J.M.J.

MARCELLIN COLLEGE RANDWICK

EXTENSION II

MATHEMATICS

2011

Weighting: 20% (Assessment Mark)

NAME:

MARK: / 60

Time Allowed: 90 minutes

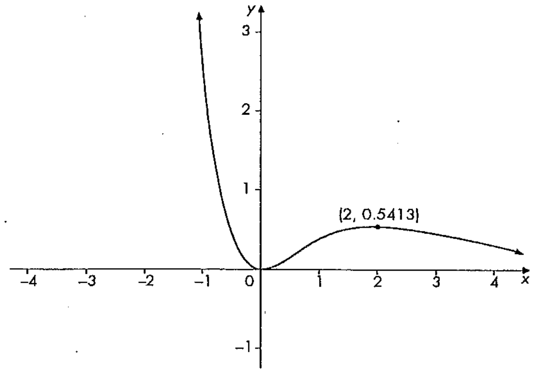
Topics: Graphs , Complex Numbers & Polynomials

Directions:

- There are **two** questions on this paper
- Marks have been allocated for each question
- Answer each questions on a separate page
- Show all necessary working
- Marks <u>may not</u> be awarded for careless or badly arranged work

Question 1(31 marks) **Marks**

(a) The diagram shows the graph of the function $y = f(x)$.



Draw separate one-third page sketches of the graphs of the following:

i. $y = f(-x)$ **1**

ii. $y = f\left(|x|\right)$ **1**

iii. $y = [f(x)]^{-1}$ **2**

iv. $y = \sqrt{f(x)}$ **2**

(b) Find the gradient of the curve $2x^3 - x^2y + y^3 = 1$ at the point $(-2,3)$. **3**

(c) The polynomial equation $P(x) = 2x^4 - 5x^3 + 3x^2 + x - 1$ has a root of multiplicity 3. Find all the zeros of this polynomial. **3**

(d) Factorise $x^4 + x^2 - 12$ over the complex field. **2**

(e) Given that $1, w, w^2$ are the cube roots of unity, i.e. the roots of $z^3 = 1$, simplify $(1-w)(1-w^2)(1-w^7)(1-w^{11})$. **2**

(f)

(i) Solve $z^5 + 1 = 0$ by De Moivre's Theorem, leaving your solutions in modulus-argument form. **2**

(ii) Prove that the solutions of $z^4 - z^3 + z^2 - z + 1 = 0$ are the non-real solutions of $z^5 + 1 = 0$. **1**

(iii) Show that if $z^4 - z^3 + z^2 - z + 1 = 0$ where $z = cis\,\theta$ then $4\cos^2\theta - 2\cos\theta - 1 = 0$. **3**

Hint: $z^4 - z^3 + z^2 - z + 1 = 0 \Rightarrow z^2 - z + 1 - \frac{1}{z} + \frac{1}{z^2} = 0$

(iv) Hence find the exact value of $\sec\frac{3\pi}{5}$. **2**

(g) The zeros of $x^3 + px^2 + qx + r$ are α, β and γ (where p, q and r are real numbers).

(i) Find $\alpha\beta + \alpha\gamma + \beta\gamma$. **1**

(ii) Find $\alpha^2 + \beta^2 + \gamma^2$. **1**

(iii) Find a cubic polynomial with integer coefficients whose zeros are $2\alpha, 2\beta$ and 2γ. **2**

(h) Given $P(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1 x + a_0$, where $a_0, a_1, \ldots, a_n$ are real prove that if α is a root of $P(x)$ then $\overline{\alpha}$ is also a root. **3**

Question 2 (29 marks)

(a) Let $z = 3 + 2i$ and $w = 2 - i$. Find, in the form $x + iy$:

(i) $z + 4w$ **1**

(ii) $z^2 w$ **1**

(iii) $\frac{2}{\overline{w}}$ **2**

(b) Sketch the region in the complex plane where the inequalities

$|z + 1 - i| < 2$ and $0 < \arg(z + 1 - i) < \frac{3\pi}{4}$ hold simultaneously. **3**

(c) Let $z = 2 + 2i$.

