Student ID: ______ __________

KAMBALA

Mathematics Extension 1

HSC Assessment Task 1

February 2011

Integration and Polynomials

Time Allowed: 50 minutes working time

INSTRUCTIONS

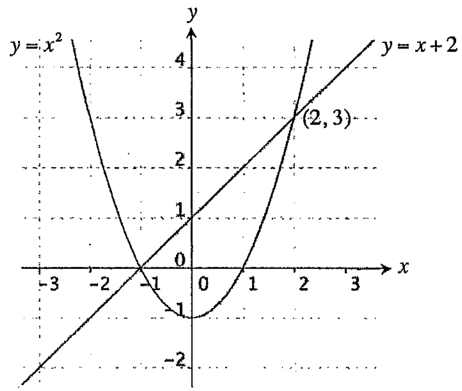
- This examination contains 3 questions of equal value. Marks for each part of each question are shown.
- Answer all questions on the writing paper provided. **Start each question on a new page.**
- Approved calculators may be used.
- Show all necessary working.
- Marks may be deducted for careless or badly arranged work.
- More marks may be awarded for questions involving higher order thinking skills.
- A table of standard integrals is attached.

Question 1 (12 Marks) *(Start a new page.)* **Marks**

(a) Find $\int (2x-5)^7 \, dx$. **1**

(b) Find the equation of the curve $y = f(x)$ given $f'(x) = \sqrt{x} + x^2$ and $f(1) = 4$. **3**

(c) The graph below shows the curve $y = x^2$ and the line $y = x + 2$. Find the area enclosed between the two curves. **3**



(d) Find $\int x^2 (x^3 - 1)^4 \, dx$ using the substitution $u = x^3 - 1$. **2**

(e) Find the value of a, where $a > 3$, if $\int_3^a (2x+4)\, dx = 39$. **3**

Question 2 (12 Marks) *(Start a new page.)* **Marks**

(a) Find the remainder when $P(x)=x^3-6x^2+4x+3$ is divided by $(x+2)$. 1

(b) State the leading term of the polynomial $P(x)=(4x^3-2)^5$. 1

(c) Given that $(x-3)$ and $(x+2)$ are factors of $P(x)=x^3-6x^2+px+q$, find the values of p and q. 3

(d) If α, β and γ are the roots of $2x^3-x^2+6x+2=0$, find:

(i) $\alpha\beta\gamma$ 1

(ii) $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}$ 2

(e) (i) Express $P(x)=x^3-7x+6$ as a product of its linear factors. 3

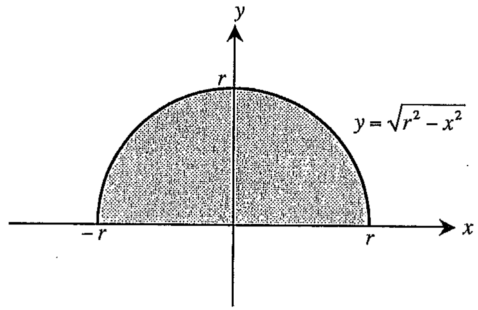
(ii) Hence, or otherwise, sketch the graph of $P(x)$ clearly showing all intercepts. 1

Question 3 (12 Marks) *(Start a new page.)* **Marks**

(a) (i) Show that the polynomial $P(x)=3x^3+7x^2-4x+3$ has one root in the interval $-3\le x\le -2$. 1

(ii) By using one application of the Halving the Interval method, determine a more accurate approximation for this root. 2

(b) The graph below shows a semi-circle with equation $y=\sqrt{r^2-x^2}$.



By rotating the shaded semi-circular area about the x-axis, verify that the volume of the solid of revolution formed is given by $V=\frac{4}{3}\pi r^3$; that is, the volume of a sphere. 3

(c) The equation $x^3-mx+2=0$ has two equal roots.

(i) Write expressions for the sum of the roots and the product of the roots. 1

(ii) Hence, or otherwise, find the value of m. 2

(d) By using the substitution $x=u^2-2$ (where $u>0$), find the area bounded by the curve $y=\frac{1}{\sqrt{2+x}}$, the x-axis and the lines $x=1$ and $x=3$. 3

End of Assessment Task

12

Well done!

