J.M.J.

MARCELLIN COLLEGE RANDWICK

EXTENSION I

MATHEMATICS

2011

Weighting: 30% (Assessment Mark)

NAME: ____________________

MARK: / 60

Time Allowed: 90 minutes

Topics: Inequalities, Graphs, Ratios, Integration, Polynomials, Trigonometry, Parametric Equations, Circle Geometry & Induction.

Directions:

- There are **five** questions on this paper
- Marks have been allocated for each question
- Answer each questions on a separate page
- Show all necessary working
- Marks may not be awarded for careless or badly arranged work

Total marks – 60
Attempt Questions 1–5
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

Question 1 (12 marks) Use a SEPARATE writing booklet.

(a) Let A be the point $(-3,8)$ and let B be the point $(5,-6)$. Find the coordinates of the point P that divides the interval AB internally in the ratio $1:3$. **2**

(b) When the polynomial $P(x)=2x^3-x^2+px-1$ is divided by $(x-3)$, the remainder is 2. Find p. **2**

(c) Solve $\dfrac{2}{x+5}\le 1$. **2**

(d) Use the substitution $u=x-1$ to evaluate $\displaystyle\int_2^4 \frac{x}{(x-1)^2}\,dx$. **2**

(e) Show that $\cos 2x = 2\cos^2 x - 1$. Hence evaluate $\displaystyle\int_0^{\pi} \cos^2 3x\,dx$. **2**

(f) Sketch the graph of $y=2\sin^{-1}3x$ showing clearly the domain and range of the function as well as any intercepts. **2**

Marks

Question 2 (12 marks) Use a SEPARATE writing booklet.

(a) Find $\frac{d}{dx}\left(3x^2\cos^{-1}x\right)$ **2**

(b) Solve the equation $\sin 2\theta = \sqrt{2}\cos\theta$ for $0 \le \theta \le 2\pi$. **2**

(c) Use mathematical induction to prove that

$$1+6+15+\ldots+n(2n-1)=\frac{1}{6}n(4n-1)(n+1)$$ **4**

for all positive integers n.

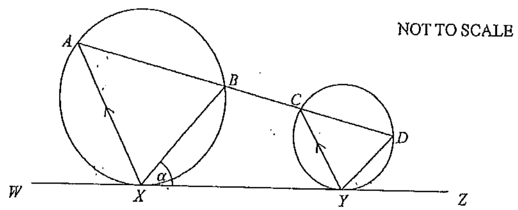
(d) Evaluate $\lim_{x\to 0}\frac{2\sin 2x}{x}$. **2**

(e) Find the obtuse angle between the lines $3x-y+5=0$ and $2x+3y-1=0$. Give your answer correct to the nearest degree. **2**

Marks

Question 3 (12 marks) Use a SEPARATE writing booklet.

(a)



In the above diagram, WZ is a common tangent to the two circles and AX is parallel to CY. AD is a straight line through B and C on the circles as shown. Let $\angle BXY = \alpha$.

Copy the diagram into your Writing Booklet.

i. Explain why BX is parallel to DY. **3**

ii. Show that $BCYX$ is a cyclic quadrilateral. **1**

(b) If α, β and γ are the roots of the equation $x^3+2x^2-11x-12=0$,

find:

(i) $\alpha+\beta+\gamma$ **1**

(ii) $\alpha\beta+\alpha\gamma+\beta\gamma$ **1**

(iii) $\alpha\beta\gamma$ **1**

(iv) $\frac{2}{\alpha\beta}+\frac{2}{\alpha\gamma}+\frac{2}{\beta\gamma}$ **1**

(c) If $t=\tan\frac{\theta}{2}$, show that $\sin\theta=\frac{2t}{1+t^2}$. **2**

(d) Find $\int\frac{2}{\sqrt{1-9x^2}}\,dx$ **2**

Marks

Question 4 (12 marks) Use a SEPARATE writing booklet.

(a) Prove that $\dfrac{1-\cos 2\theta-\sin\theta}{\sin 2\theta-\cos\theta}=\tan\theta$ 3

(b) $P(6p,3p^2)$ is a point on the parabola $x^2=12y$. 9

(i) Show that equation of the normal at P is:

$x+py=6p+3p^3$

(ii) Q is the point where this normal meets the $y-$axis.

Find the coordinates of Q.

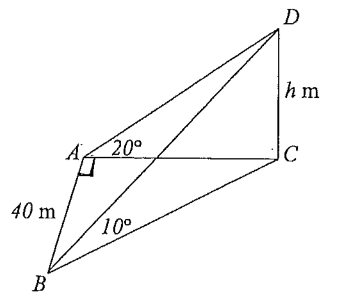
(iii) Show the coordinates of R which divides PQ externally in the ratio 2:1 is $(-6p,3p^2+12)$.

