Year 12 2U Mathematics

Geometrical Applications of Calculus, Integration

Term 1, 2011 | Week 6 *Time Allowed*: 50 mins *Marks*: 36

Show all working to gain maximum marks. Marks will be deducted for poor or illegible work

Name: ____________________ *Teacher*: HRK GHW RABS CRA

PART A – Curve Sketching (9 Marks) Marked by HRK

1. Consider the function $f(x) = 1 - 3x + x^3$, in the domain $-2 \le x \le 3$.

   a) There are two turning points for $f(x)$. Find their co-ordinates and determine their nature. **4**

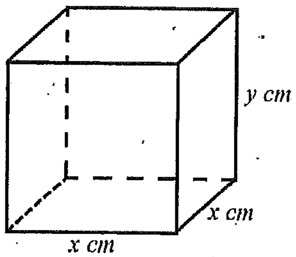
   b) Find any points of inflexion **2**

   c) Draw a neat sketch of the curve $y = f(x)$ in the domain $-2 \le x \le 3$, clearly showing all its essential features. **2**

   d) What is the maximum value of the function $f(x)$ in the domain $-2 \le x \le 3$? **1**

PART B – Maxima and Minima Problems (6 Marks) Marked by CRA

1.



A box is to have a square base. Its combined length, breadth and depth add up to 48cm.

   a) Show that the volume of the box in terms of the base edge x, is given by $V = 48x^2 - 2x^3$ **2**

   b) Hence, determine the maximum volume of the box. **4**

PART C – Integration (11 Marks) Marked by RABS

1. Find $f(x)$ given that $f''(x) = 24x^2 + 6$, $f'(2) = 72$ and $f(2) = 37$. **3**

2. Find the following integrals, leaving your answer with positive, non-fractional indices where necessary.

   a) $\int \frac{dx}{(3-6x)^3}$ **2**

   b) $\int \sqrt{2x-1}\, dx$ **2**

3. Evaluate the following integrals, correct to 2.d.p.

   a) $\int_{-1}^{4} 4x^2 + 3\, dx$ **2**

   b) $\int_{1}^{3} x\sqrt{x}\, dx$ **2**

PART D – Integration (10 Marks) Marked by GHW

1. $y = f(x)$ is known to be a continuous function, and experimentally the following table of results was recorded:

x	2.0	2.5	3.0	3.5	4.0	4.5	5.0
$y = f(x)$	3.1	3.8	4.0	3.6	2.6	2.5	2.4

By means of Simpson's rule, find an approximate value for $\int_{2}^{5} f(x)\, dx$, correct to 1 d.p. **3**

2. Find the area bounded by the curve $y = x^3$ and the line $y = x$ **3**

3. A mould for producing glasses is made by rotating the area bounded by $y = x^3$, the y axis and the lines $y = 1$ and $y = 8$ about the y axis.

   What would be the volume of the glass in cubic centimetres, as an exact value? **4**

√ = 1 mark

Q1 $f(x) = 1 - 3x + x^3$

$f'(x) = -3 + 3x^2$

$f''(x) = 6x$

a) (4) For st pts $f'(x) = 0$ ✓

ie $-3 + 3x^2 = 0$

$x = \pm 1$ ✓

then $y = -1, 3$

$f''(1) = 6 > 0 \therefore \cup \therefore$ min at $(1, -1)$ ✓

$f''(-1) = -6 < 0 \therefore \cap \therefore$ max at $(-1, 3)$. ✓

b) (2) $f''(x) = 6x = 0$

$x = 0,\ y = 1$ ✓

$\therefore$ possible inflexion at $(0, 1)$

*REMEMBER TO TEST concavity either side

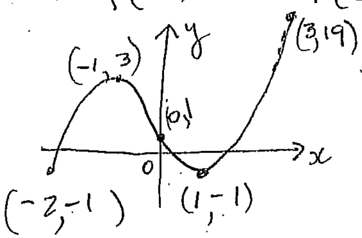
x	-1	0	1
y''	-6	0	$+6$

y'' has change in sign

$\therefore$ Point of inflexion exists at $(0, 1)$ ✓

c) *YOU MUST FIND ENDPOINTS*

$f(-2) = -1 \quad f(3) = 19$



NB make sure (3,19) is higher up the page ✓✓

d) HERE give value - ie y value NOT THE POINT

MAXIMUM VALUE = 19 ✓

NOTE: this is a change in concavity (NO MORE BUMPS are needed!)

PART B - CRA | **MARKS** | **COMMENTS**

① $l + b + d = 48$cm

(a) $V = x \times x \times y$

$\therefore V = x^2 y$ — ①

Since $l + b + d = 48$cm (from question)

$x + x + y = 48$

$2x + y = 48$

$y = 48 - 2x$ — ② ✓

sub ② into ①

$\therefore V = x^2(48 - 2x)$

$\therefore V = 48x^2 - 2x^3$ ✓

Comments:
- Done well on the whole
- For many - it was very unclear what you were doing.
  ↳ When solving simultaneous equations, be very clear what you are doing.

(b) $V = 48x^2 - 2x^3$

$V' = 96x - 6x^2$ ✓

$0 = 96x - 6x^2$

$6x^2 - 96x = 0$

$x^2 - 16x = 0$

$x(x - 16) = 0$

$\therefore x = 0, 16$ ✓

↳ $x > 0 \therefore$ test $x = 16$.