Question One

a) $\int (2x-5)^7\,dx = \frac{(2x-5)^8}{2\cdot 8} + C$

$= \frac{(2x-5)^8}{16} + C$

1

b) $f'(x) = x^{\frac{1}{2}} + x^2$

$\int (x^{\frac{1}{2}} + x^2)\,dx = \frac{x^{\frac{3}{2}}}{3/2} + \frac{x^3}{3} + C$

$= \frac{2x^{\frac{3}{2}}}{3} + \frac{1}{3}x^3 + c$

$f(1) = 4$

$\frac{2}{3} + \frac{1}{3} + C = 4$

$1 + C = 4$

$\boxed{C = 3}$

$\therefore f(x) = \frac{2}{3}x^{\frac{3}{2}} + \frac{1}{3}x^3 + 3$

3

c) $\int_{-1}^{2} (x+2)\,dx - \int_{-1}^{2} (x^2)\,dx$

$= \left[\frac{x^2}{2} + 2x\right]_{-1}^{2} - \left[\frac{x^3}{3}\right]_{-1}^{2}$

$= \left[\left(\frac{2^2}{2} + 2(2)\right) - \left(\frac{(-1)^2}{2} + 2(-1)\right)\right] - \left(\frac{2^3}{3} - \frac{(-1)^3}{3}\right)$

$= 6\frac{3}{2} - 3$?

$= 3\frac{3}{2}$ units^2 ? Write as $4\frac{1}{2}$

3

d) $\int x^2(x^3-1)^4\,dx$

$u = x^3 - 1$

$\frac{du}{dx} = 3x^2$

$\frac{du}{3} = x^2\,dx$

$\to \frac{1}{3}\int (u^4)\,du$

$\frac{1}{3}\left[\frac{u^5}{5} + c\right]$

$= \frac{u^5}{15} + C$

$= \frac{(x^3-1)^5}{15} + c$

2

Q: 1 (cont.)

e) $\int_{3}^{a} (2x+4)\,dx = 39$

$\left[\frac{2x^2}{2} + 4x\right]_{3}^{a} = 39$

$\left[x^2 + 4x\right]_{3}^{a} = 39$

$a^2 + 4a - (9+12) = 39$

$a^2 + 4a = 60$

$a^2 + 4a - 60 = 0$

$(a+10)(a-6) = 0$

$\therefore a = -10,\ a = 6$

BUT, given that $a > 3$,

$\boxed{\therefore a = 6}$

1	60
2	30
3	20
4	15
5	12
6	10

3

7

Question Two

$\frac{12}{12}$

a) $P(x) = x^3 - 6x^2 + 4x + 3 \quad \div \quad (x+2)$

$P(-2) = (-2)^3 - 6(-2)^2 + 4(-2) + 3$

$= -8 - 24 - 8 + 3$

$= -37$

$\boxed{\therefore \text{ Remainder is } 37}$ 1

b) $1024x^{15}$ is the leading term. 1

c) $(x-3)(x+2) \rightarrow P(x) = x^3 - 6x^2 + px + q$

$P(3) = 3^3 - 6(3)^2 + p(3) + q$

$0 = 3^3 - 6(3)^2 + 3p + q$

$\boxed{27 = 3p + q}$ ①

$P(-2) = (-2)^3 - 6(-2)^2 + p(-2) + q$

$0 = -8 - 24 - 2p + q$

$\boxed{32 = -2p + q}$ ②

$27 - 3p = 32 + 2p$

$-5 = 5p$

$\boxed{\therefore p = -1}$

$27 = -3 + q$

$\boxed{30 = q}$ 3.

d) $\alpha, \beta, \gamma \rightarrow 2x^3 - x^2 + 6x + 2 = 0$

i) $\alpha\beta\gamma = \frac{-d}{a}$

$= \frac{-2}{2}$

$\boxed{\alpha\beta\gamma = -1}$ 1

ii) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\alpha\beta + \alpha\gamma + \gamma\beta}{\alpha\beta\gamma}$ $\rightarrow \frac{c}{a}$, $\rightarrow -\frac{d}{a}$

$= \frac{3}{-1}$

$\boxed{\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = -3.}$ 2.

e) i) $P(x) = x^3 - 7x + 6 \rightarrow \pm 1, 2, 3, 6$

$P(1) = 1 - 7 + 6$

$= 0$

$\therefore (x-1)$ is a factor.

