Student Number

ASCHAM SCHOOL

2011
YEAR 12

HIGHER SCHOOL CERTIFICATE

TRIAL EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes.
- Working time – 3 hours.
- Write using blue or black pen.
- Board-approved calculators may be used.
- A table of standard integrals is provided.
- All necessary working should be shown in every question.

Total marks – 120

- Attempt Questions 1-8.
- All questions are of equal value.
- Answer each question in a new booklet.

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Total marks – 120
Attempt Questions 1-8
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

QUESTION 1 (15 marks)
Use a SEPARATE writing booklet.

(a) Find $\int \sin^3\theta \, d\theta$. **2**

(b) (i) Express $\frac{3x+1}{(x+1)(x^2+1)}$ in the form $\frac{a}{x+1}+\frac{bx+c}{x^2+1}$. **2**

(ii) Hence find $\int \frac{3x+1}{(x+1)(x^2+1)}\,dx$. **2**

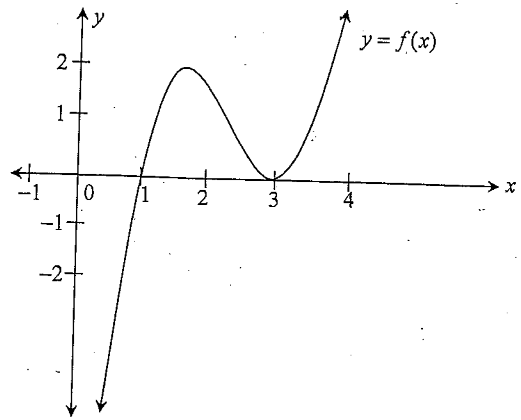
(c) Use the substitution $x = 2\sin\theta$, or otherwise, to evaluate $\int_1^{\sqrt{3}} \frac{x^2}{\sqrt{4-x^2}}\,dx$. **3**

(d) Find $\int x^2\sqrt{3-x}\,dx$. **3**

(e) Evaluate $\int_0^1 \tan^{-1}\theta \, d\theta$. **3**

QUESTION 2 (15 marks) **Marks**
Start a new writing booklet.

(a)



The diagram above is a sketch of the function $y = f(x)$.

On separate diagrams sketch:

(i) $y = (f(x))^2$ **2**

(ii) $y = \sqrt{f(x)}$ **2**

(iii) $y = \ln[f(x)]$ **2**

(iv) $y^2 = f(x)$ **2**

(b) (i) If $f'(x) = \frac{2-x}{x^2}$ and $f(1) = 0$, find $f''(x)$ and $f(x)$. **3**

(ii) Explain why the graph of $f(x)$ has only one turning point and find the value of the function at that point, stating whether it is a maximum or a minimum value. **2**

(iii) Show that $f(4)$ and $f(5)$ have opposite signs and draw a sketch of $f(x)$. **2**

QUESTION 3 (15 marks) Marks
Start a new writing booklet.

(a) Express $(\sqrt{3}+i)^8$ in the form $x+iy$. 3

(b) On an Argand diagram, sketch the region where the inequalities 3

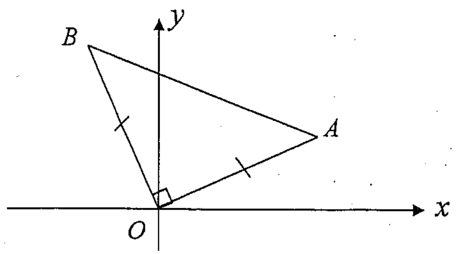
$$|z| \le 3 \text{ and } -\frac{2\pi}{3} \le \arg(z+2) \le \frac{\pi}{6} \text{ both hold.}$$

(c) Show that $\dfrac{1+\sin\theta+i\cos\theta}{1+\sin\theta-i\cos\theta}=\sin\theta+i\cos\theta$. 3

(d) (i) Express $z=\dfrac{-1+i}{\sqrt{3}+i}$ in modulus-argument form. 2

(ii) Hence evaluate $\cos\dfrac{7\pi}{12}$ in surd form. 2

(e) The Argand diagram below shows the points A and B which represent the complex numbers z_1 and z_2 respectively.



Given that ΔBOA is a right-angled isosceles triangle, show that $(z_1+z_2)^2=2z_1z_2$. 2

QUESTION 4 (15 marks) Marks
Start a new writing booklet.

(a) If $z=1+i$ is a root of the equation $z^3+pz^2+qz+6=0$ where p and q are real, find p and q. 3

(b) Show that if the polynomial $f(x)=x^3+px+q$ has a multiple root, then $4p^3+27q^2=0$. 3

(c) The base of a solid is the region in the first quadrant bounded by the curve $y=\sin x$, the x-axis and the line $x=\dfrac{\pi}{2}$. 3

Find the volume of the solid if every cross-section perpendicular to the base and the x-axis is a square.

