THE SCOTS COLLEGE

Extension 1 Mathematics

HSC Assessment 2

25th February 2011

Time Allowed: 45 minutes

Instructions:

- Show all necessary workings
- Approved non-programmable calculators may be used
- Begin a new sheet of paper for each question

Outcomes to be assessed:

Methods of Integration: P2, P8, H4, H5,H8, PE2, HE6 - E11.5, E13.6
Polynomials: PE3 - 16.1 – 16.4

Calculus	Q1, Q2	/20
Functions	Q3	/12
	TOTAL	/32

QUESTION ONE (10 MARKS) BEGIN A NEW SHEET OF PAPER

a) *i)* Differentiate $y = x\sin 3x$ *(1)*

ii) Hence find $\int x\cos(3x)\,dx$ *(2)*

b) Using the substitution $u = 2 + x$, find $\int x(2+x)^4\,dx$ *(3)*

c) Using the substitution $x = 2\cos\theta$, find $\int_0^1 \sqrt{4 - x^2}\,dx$ *(4)*

QUESTION TWO (10 MARKS) BEGIN A NEW SHEET OF PAPER

a) *(i)* Show that the curve $f(x) = 3 - x\log_e x$ has an *x-intercept* between $x = 2$ and $x = 3$. *(2)*

(ii) Use two applications of the halving the interval method to find between which two numbers the curve cuts the x-axis, with a difference of 0.25. *(3)*

b) The curve $y = 2\sin\left(\frac{x}{2}\right)$ and the line $y = \frac{x}{3}$ are on the same number plane.

(i) Show that the *x-coordinates* of their points of intersection satisfy the equation $6\sin\left(\frac{x}{2}\right) - x = 0$. *(1)*

(ii) By sketching the graphs of the two functions, show that the equation in part *(i)* has a root between 4 and 5. *(2)*

(iii) Using 4.5 as a first approximation, use Newton's method once to find another approximation to the root correct to one decimal place. *(2)*

QUESTION THREE (12 MARKS) BEGIN A NEW SHEET OF PAPER

a) *(i)* Using long division, prove that $(2x - 3)$ is a factor of $P(x) = 6x^3 + 5x^2 - 33x + 18$. *(2)*

(ii) Hence completely factorise $P(x)$. *(2)*

b) $(x - k)$ is a factor of $x^2 - 5x + (2k + 2)$. Using the factor theorem, find the values of k. *(2)*

c) When the polynomial $P(x) = ax^3 + ax^2 + 3x + 3$ is divided by $(x + 2)$, the remainder is 5. Find the value of a, using the remainder theorem. *(2)*

d) If α, β, γ are the roots of $x^3 - 2x^2 + 3x + 7 = 0$, find the values of:

(i) $\frac{2}{\alpha} + \frac{2}{\beta} + \frac{2}{\gamma}$ *(2)*

(ii) $\alpha^2 + \beta^2 + \gamma^2$ *(2)*

ANSWER SHEET

$\frac{8}{10}$

Name:

Teacher:

Question No. 1

a) i) $y = x\sin 3x$

$u = x \quad v = \sin 3x$

$u' = 1 \quad v' = 3\cos 3x$

$\frac{dy}{dx} = vu' + uv'$

$= \sin 3x \times 1 + x \times 3\cos 3x$

$= \sin 3x + 3x\cos 3x$

$= \sin 3x + 3x\cos 3x$ ✓

1

ii) $\sin 3x + 3x\cos 3x = y'$ $\therefore x\cos 3x = \frac{y' - \sin 3x}{3}$

$\int x\cos 3x\, dx = \int \frac{-\sin 3x}{3} + x\sin 3x + C$

$\int x\cos 3x\, dx$

$= x\sin 3x - \frac{\sin 3x}{3} + C$

$= \frac{x^2 \sin 3x}{2} + C$?

$\int x\cos 3x\, dx$ let $u = 3x$

$\frac{du}{dx} = 3$

$du = 3\,dx$

$x = \frac{u}{3}$

$\int \frac{u}{9}\cos u\, du$

$= \frac{u^2}{18}\sin u + C$

$= \frac{(3x)^2}{18}\sin 3x + C$

$= \frac{9x^2}{18}\sin 3x + C$

$= \frac{x^2 \sin 3x}{2}$

b) $\int x(2+x)^4\, dx$

let $u = 2+x$

$\frac{du}{dx} = 1$

$du = dx$

$u = 2+x$

$u - 2 = x$

$\therefore x = u-2$

$= \int (u-2)(u)^4\, du$

$= \int u^5 - 2u^4\, du$

$= \frac{u^6}{6} - \frac{2u^5}{5} + C$

$= \frac{(2+x)^6}{5} - \frac{2(2+x)^5}{5} + C$

3

c) is over the page

c) $\int_0^1 \sqrt{4 - x^2}\, dx$

let $x = 2\cos\theta$

$\frac{dx}{d\theta} = -2\sin\theta$

$dx = -2\sin\theta\, d\theta$

when $x = 1$, $\theta = \frac{\pi}{3}$

$x = 0$, $\theta = \frac{\pi}{2}$ ✓

$\frac{x}{2} = \cos\theta$

$\theta = \cos^{-1}\left(\frac{x}{2}\right)$

$= \int_{\frac{\pi}{2}}^{\frac{\pi}{3}} \sqrt{4 - (2\cos\theta)^2} \times -2\sin\theta\, d\theta$