$P(2) = 2^3 - 7(2) + 6$

$= 8 - 14 + 6$

$= 0$

$\therefore (x-2)$ is a factor

$\therefore (x-1)(x-2)$ must be a factor

$\rightarrow x^2 - 3x + 2$

Q: 2 (cont.)

e) i) CONT.

$x^2 - 3x + 2 \overline{) \, x^3 - 7x + 6}$

$\rightarrow$ $x^2 - 3x + 2 \overline{) \, x^3 + 0x^2 - 7x + 6}$, quotient $x + 3$

$x^3 - 3x^2 + 2x$

$3x^2 - 9x + 6$

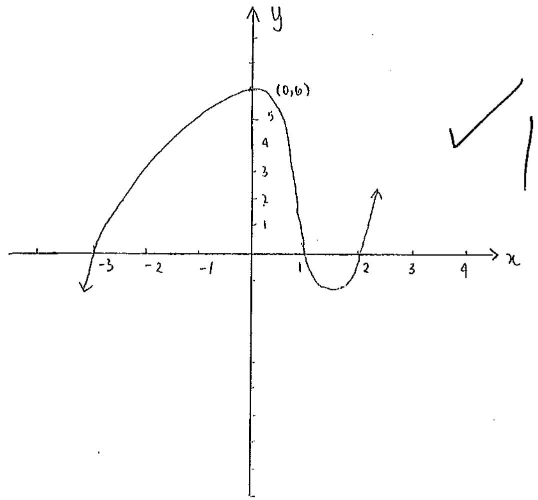
$3x^2 - 9x + 6$

0

$\therefore (x+3)$ is the final factor. 3

$\boxed{\therefore P(x) = (x-1)(x-2)(x+3)}$

ii)



1

9/12

Question Three

a) i. $P(x) = 3x^3 + 7x^2 - 4x + 3$ root in $-3 \leq x \leq -2$.

$P(-3) = 3(-3)^3 + 7(-3)^2 - 4(-3) + 3$

$= -81 + 63 + 12 + 3$

$= -3$

① $\boxed{\therefore P(-3) < 0}$

$P(-2) = 3(-2)^3 + 7(-2)^2 - 4(-2) + 3$

$= -24 + 28 + 8 + 3$

$= 15$

② $\boxed{\therefore P(-2) > 0}$ ✓

→ ① and ② are of opposite sign, ∴ a root lies between $-3 \leq x \leq -2$

Part ii →

b) $y = \sqrt{r^2 - x^2}$

$V = \pi \int_{-r}^{r} (r^2 - x^2)\, dx$. ✓

$= \pi \left[xr^2 - \frac{x^3}{3} \right]_{-r}^{r}$

$= \pi \left[\left(r(r^2) - \frac{r^3}{3} \right) - \left((-r)(r^2) - \frac{(-r)^3}{3} \right) \right]$ ✓

$= \pi \left(\left(r^3 - \frac{r^3}{3} \right) - \left(-r^3 + \frac{r^3}{3} \right) \right)$

$= \pi \left(\frac{2r^3}{3} - \left(-\frac{2r^3}{3} \right) \right)$

$= \pi \left(\frac{2r^3}{3} + \frac{2r^3}{3} \right)$

$= \pi \left(\frac{4r^3}{3} \right)$

$= \frac{4}{3} \pi r^3$. units^3

∴ Volume is $\frac{4}{3}\pi r^3$, i.e. volume of a sphere. ✓

3/3

Q: 3 a)ii)

3. a) ii) $\frac{-3-2}{2} = \frac{-5}{2}$

$P\left(-\frac{5}{2}\right) = 3\left(-\frac{5}{2}\right)^3 + 7\left(-\frac{5}{2}\right)^2 - 4\left(-\frac{5}{2}\right) + 3$

$= \frac{1543}{36} > 0$. ✓

In i), $P(-3) < 0$.