(iv) Find the Cartesian equation of the locus of R.

Marks

Question 5 (12 marks) Use a SEPARATE writing booklet.

(a)



A vertical flagpole CD of height h metres stands with its base C on horizontal ground. A is a point on the ground due West of C and B is a point on the ground 40 metres due South of A. From A and B the angles of elevation of the top D of the flagpole are $20°$ and $10°$ respectively.

i. Show that AC=$\dfrac{h}{\tan 20}$ and BC=$\dfrac{h}{\tan 10}$. 1

ii. Hence, show that $h=\sqrt{\dfrac{1600}{\cot^2 10-\cot^2 20}}$. 2

(b) Prove by Mathematical Induction that $3^{3n}+2^{n+2}$ is divisible by 5, for all positive integers n. 4

(c) Consider the function given by $f(x)=\dfrac{e^x}{x-1}$.

i. Determine all vertical and horizontal asymptotes of the graph $y=f(x)$. 2

ii. Find any stationary point(s) and sketch the graph of $y=f(x)$ including any intercepts with the coordinate axes. 3

1

(a) $(-3,8)$ $(5,-6)$

$1:3$

$m = \dfrac{3(-3)+(1)(5)}{1+3}, \dfrac{3(8)+1(-6)}{1+3}$ (1)

$= \left(-\dfrac{4}{4}, \dfrac{18}{4}\right)$

$= \left(-1, 4\tfrac{1}{2}\right)$ (1)

(b) $P(3) = 2(3)^3 - (3)^2 + p(3) - 1$ (1)

$2 = 54 - 9 + 3p - 1$

$2 = 44 + 3p$

$3p = -42$

$p = -14$ (1)

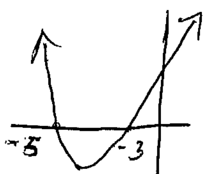
(c) $\dfrac{2}{x+5} \le 1$ $x \ne -5$

$2(x+5) \le (x+5)^2$

$(x+5)^2 - 2(x+5) \ge 0$ (1)

$(x+5)[x+5-2] \ge 0$

$(x+5)(x+3) \ge 0$



$x < -5$ & $x \ge -3$ (1)

(d) $\displaystyle\int_2^4 \frac{x}{(x-1)^2}\,dx$.

$x = u+1$

$u = x-1$

$\dfrac{du}{dx} = 1$

$du = dx$

$x=4$ $u=3$

$x=2$ $u=1$

$\displaystyle\int_1^3 \frac{u+1}{u^2}\,du$. (1)

$\displaystyle\int_1^3 (u^{-1}+u^{-2})\,du$

$= \left[\ln u + \dfrac{u^{-1}}{-1}\right]_1^3$

$= \left[\ln 3 + -\dfrac{1}{3}\right] - \left[\ln 1 - 1\right]$.

$= \ln 3 + \dfrac{2}{3}$ (1)

(e) $\cos(x+x) = \cos^2 x - \sin^2 x$

$= \cos^2 x - (1-\cos^2 x)$

$= 2\cos^2 x - 1$

$\cos^2 x = \frac{1}{2}(\cos 2x + 1)$ (1)

$\cos^2 3x = \frac{1}{2}(\cos 6x + 1)$

$\dfrac{1}{2}\displaystyle\int_0^{\pi} \cos 6x + 1\,dx$

$= \dfrac{1}{2}\left[\dfrac{1}{6}\sin 6x + x\right]_0^{\pi}$

$= \dfrac{1}{2}\left[\dfrac{1}{6}\sin 6\pi + \pi\right] - \dfrac{1}{2}\left[\dfrac{1}{6}\sin 0 + 0\right]$

$= \dfrac{\pi}{2}$. (1)

(f) $y = 2\sin^{-1} 3x$

$-1 \le 3x \le 1$

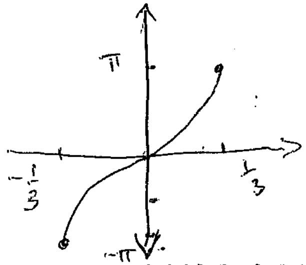
$-\frac{1}{3} \le x \le \frac{1}{3}$

$y = \sin^{-1} 3x$ Range of

$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

$\therefore$ $y = 2\sin^{-1} 3x$ has a range of

$-\pi \le y \le \pi$



(2)

2 (a) $\dfrac{d}{dx}(3x^2 \cos^{-1} x)$ (1)

$= 6x\cos^{-1} x + 3x^2 \times \dfrac{-1}{\sqrt{1-x^2}}$

$= 6x\cos^{-1} x - \dfrac{3x^2}{\sqrt{1-x^2}}$ (1)

(b) $\sin 2\theta = \sqrt{2}\cos\theta$

$2\sin\theta\cos\theta = \sqrt{2}\cos\theta$.