$= \int_{\frac{\pi}{2}}^{\frac{\pi}{3}} \sqrt{4 - 4\cos^2\theta} \times -\sin\theta\, d\theta$

$= \int_{\frac{\pi}{2}}^{\frac{\pi}{3}} \sqrt{4\sin^2\theta} \times -\sin\theta\, d\theta$ $\boxed{4\cos^2\theta + 4\sin^2\theta = 4}$

$= \int_{\frac{\pi}{2}}^{\frac{\pi}{3}} 2\sin\theta \times -\sin\theta\, d\theta \longrightarrow \int_{\frac{\pi}{2}}^{\frac{\pi}{3}} -2\sin^2\theta\, d\theta$

~~$= \left[-2\cos\theta\right]_{\frac{\pi}{2}}^{\frac{\pi}{3}}$~~

~~$= \left[\left(-2\cos\frac{\pi}{3}\right) - \left(-2\cos\frac{\pi}{2}\right)\right]$~~

~~$= -2 \times \frac{1}{2} + 2 \times 0$~~

~~$= -\frac{2}{2}$~~

~~$= -1$~~

$\cos 2x = 1 - 2\sin^2 x$

$2\sin^2 x = 1 - \cos 2x$

$= -\int_{\frac{\pi}{2}}^{\frac{\pi}{3}} 2\sin^2\theta\, d\theta$

$= -\int_{\frac{\pi}{2}}^{\frac{\pi}{3}} 1 - \cos 2\theta\, d\theta$

$= -\left[\theta - \frac{1}{2}\sin\theta\right]_{\frac{\pi}{2}}^{\frac{\pi}{3}}$

$= -\left[\left(\frac{\pi}{3} - \frac{1}{2}\sin\frac{\pi}{3}\right) - \left(\frac{\pi}{2} - \frac{1}{2}\sin\frac{\pi}{2}\right)\right]$

$= -\left[\frac{\pi}{3} - \frac{1}{2} \times \frac{\sqrt{3}}{2} - \frac{\pi}{2} + \frac{1}{2} \times 1\right]$

$= -\left(\frac{\pi}{3} - \frac{\pi}{2} - \frac{\sqrt{3}}{4} + \frac{1}{2}\right]$

3

$= -\left(-\frac{1}{6} - \frac{\sqrt{3}}{4} + \frac{2}{4}\right]$

$= -\left(-\frac{1}{6} - \left(\frac{\sqrt{3} - 2}{4}\right)\right]$

$= \frac{1}{6} + \frac{\sqrt{3} - 2}{4}$

$= \frac{-4\sqrt{3}}{12}$

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ANSWER BOOKLET

Name:

Teacher:

Question No. 2

d) i) $f(x) = 3 - x\log_e x$

$f(2) = 3 - 2\log_e 2$ $\quad f(3) = 3 - 3\log_e 3$

$= 1.613\ldots > 0$ $\quad = -0.2958\ldots < 0$

$\therefore$ since $f(2) > 0$ & $f(3) < 0$ a zero exists between them. Therefore $y = 0$ between $f(2)$ & $f(3)$ as ~~one~~ side of zero is +ve and the other side is −ve $\therefore$ an x-intercept exists between $x = 2$ & $x = 3$

ii)

	$f(a) > 0$ a	$f(b) < 0$ b	$c = \frac{a+b}{2}$	$f(c)$
1	2	3	2.5	+ve
2	2.5	3	2.75	+ve
	2.75	3		

$\therefore$ after 2 applications of halving the interval curve cuts the x-axis between $x = 2.75$ and $x = 3$ $\boxed{\text{as } 3 - 2.75 = 0.25}$