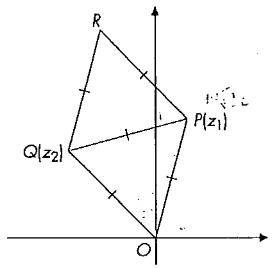
(i) Express z in the form $r(\cos\theta + i\sin\theta)$. **2**

(ii) Hence express z^{16} in the form $a + ib$ where a and b are real. **2**

(d) If $1 - 2i$ is root of $x^3 - Ax^2 + Bx - 25 = 0$, where A and B are real, find the values other roots and the values of A and B. **2**

(e) If z is a complex number $x + iy$, where x and y are real, show that $z^2 - (\overline{z})^2$ is purely imaginary. **2**

(f)



The point P on the Argand diagram corresponds to the complex number $z_1 = 1 + \sqrt{2}i$. The triangles OPQ and PQR are equilateral triangles.

Show that $z_2 = \dfrac{1-\sqrt{6}}{2} + i\left(\dfrac{\sqrt{3}+\sqrt{2}}{2}\right)$. **2**

(g) Given $z = \dfrac{\sqrt{3}+i}{1+i}$,

(i) Find the argument and modulus of z. **2**

(ii) Find the smallest positive integer n such that z^n is real. **1**

(h) The complex numbers represented by 0, z, $z + \dfrac{1}{z}$, and $\dfrac{1}{z}$ form a parallelogram. **2**
Describe the locus of z if this parallelogram is a rhombus.

(i) By applying De Moivre's theorem and also expanding $(\cos\theta + i\sin\theta)^3$, **3**
express $\cos 3\theta$ as a polynomial in $\cos\theta$.

(j) Prove by induction that $(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$ for all **4**
integers $n \geq 1$.

SOLUTIONS
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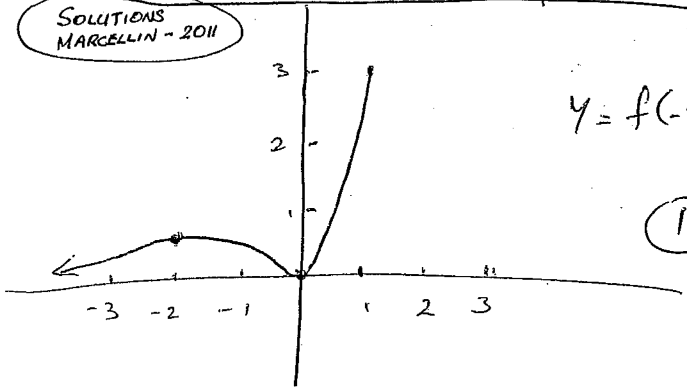
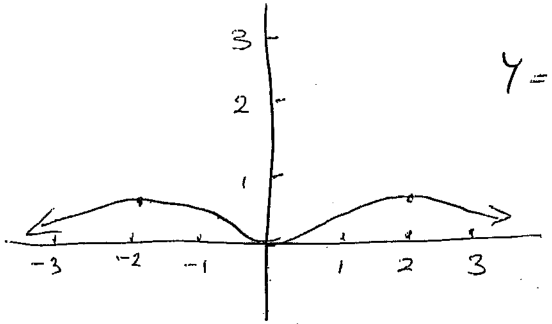
(a) i)



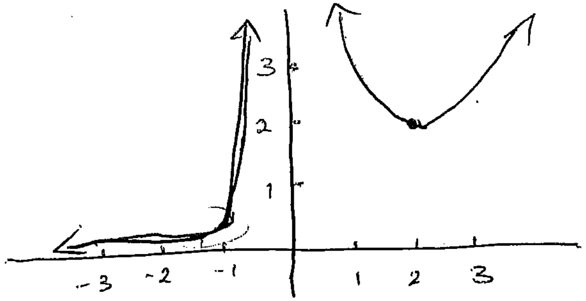
$y = f(-x)$ (1)

ii)