$V'' = 96 - 12x$

when $x = 16$, $V'' < 0 \therefore$ max when $x = 16$ ✓

when $x = 16$, $V = 48(16)^2 - 2(16)^3$

$= 4096$ cm^3 ✓

Comments:
- Done well on the whole.
- You had to ensure you tested your x-value 16 to ensure it was a max.
  ↳ many students lost a mark for not testing.

PART C ... Integration

1. $f''(x) = 24x^2 + 6$

$f'(x) = 8x^3 + 6x + c$ ← DON'T FORGET C.

$f'(2) = 72 = 64 + 12 + c$

$72 = 76 + c$

$\therefore c = -4$

$f'(x) = 8x^3 + 6x - 4$ ✓

$f(x) = 2x^4 + 3x^2 - 4x + c$

$f(2) = 37 = 32 + 12 - 8 + c$

$37 = 36 + c$

$\therefore c = 1$

$f(x) = 2x^4 + 3x^2 - 4x + 1$ ✓ + 1 for working.

- MANY FORGOT +C & THUS DID NOT KNOW WHAT TO DO WITH $f'(2)$.

2. (a) $\int (3-6x)^{-3}\,dx = \dfrac{(3-6x)^{-2}}{-6 \times -2} + c$

$= \dfrac{(3-6x)^{-2}}{12} + c$ ✓

$= \frac{1}{12} \times \dfrac{1}{(3-6x)^2} + c$

$= \dfrac{1}{12(3-6x)^2} + c$ ✓

- SHOULD HAVE BEEN ANSWERED BETTER.
- INDEX LAWS $\left(a^{-m} = \frac{1}{a^m}\right)$ PROVED TO BE CHALLENGING.
- MANY FORGOT TO "+1 TO THE POWER"

(b) $\int (2x-1)^{1/2}\,dx = \dfrac{(2x-1)^{3/2}}{2 \times 3/2} + c$ ✓

$= \dfrac{\sqrt{(2x-1)^3}}{3} + c$ ✓

- A LOT DID NOT WRITE "WITH A POSITIVE, NON-FRACTIONAL INDEX".

3. (a) $\int_{-1}^{4} 4x^2 + 3\,dx = \left[\dfrac{4x^3}{3} + 3x\right]_{-1}^{4}$ ✓

$= \left(\frac{256}{3} + 12\right) - \left(\frac{-4}{3} - 3\right)$

$= \frac{293}{3} + \frac{13}{3}$

$= \frac{305}{3} = 101.67$ ✓

REM: "TO 2 DPS!"

- MAKE SURE YOU DON'T GET SLOPPY WITH YOUR INTEGRAL SYMBOLS & TAKE CARE WITH DOUBLE NEGATIVES, etc...

(b) $\int_1^3 x \cdot x^{1/2}\,dx = \int_1^3 x^{3/2}\,dx$ ← USE INDEX LAWS!

$= \left[\frac{2x^{5/2}}{5}\right]_1^3$ ✓

$= (6.23\ldots) - (2/5)$

$= 5.84$ ✓

- SEE ABOVE NOTES, PLUS...
- MANY STUDENTS MADE UP SOME NEW INTEGRATION TECHNIQUE:

$$\int x \cdot x^{1/2}\,dx \neq x^2 \cdot \frac{2x^{3/2}}{3}$$

YUCK!

- USE INDEX LAWS TO MAKE UNUSUAL INTEGRALS FAMILIAR!

PART D - GHW

Must be careful to times correct numbers by 2 and 4 respectively.

1. $A \approx \frac{h}{3}\left[y_0 + y_n + 2(y_2+y_4+\ldots) + 4(y_1+y_3+\ldots)\right]$

If using $h = \frac{b-a}{n}$, n is number of intervals

$h = 0.5$ from table

$\therefore \int_2^5 f(x)\,dx = \frac{0.5}{3}\left[3.1 + 2.4 + 2(4.0+2.6) + 4(3.8+3.6+2.5)\right]$ (1) (1)

$= \frac{1}{6} \times 58.3$

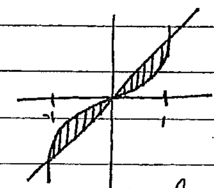
$= 9.7$ (2dp) (1)

2. $x^3 = x$

$x^3 - x = 0$

$x(x^2-1) = 0$

$\therefore x = 0, \pm 1$ (1)



odd fn.

$A = 2\int_0^1 (x - x^3)\,dx$

$= 2\left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1$ (1)

$= 2 \times \left(\frac{1}{2} - \frac{1}{4}\right)$

$= \frac{1}{2}\,u^2$ (1)

3. $y = x^3$

$\therefore \quad x = \sqrt[3]{y}$

so $x^2 = \left(\sqrt[3]{y}\right)^2 = y^{\frac{2}{3}}$ (1)

Must remember to find x in terms of y.

$V = \pi \int_1^8 y^{\frac{2}{3}} \, dy$ (1)

$= \pi \left[\frac{3}{5} y^{5/3} \right]_1^8$ (1)

$= \pi \left(\frac{32 \times 3}{5} - \frac{3}{5} \right)$

$= \frac{93\pi}{5} \text{ u}^3$ (1)

Must leave as an exact value as asked in qn.