$\sqrt{2}\sin\theta\cos\theta - \cos\theta = 0$

$\cos\theta(\sqrt{2}\sin\theta - 1) = 0$ (1)

$\cos\theta = 0$ $\sin\theta = \dfrac{1}{\sqrt{2}}$

$\theta = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$ $\theta = \dfrac{\pi}{4},$

(1)

(c) ① Prove true $n=1$

LHS $= 1$ RHS $= \frac{1}{6}(1)(3)(2)$

$= 1$

$\therefore$ LHS = RHS. (1)

② Assume true for $n=k$.

$1+6+15+\ldots+k(2k-1) = \frac{1}{6}k(4k-1)($

③ Prove true for $n=k+1$

$1+6+15+\ldots+k(2k-1)+(k+1)(2k+1)$

$= \frac{1}{6}(k+1)(4k+3)(k+2$

(1)

LHS $= \frac{1}{6}k(4k-1)(k+1) + (k+1)(2k$

$= \frac{1}{6}(k+1)\left[k(4k-1) + 6(2k+1)\right]$

$= \frac{1}{6}(k+1)(4k^2 - k + 12k + 6)$

$= \frac{1}{6}(k+1)(4k^2 + 11k + 6)$

$= \frac{1}{6}(k+1)(4k+3)(k+2)$. (1)

$=$ RHS.

④ If true for $n=k$, then true for $n=k+1$. (1)

Since true for $n=1$, then true for $n=2$ etc.

$\therefore$ true for all positive integers

(d) $2\lim_{x\to 0}\frac{\sin 2x}{x}$

$$2\lim_{x\to 0} 2x\,\frac{\sin 2x}{2x} \quad (1)$$

$$= 4\times 1$$

$$= 4. \quad (1)$$

(e)

$3x-y+5=0 \qquad 2x+3y-1=0$

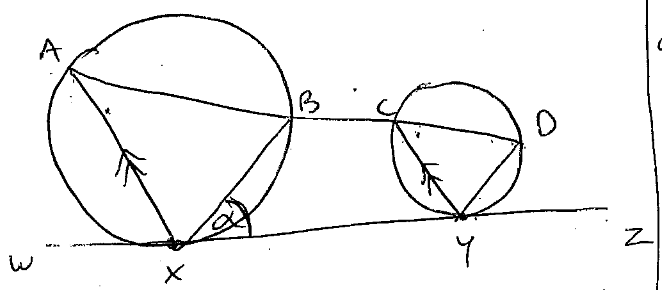
$m_1 = 3 \qquad m_2 = -\frac{2}{3}. \quad (1)$

$$\tan\theta = \left|\frac{3+\frac{2}{3}}{1+3\left(-\frac{2}{3}\right)}\right|$$

$\theta = 75$

$\therefore$ obtuse angle $= 105^\circ$ (1)

(3)



(i) $\angle XAB = \alpha$ (∠ in the alternate segment equals the ∠ between chord & tangent).

$\therefore \angle YCD = \alpha$ (corresponding ∠'s).

$\therefore \angle DYZ = \alpha$ (∠ in the alternate segment equals the ∠ between the chord & tangent)

$\therefore BX \parallel DY$ (... ∠'s)

ii) $\angle DCY = \alpha$

$\therefore \angle BCY = 180-\alpha$ (straight line)

$\therefore BCYX$ is cyclic quad.

opposite ∠'s add up to 180°.

(b) (i) $\alpha+\beta+\gamma = -\frac{b}{a}$

$= -\frac{2}{1}$

$= -2.$ (1)

ii) $\alpha\beta+\alpha\gamma+\beta\gamma = \frac{c}{a}$

$= -11$ (1)

iii) $\alpha\beta\gamma = -\frac{d}{a}$

$= 12$ (1)

iv) $\frac{2\gamma+2\beta+2\alpha}{\alpha\beta\gamma}$

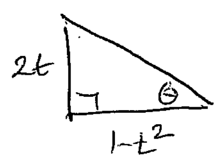
$= \frac{2(\gamma+\beta+\alpha)}{\alpha\beta\gamma}$

$= \frac{2(-2)}{12}$

$= -\frac{1}{3}.$ (1)

(c) $t = \tan\frac{\theta}{2}.$

$\tan\theta = \frac{2t}{1-t^2}.$



$c^2 = 4t^2 + (1-t^2)^2$ (1)

$= 4t^2+1-2t^2+t^4$

$= t^4+2t^2+1$

$c^2 = (t^2+1)^2$

$\therefore c = t^2+1$

$\therefore \sin\theta = \frac{2t}{1+t^2}$ (1)