∴ Root lies approx. between $-3 \leq x \leq \frac{-5}{2}$. ✗

Find a better ROOT.

∴, it is $\frac{-3 + \frac{-5}{2}}{2} = -2.75$

2/3

Q: 3 (cont.)

c) $x^3 - mx + 2 = 0$

Let the roots be α, α, β.

i) Sum: $\alpha + \alpha + \beta = \frac{-b}{a}$

$\boxed{2\alpha + \beta = 0}$

$2\alpha + \beta = 0$
$\alpha + \beta = -\alpha$

Product: $\alpha \times \alpha \times \beta = \frac{-d}{a}$

$\boxed{\alpha^2\beta = -2} \rightarrow \alpha^2 = \frac{-2}{\beta}$ ✓

ii) Sum of Pairs: $\alpha\beta + \alpha\beta + \alpha^2 = \frac{c}{a}$

$x^3 + 0x^2 - mx + 2 = 0$

$2\alpha\beta + \alpha^2 = -m.$

$-\beta \times \beta$? $(-\beta)(\beta) + \alpha^2 = -m$

$\alpha^2 - \beta^2 = -m$

$(\alpha+\beta)(\alpha-\beta) = -m$

? $-\alpha(\alpha-\beta) = -m$

$\left(\begin{array}{l} 2\alpha+\beta = 0 \\ \alpha+\beta = -\alpha \end{array}\right)$

$\alpha(\alpha-\beta) = m$

$\boxed{\alpha^2 - \alpha\beta = m}$

$2\alpha\beta + \alpha^2 = -m$

$2\alpha + \beta = 0$
$\alpha^2\beta = -2$

$\frac{-2}{\beta} \quad \frac{-m-\alpha^2}{2} = m$

$\frac{-4-\beta m-\alpha^2\beta}{2\beta} = m$ ✗ ✗

1/3

ii) $\beta = 2\alpha \qquad \alpha^2\beta = -2$

$\alpha^2(-2\alpha) = -2$

$-2\alpha^3 = -2$

$\alpha^3 = 1$

$\boxed{\alpha = 1}$

$\therefore P(1) = 0$

$x^3 - mx + 2 = 0$

$1 - m + 2 = 0$

$\therefore \boxed{m = 3}$

Q: 3 (cont.)

d) $y = (2+x)^{-\frac{1}{2}}$

$\frac{dx}{du} = 2u$

$dx = 2u \, . \, du$

$x = u^2 - 2$

$u^2 = x + 2$

$\boxed{u > 0}$

$3 = u^2 - 2 \qquad 1 = u^2 - 2$

$u^2 = 5 \qquad u^2 = 3$

$u = \sqrt{5} \qquad u = \sqrt{3}$

since $u > 0$. ✓

Area $= \int_1^3 f(x)\, dx$

~~$= \int_1^3 (u^2)^{-\frac{1}{2}} . 2u\, du$~~

~~$= \int_1^3 u^{-1} 2u^1 du$~~

~~$= \int_1^3 2\, du$~~

~~$= [2u]_1^3$~~

~~$= 2(3) - 2(1)$~~

~~$= 6 - 2$~~

~~$= 4$ units^2.~~

$= \int_{\sqrt{3}}^{\sqrt{5}} (u^2)^{-\frac{1}{2}} . 2u\, du$ ✓

$= \int_{\sqrt{3}}^{\sqrt{5}} 2\, du$ since $u^{-1} \times u^1 = u^0$

$= [2u]_{\sqrt{3}}^{\sqrt{5}}$

$= 2\sqrt{5} - 2\sqrt{3}$ units^2. ✓

3/3