(d) (i) Find the five roots of the equation $z^5=1$. Give the roots in modulus-argument form. 2

(ii) Show that z^5-1 can be factorised in the form :

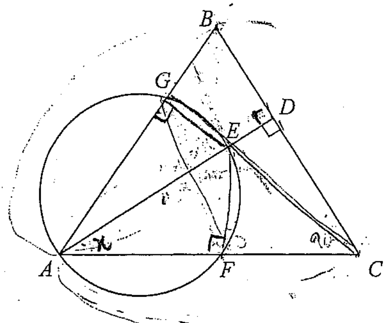
$$z^5-1=(z-1)(z^2-2z\cos\frac{2\pi}{5}+1)(z^2-2z\cos\frac{4\pi}{5}+1)$$ 2

(iii) Hence show that $\cos\dfrac{2\pi}{5}+\cos\dfrac{4\pi}{5}=-\dfrac{1}{2}$. 2

QUESTION 5 (15 marks) **Marks**
Start a new writing booklet.

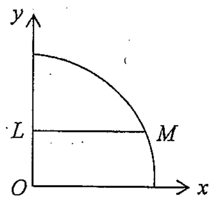
(a) The ellipse $(x-1)^2+\frac{y^2}{4}=1$ is rotated about the y-axis.
Use the method of slicing to find the volume of the solid formed by the rotation. **4**

(b) In the triangle ABC, AD is the perpendicular from A to BC. E is any point on AD and the circle drawn with AE as diameter cuts AC at F and AB at G. **4**



Prove B, G, F and C are concyclic.

(c) The diagram below shows the part of the circle $x^2+y^2=a^2$ in the first quadrant.



(i) If the horizontal line LM through $L(0,b)$, where $0<b<a$, divides the area between the curve and the coordinates axes into two equal parts, show that

$$\sin^{-1}\frac{b}{a}+\frac{b\sqrt{a^2-b^2}}{a^2}=\frac{\pi}{4}.$$ **3**

(ii) If the radius of the circle is 1 unit, show that b can be found by solving the equation $\sin 2\theta=\frac{\pi}{2}-2\theta$, where $\theta=\sin^{-1}b$. **3**

(iii) Without attempting to solve the equation, how could θ (and hence b) be approximated? **1**

QUESTION 6 (15 marks) **Marks**
Start a new writing booklet.

(a) An ellipse has equation $\frac{x^2}{4}+\frac{y^2}{3}=1$ with vertices $A(2,0)$ and $A'(-2,0)$. P is a point (x_1,y_1) on the ellipse.

(i) Find its eccentricity, coordinates of its foci, S and S', and the equations of its directrices. **3**

(ii) Prove that the sum of the distances SP and $S'P$ is independent of the position of P. **3**

(iii) Show that the equation of the tangent to the ellipse at P is $\frac{xx_1}{4}+\frac{yy_1}{3}=1$. **3**

(iv) The tangent at $P(x_1,y_1)$ meets the directrix at T. Prove that angle PST is a right angle. **3**

(b) If $a+b+c=1$,

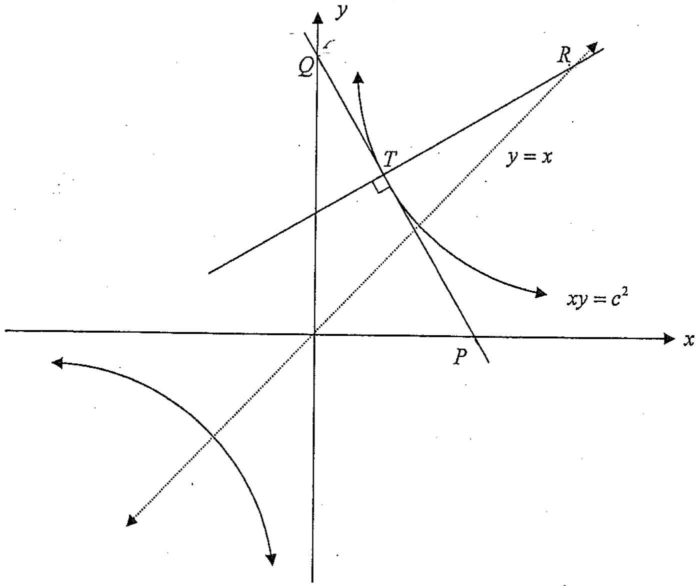
(i) Prove $a^2+b^2\geq 2ab$. **1**

(ii) Prove $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\geq 9$. **2**

QUESTION 7 (15 marks) **Marks**
Start a new writing booklet.

(a) The point $T(ct,\frac{c}{t})$ lies on the hyperbola $xy=c^2$.

The tangent at T meets the $x-$axis at P and the $y-$axis at Q.
The normal at T meets the line $y=x$ at R.



not to scale

You may assume that the tangent at T has equation $x+t^2y=2ct$.

(i) Find the coordinates of P and Q. **2**

(ii) Find the equation of the normal at T. **2**

(iii) Show that the $x-$coordinate of R is $x=\frac{c}{t}(t^2+1)$. **2**

(iv) Prove that ΔPQR is isosceles. **3**

(b) (i) If $I_n=\int\frac{dx}{(x^2+1)^n}$ prove that $I_n=\frac{1}{2(n-1)}\left[\frac{x}{(x^2+1)^{n-1}}+(2n-3)I_{n-1}\right]$. **4**

(ii) Hence evaluate $\int_0^1\frac{dx}{(x^2+1)^2}$. **2**

QUESTION 8 (15 marks) **Marks**
Start a new writing booklet.