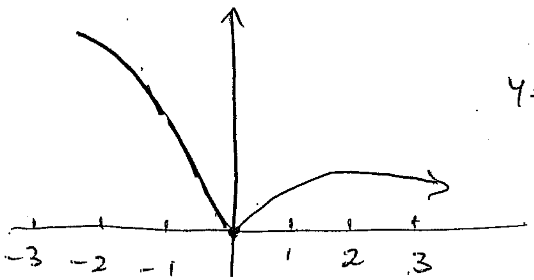
$y = f(|x|)$ (1)

iii)



$y = [f(x)]^{-1}$ (2)

(iv)



$y = \sqrt{f(x)}$ (2)

(b) $6x^2 - 2xy + (-x^2)\dfrac{dy}{dx} + 3y^2 \cdot \dfrac{dy}{dx} = 0$ (1)

$\dfrac{dy}{dx}(-x^2 + 3y^2) = 2xy - 6x^2$

$\dfrac{dy}{dx} = \dfrac{2xy - 6x^2}{-x^2 + 3y^2}$ (1)

at $(-2, 3)$

$\dfrac{dy}{dx} = \dfrac{2(-2)(3) - 6(-2)^2}{-(-2)^2 + 3(3)^2}$

$= -\dfrac{36}{23}$ (1)

(c) $P(x) = 2x^4 - 5x^3 + 3x^2 + x - 1$

$P'(x) = 8x^3 - 15x^2 + 6x + 1$

$P''(x) = 24x^2 - 30x + 6$

$6(4x^2 - 5x + 1) = 0$

$6(x-1)(4x-1) = 0$ (1)

$x = 1 \quad x = \frac{1}{4}$

$P''(1) = 0$

$P'(1) = 0$

$P(1) = 0$

$P(x) = (x-1)^3(2x+1)$ (1)

$\therefore$ 1 is a triple root

$\therefore$ roots are $1, 1, 1$ & $-\frac{1}{2}$. (1)

(d) $x^4 + x^2 - 12 = (x^2+4)(x^2-3)$ (1)

$= (x+2i)(x-2i)(x+\sqrt{3})(x-\sqrt{3})$ (1)

(e) $(1-w)(1-w^2)(1-w^7)(1-w^{11})$

$= (1-w)(1-w^2)(1-w)(1-w^2)$ (1)

$= (1-w)^2(1-w^2)^2$

$= (1-2w+w^2)(1-2w^2+w^4)$

$= (1+w+w^2-3w)(1-2w^2+w)$

$= (-3w)(-w^2-2w^2)$

$= (-3w)(-3w^2)$ (1)

$= 9$

$w^2+w+1=0$

$w^2 = -w-1$

$z^3-1 = (z-1)(z^2+z+1)$

$0 = (z-1)(z^2+z+1)$

$\therefore z^2+z+1=0.$

(f) $z^5+1=0$

i) $z^5=-1$

$z^5 = \text{cis}(\pi+2k\pi)$

$z = \text{cis}\left(\frac{\pi(2k+1)}{5}\right)$ (1)

$z_1 = \text{cis}\,\frac{\pi}{5}$

$z_2 = \text{cis}\,\frac{3\pi}{5}$

$z_3 = \text{cis}\,\pi = -1$

$z_4 = \text{cis}\,-\frac{\pi}{5}$ (1)

$z_5 = \text{cis}\,-\frac{3\pi}{5}$

ii) $z^5+1 = (z+1)(z^4-z^3+z^2-z+1)$

The only real root $z=-1$

All 4 roots are non real. (1)

iii) $z^4-z^3+z^2-z+1=0$

$\div z^2$

$z^2-z+1-\frac{1}{z}+\frac{1}{z^2}=0$

$z^2+\frac{1}{z^2}-\left(z+\frac{1}{z}\right)+1=0$ (1)

$2\cos 2\theta - 2\cos\theta + 1 = 0.$ (1)

$2(2\cos^2\theta-1)-2\cos\theta+1=0$

$4\cos^2\theta-2\cos\theta-1=0$ (1)

where $z = \cos\theta + i\sin\theta$.

