$$4\theta = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \frac{13\pi}{4}, \frac{17\pi}{4}, \frac{21\pi}{4}, \frac{25\pi}{4}, \frac{29\pi}{4}$$
$$\theta = \frac{\pi}{16}, \frac{5\pi}{16}, \frac{9\pi}{16}, \frac{13\pi}{16}, \frac{17\pi}{16}, \frac{21\pi}{16}, \frac{25\pi}{16}, \frac{29\pi}{16}$$
$$x^4 + 4x^3 - 6x^2 - 4x + 1 = 0$$
Let $x = \tan\theta$
$$\tan^4\theta + 4\tan^3\theta - 6\tan^2\theta - 4\tan\theta + 1 = 0$$
Let $t = \tan\theta$
$$t^4 + 4t^3 - 6t^2 - 4t + 1 = 0$$
$$t^4 - 6t^2 + 1 = 4t - 4t^3$$
$$\frac{4t - 4t^3}{t^4 - 6t^2 + 1} = 1$$
$$\tan 4\theta = 1$$
$$\theta = \frac{\pi}{16}, \frac{5\pi}{16}, \frac{9\pi}{16}, \frac{13\pi}{16}$$
$$x = \tan\frac{\pi}{16}, \tan\frac{5\pi}{16}, \tan\frac{9\pi}{16}, \tan\frac{13\pi}{16}$$

(iv)

$$\tan\frac{\pi}{16}\times\tan\frac{5\pi}{16}\times\tan\frac{9\pi}{16}\times\tan\frac{13\pi}{16}=1 \quad \text{(product of roots)}$$

$$\tan\frac{\pi}{16}\times\tan\frac{5\pi}{16}\times\tan\left(\pi-\frac{7\pi}{16}\right)\times\tan\left(\pi-\frac{3\pi}{16}\right)=\tan\frac{\pi}{16}\times\tan\frac{5\pi}{16}\times-\tan\frac{7\pi}{16}\times-\tan\frac{3\pi}{16}$$

$$=\tan\frac{\pi}{16}\times\tan\frac{5\pi}{16}\times\tan\frac{7\pi}{16}\times\tan\frac{3\pi}{16}$$

$$\text{So } \tan\frac{\pi}{16}\times\tan\frac{3\pi}{16}\times\tan\frac{5\pi}{16}\times\tan\frac{7\pi}{16}=1$$

d)   (i) $C=-iz$

(ii) $\overrightarrow{OA}+\overrightarrow{OC}=\overrightarrow{OB}$

$$z-iz=4+6i$$

$$z(1-i)=4+6i$$

$$z=\frac{4+6i}{1-i}\times\frac{1+i}{1+i}$$

$$=\frac{4+10i-6}{2}$$

$$=\frac{-2+10i}{2}$$

$$=-1+5i$$

(iii)

$$\left|z-\frac{3}{z}\right|=5$$

Using the triangular inequality $|z_1-z_2|\geq|z_1|+|z_2|$

$$\left|z-\frac{3}{z}\right|\geq|z|-\left|\frac{3}{z}\right|$$

$$5\geq|z|-\left|\frac{3}{z}\right|$$

$$|z|-\left|\frac{3}{z}\right|\leq5$$

$$|z|-\frac{3}{|z|}\leq5$$

$$|z|^2-3\leq5|z|$$

$$|z|^2-5|z|\leq3$$

Completing the square:

$$|z|^2-5|z|+\left(-\frac{5}{2}\right)^2\leq3+\left(-\frac{5}{2}\right)^2$$

$$\left(|z|-\frac{5}{2}\right)^2\leq\frac{37}{4}$$

$$|z|-\frac{5}{2}\leq\frac{\sqrt{37}}{2} \quad \text{since modulus}>0$$

$$|z|\leq\frac{\sqrt{37}+5}{2}$$

Maximum value of $|z|=\dfrac{\sqrt{37}+5}{2}$