(d) $\int \frac{2}{\sqrt{1-9x^2}}\,dx$

$\int \frac{2}{\sqrt{9(\frac{1}{9}-x^2)}}\,dx$

$= \frac{2}{3}\int \frac{1}{\sqrt{(\frac{1}{3})^2-x^2}}\,dx$ (1)

$= \frac{2}{3}\sin^{-1}\frac{x}{\frac{1}{3}} + c$

$= \frac{2}{3}\sin^{-1}3x + c$ (1)

(4) (a)

$$LHS = \frac{1-(1-2\sin^2\theta)-\sin\theta}{\cos\theta(2\sin\theta-1)}$$ (1)

$$= \frac{2\sin^2\theta-\sin\theta}{\cos\theta(2\sin\theta-1)}$$ (1)

$$= \frac{\sin\theta\cancel{(2\sin\theta-1)}}{\cos\theta\cancel{(2\sin\theta-1)}}$$ (1)

$= \tan\theta$

$= RHS$.

(b) $y = \frac{x^2}{12}$

i) $y' = \frac{x}{6}$ (1)

$m_1 = \frac{6p}{6}$

$m_1 = p$ (1)

$\therefore m_2 = -\frac{1}{p}$.

$y-3p^2 = -\frac{1}{p}(x-6p)$

$-py+3p^3 = x-6p$

$x+py = 6p+3p^3$ (1)

ii) $x = 0$

$py = 6p+3p^3$ (1)

$y = 6+3p^2$

$Q(0, 6+3p^2)$ (1)

iii) $P(6p, 3p^2)$ $Q(0, 6+3p^2)$

$2 : -1$ (1)

$$R = \left(\frac{(-1)(6p)+(2)(0)}{2-1}, \frac{(-1)(3p^2)+(2)(6+3p^2)}{2-1}\right)$$

$= (-6p, -3p^2+12+6p^2)$

$= (-6p, 3p^2+12)$ (1)

iv) $x = -6p$ $y = 3p^2+12$

$p = \frac{x}{-6}$

$\therefore y = 3\left(\frac{x^2}{36}\right)+12$ (1)

$y = \frac{x^2}{12}+12$

$12y = x^2+144$

$x^2 = 12(y-12)$ (1)

(5) (a)

i) $\tan 20 = \frac{h}{AC}$ $\tan 10 = \frac{h}{BC}$ (1)

$\therefore AC = \frac{h}{\tan 20}$ $\therefore BC = \frac{h}{\tan 10}$

ii) $$\left(\frac{h}{\tan 10}\right)^2 = 40^2 + \left(\frac{h}{\tan 20}\right)^2$$ (1)

$1600 = h^2(\cot^2 10 - \cot^2 20)$ (1)

$$h = \sqrt{\frac{1600}{\cot^2 10-\cot^2 20}}$$

(b) Prove true for $n=1$

$3^3+2^3 = 27+8$

$= 35$

which is divisible by 5 (1)

Assume true for $n=k$.

$3^{3k}+2^{k+2} = 5M$, where M is an integer

$3^k = 5M - 2^{k+2}$

Prove true for $n=k+1$

$3^{3k+3} + 2^{k+3}$

$3^3 . (5M - 2^{k+2}) + 2.2^{k+2}$ (1)

$27.5M - 27.2^{k+2} + 2.2^{k+2}$

$27.5M - 25.2^{k+2}$.

$5[27M - 5.2^{k+2}]$ (1)

which is divisible by 5.

Since true for n=k then true for n=k+1
Since true for n=1 then true for n=2 etc. (1)

$\therefore$ true for all positive integers.

(c) $f(x) = \dfrac{e^x}{x-1}$ $\begin{matrix} u \\ v \end{matrix}$ $u'v - v'u$

(i) $x \neq 1$ (1)

$x \to \infty$ $f(x) \to \infty$
$x \to -\infty$ $f(x) \to 0^-$

$\therefore y = 0$ is a horizontal asymptote. (1)

(ii) $f'(x) = \dfrac{(x-1)e^x - e^x}{(x-1)^2}$

$= \dfrac{e^x(x-2)}{(x-1)^2}$

$f'(x) = 0$.
when $x = 2$

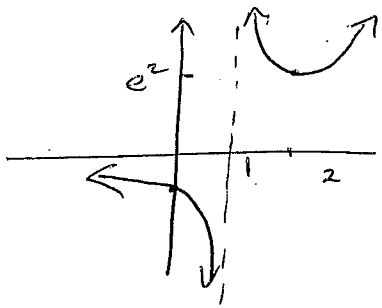
x	2^-	2	2^+
$f'(x)$	$-$	0	$+$

(1)

$\setminus \; - \; /$

$\therefore (2, e^2)$ is a min

when $x = 0$
$y = -1$ (1)



(1)