(iv) $z_2 = \text{cis}\,\frac{3\pi}{5}$ is a solution of

$z^4-z^3+z^2-z+1=0$, So $\theta=\frac{3\pi}{5}$

is a solution of $4\cos^2\theta-2\cos\theta-1=0$

$\cos\theta = \dfrac{2\pm\sqrt{4-4(4)(-1)}}{8}$ (1)

$= \dfrac{1\pm\sqrt{5}}{4}$

$\cos\frac{3\pi}{5} < 0.$

$\therefore \sec\frac{3\pi}{5} = \dfrac{4}{1-\sqrt{5}}$ (1)

$= -(1+\sqrt{5})$

(g) $x^3 + px^2 + qx + r$

(i) $\alpha\beta + \alpha\gamma + \beta\gamma = q.$ (1)

(ii) $\alpha^2 + \beta^2 + \gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma$

$= (-p)^2 - 2(q)$

$= p^2 - 2q$ (1)

(iii) If $2\alpha + 2\beta + 2\gamma$ are zeros then

$\left(\frac{y}{2}\right)^3 + p\left(\frac{y}{2}\right)^2 + q\left(\frac{y}{2}\right) + r$ (1)

$y^3 + 2py^2 + 4qy + 8r$ (1)

(h) $P(\alpha) = 0$

$\overline{a_n\alpha^n + a_{n-1}\alpha^{n-1} + \ldots + a_1\alpha + a_0} = \overline{0}$

$\overline{a_n\alpha^n} + \overline{a_{n-1}\alpha^{n-1}} + \ldots + \overline{a_1\alpha} + \overline{a_0} = 0$ (1)

$\overline{z_1 z_2} = \overline{z_1} \times \overline{z_2}$ & $a_0 = \overline{a_0}$

$\therefore \overline{a_n} \times \overline{\alpha^n} + \overline{a_{n-1}} \times \overline{\alpha^{n-1}} + \ldots + \overline{a_1} \times \overline{\alpha} + a_0 = 0$ (1)

$\overline{\alpha^n} = \overline{\alpha} \times \overline{\alpha} \times \overline{\alpha} \ldots$

$a_n \times (\overline{\alpha})^n + a_{n-1} \times (\overline{\alpha})^{n-1} + \ldots + a_1\overline{\alpha} + a_0 = 0.$

$\therefore P(\overline{\alpha}) = 0$ (1)

$\therefore \overline{\alpha}$ is a root.

2 (a) i) $3+2i + 4(2-i)$

$= 3 + 2i + 8 - 4i$ (1)

$= 11 - 2i$

ii) $(3+2i)^2(2-i)$

$= (9 + 12i - 4)(2-i)$

$= (5 + 12i)(2-i)$

$= 10 + 24i - 5i + 12$ (1)

$= 22 + 19i$

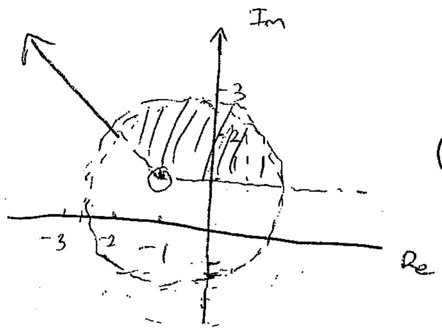
iii) $\frac{2}{2+i} \times \frac{2-i}{2-i}$ (1)

$= \frac{4-2i}{4+1}$

$= \frac{4-2i}{5}$ $\quad \frac{4}{5} - \frac{2i}{5}$ (1)

(b) $|z - (i-1)| < 2$

$0 < \arg[z - (i-1)] < \frac{3\pi}{4}$ (1)



(2)

(c)

(i) $z = 2 + 2i$

$|z| = \sqrt{2^2 + 2^2}$

$= 2\sqrt{2}.$ (1)

$\tan\theta = \frac{2}{2}.$

$\theta = \frac{\pi}{4}.$

$z = 2\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$ (1)

ii) $z^{16} = (2\sqrt{2})^{16}\left(\cos\frac{16\pi}{4} + i\frac{16\pi}{4}\right)$