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Question No. 2

b) i) $y = 2\sin\left(\frac{x}{2}\right)$ ① $\quad y = \frac{x}{3}$ ②

Points of intersection are ① − ②

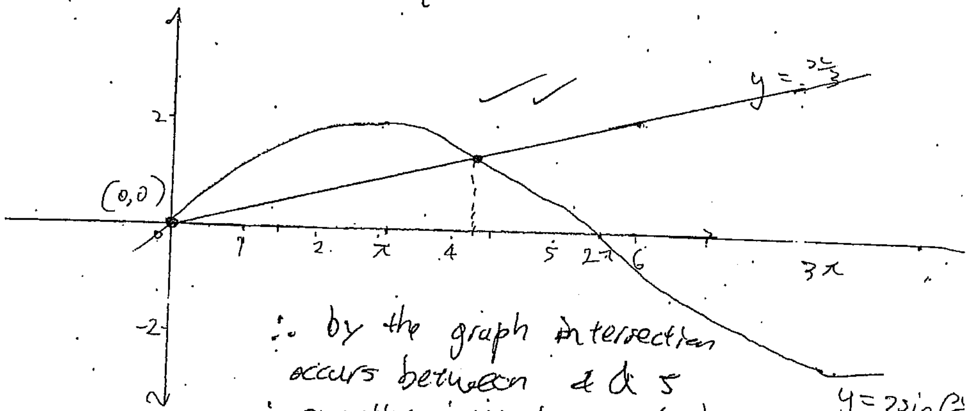
$2\sin\left(\frac{x}{2}\right) = \frac{x}{3}$

$6\sin\left(\frac{x}{2}\right) = x$

$6\sin\left(\frac{x}{2}\right) - x = 0$

$\therefore$ x-coords of point of intersection must satisfy equation above to equal zero.

ii)



$\therefore$ by the graph intersection occurs between 4 & 5

$\therefore$ equation in (i) has roots between 4 & 5

Question No. 2

ii) $f(x) = 6\sin\left(\frac{x}{2}\right) - x$

$f'(x) = 3\cos\left(\frac{x}{2}\right) - 1$

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$= 4.5 - \frac{f(4.5)}{f'(4.5)}$

$= 4.5 - \frac{\left(6\sin\left(\frac{4.5}{2}\right) - 4.5\right)}{\left(3\cos\left(\frac{4.5}{2}\right) - 1\right)}$

$= 4.5583\ldots$

$\doteqdot 4.6$ (to 1 d.p.)

12/12 V. Good!

ANSWER BOOKLET

Name:

Teacher:

Question No. 3

a) i) $P(x) = 6x^3 + 5x^2 - 33x + 18$

$$
\begin{array}{r}
3x^2 + 7x - 6 \\
2x - 3 \overline{\big)\, 6x^3 + 5x^2 - 33x + 18} \\
-\underline{6x^3 \pm 9x^2} \\
14x^2 - 33x \\
-\underline{14x^2 \pm 21x} \\
-12x + 18 \\
-\underline{12x + 18} \\
0
\end{array}
$$

$\therefore (2x-3)$ is a factor ~~because~~ as long ~~poly~~ division gave no remainder

ii) $P(x) = (2x-3)(3x^2 + 7x - 6)$

$= (2x-3)(3x-2)(x+3)$

b) $f(x) = x^2 - 5x + (2k+2)$

~~$f(k) =$~~ since $(x-k)$ is a factor

$f(k) = 0$

$f(k) = k^2 - 5k + (2k+2)$

$0 = k^2 - 3k + 2$

$0 = (k-1)(k-2)$

$\therefore k = 1$ or $k = 2$

Prove if time

Question No. 3

c) $p(x) = ax^3 + ax^2 + 3x + 3$

$(x+2)$ gives a remainder of 5

$\therefore P(-2) = 5$

$p(-2) = a(-2)^3 + a(-2)^2 + 3(-2) + 3$

$5 = -8a + 4a - 6 + 3$

$5 = -4a - 3$

$8 = -4a$

$-2 = a$

$\therefore a = -2$

d) $x^3 - 2x^2 + 3x + 7 = 0$, if α, β, γ are roots

~~if $\alpha + \beta$~~

then $\alpha + \beta + \gamma = -\frac{b}{a}$

$= \frac{--2}{1}$

$= 2$

$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}$

$= \frac{3}{1}$

$= 3$

$\alpha\beta\gamma = \frac{-d}{a}$

$= \frac{-7}{1} = -7$

Question No. 3

d) i) $\frac{2}{\alpha} + \frac{2}{\beta} + \frac{2}{\gamma} = \frac{2\beta\gamma + 2\alpha\gamma + 2\alpha\beta}{\alpha\beta\gamma}$

$= \frac{2(\alpha\beta + \beta\gamma + \alpha\gamma)}{\alpha\beta\gamma}$

$= \frac{2(3)}{-7}$ $\boxed{\frac{2\times(\frac{c}{a})}{(-\frac{d}{a})}} = \boxed{2\times -\frac{c}{d}}$

$= \frac{6}{-7}$

ii) $\alpha^2 + \beta^2 + \gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$

$= (2)^2 - 2(3)$

$= 4 - 6$

$= -2$