$= 8^8(\cos\pi + i\sin\pi)$

$= 8^8[1 + i(0)]$

$= 8^8$ (1)

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(d) $1-2i$ then $1+2i$ is another root $\rightarrow$ coeff are real

let $\alpha = 3^{rd}$ root

$(1-2i)(1+2i)(\alpha) = 25$

$\therefore 5\alpha = 25$

$\alpha = 5$

$1-2i + 1+2i + 5 = A$

$\therefore A = 7$ (1)

$(1+2i)(1-2i) + 5(1+2i) + 5(1-2i) = B$

$5 + 5 + \cancel{10i} + 5 - \cancel{10i} = B$

$\therefore B = 15$ (1)

(e) $z^2 - (\bar{z})^2 = (x^2 - y^2 + i2xy) - (x^2 - y^2 - i2xy)$ (1)

$(x+iy)^2$

$(x-iy)^2$ $= i4xy$ (1)

which is purely imaginary

(f) $\triangle OPQ$ is quilateral ie $\angle POQ = \frac{\pi}{3}$

The coordinates of $z_2 = z_1 \times \left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$ (1)

$= (1 + i\sqrt{2})\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$

$= \frac{1-\sqrt{6}}{2} + i\left(\frac{\sqrt{3}+\sqrt{2}}{2}\right)$ (1)

(g) (i) $z = \frac{\sqrt{3}+i}{1+i} \times \frac{(1-i)}{(1-i)}$

$= \frac{\sqrt{3} + (1-\sqrt{3})i + 1}{2}$

$= \frac{1+\sqrt{3}}{2} + \frac{(1-\sqrt{3})}{2}i$ (1)

$|z| = \sqrt{\left(\frac{1+\sqrt{3}}{2}\right)^2 + \left(\frac{1-\sqrt{3}}{2}\right)^2}$

$= \sqrt{\frac{1+2\sqrt{3}+3+1-2\sqrt{3}+3}{4}}$

$= \sqrt{2}$

$\arg(z) = \tan\theta = \frac{\frac{1-\sqrt{3}}{2}}{\frac{1+\sqrt{3}}{2}}$ (1)

$\theta = -\frac{\pi}{12}$

(ii) z^n is real

$z = \sqrt{2}\operatorname{cis}\left(-\frac{\pi}{12}\right)$

$z^n = (\sqrt{2})^n \operatorname{cic}\left(-\frac{n\pi}{12}\right)$

$\therefore \sin\left(-\frac{n\pi}{12}\right) = 0$

$z^{12} = (\sqrt{2})^2 \operatorname{cis}(-\pi)$

$= -2^6$

$\therefore n = 12.$ (1)

(h) Adjacent sides are equal

$$|z| = \left|\frac{1}{z}\right| \qquad 1$$

$$|z|^2 = 1$$

$$|z| = 1 \qquad 1$$

Locus of z is a circle centre $(0,0)$ & radius 1.

(i) $(\cos\theta + i\sin\theta)^3 = \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin$ 1

& $(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$

$$\therefore \cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta \qquad 1$$

$$= \cos^3\theta - 3\cos\theta(1-\cos^2\theta) \qquad 1$$

$$= 4\cos^3\theta - 3\cos\theta$$

(j) $(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$

1 $n = 1$

LHS $= \cos\theta + i\sin\theta$ 1

$=$ RHS

2 Assume true for $n=k$

$(\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta$ 1

3 Prove true for $n=k+1$

LHS $= (\cos\theta + i\sin\theta)^{k+1}$

$= (\cos\theta + i\sin\theta)^k \cdot (\cos\theta + i\sin\theta)$ 1

$= (\cos k\theta + i\sin k\theta) \cdot (\cos\theta + i\sin\theta)$

$= (\cos k\theta . \cos\theta - \sin k\theta . \sin\theta) + i(\cos k\theta\sin\theta + \sin k\theta .$ $\cos$

$= \cos(k+1)\theta + i\sin(k+1)\theta$ = RHS