(a) A plane of mass M kg on landing, experiences a variable resistive force due to air resistance of magnitude Bv^2 newtons, where v is the speed of the plane. That is, $M\ddot{x}=-Bv^2$.

(i) Show that the distance (D_1) travelled in slowing the plane from speed V to speed U under the effect of air resistance only, is given by: **4**

$$D_1=\frac{M}{B}\ln(\frac{V}{U})$$

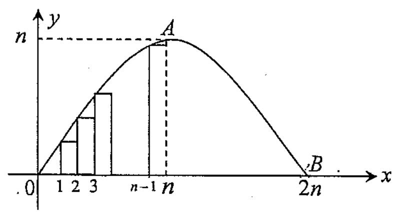
After the brakes are applied, the plane experiences a constant resistive force of A Newtons (due to brakes) as well as a variable resistive force, Bv^2. That is, $M\ddot{x}=-(A+Bv^2)$.

(ii) After the brakes are applied when the plane is travelling at speed U, show that the distance D_2 required to come to rest is given by: **4**

$$D_2=\frac{M}{2B}\ln\left[1+\frac{B}{A}U^2\right].$$

(iii) Use the above information to estimate the total stopping distance after landing, for a 100 tonne plane if it slows from $90\,\text{m/s}$ to $60\,\text{m/s}$ under a resistive force of $125v^2$ Newtons and is finally brought to rest with the assistance of a constant braking force of magnitude 75 000 Newtons. (Note: 1 Newton (N) = 1 kg. m/s^2) **2**

(b)



The diagram above represents the curve $y=n\sin\frac{\pi x}{2n}$, $0\le x\le 2n$, where n is any integer $n\ge 2$.
The points $O(0,0)$, $A(n,n)$ and $B(2n,0)$ lie on this curve.

(i) By considering the areas of the lower rectangles of width 1 from $x=0$ to $x=n$, prove that **3**

$$\sin\frac{\pi}{2n}+\sin\frac{2\pi}{2n}+\sin\frac{3\pi}{2n}+....+\sin\frac{\pi(n-1)}{2n}<\frac{2n}{\pi}.$$

(ii) Hence or otherwise, explain why $2n\sum_{r=1}^{n-1}\sin\frac{\pi r}{2n}<\frac{\pi n^2}{2}$. **2**

END OF PAPER

ASCHAM SCHOOL

MATHEMATICS WRITING BOOKLET

Name:	
Form Class:	Teacher's Initials

Question Number: 1

- Enter the information requested in each of the boxes above
- Take a new writing booklet for each question
- You may ask for an extra writing booklet if you need one. Label it as above and put it inside the first booklet for the question.
- If you do not attempt a question still hand in the booklet with "NOT ATTEMPTED" written on the front.
- Write the question part in the margin
- Write on the ruled lines in black or blue ink

Do not write in this box: 13

a) $\int \sin^3\theta \, d\theta = \int \sin^2\theta \cdot \sin\theta \, d\theta$

$= \int (1-\cos^2\theta)\sin\theta \, d\theta$

$= \int \sin\theta - \sin\theta\cos^2\theta \, d\theta$

$= -\cos\theta + \int -\sin\theta\cos^2\theta \, d\theta$

$\therefore -\cos\theta + \frac{\cos^3\theta}{3} + c$

2/2

b) i) $3x+1 = a(x^2+1) + (bx+c)(x+1)$

Let $x=-1$. $-3+1 = 2a$

$-2 = 2a$

$a = -1$

$3x+1 = -1(x^2+1) + (bx+c)(x+1)$

Let $x = 0$

$0+1 = -(0^2+1) + (0+c)(0+1)$

$1 = -1+c$

$c = 2$

Let $x = 1$.

$3+1 = -(1^2+1) + (b(1)+2)(1+1)$

$4 = -2 + (2+b)2$

$= -2 + 4 + 2b$

$6 = 4+2b$

$2 = 2b$

$b = 1$

$\therefore \frac{3x+1}{(x+1)(x^2+1)} = \frac{-1}{x+1} + \frac{x+2}{x^2+1}$

2/2

ii) $\int \frac{3x+1}{(x+1)(x^2+1)} dx = \int -\frac{1}{x+1} + \frac{x}{x^2+1} + \frac{2}{x^2+1} dx$

$= -\ln(x+1| + \frac{1}{2}\ln|x^2+1| + 2\tan^{-1}x + C$

$= \ln\left|\frac{\sqrt{x^2+1}}{x+1}\right| + 2\tan^{-1}x + C$.

2/2

c) $\int_1^{\sqrt3} \frac{x^2}{\sqrt{4-x^2}}\,dx = \int_{\pi/6}^{\pi/3} \frac{4\sin^2\theta}{\sqrt{4-4\sin^2\theta}} \times 2\cos\theta\, d\theta$ Let $x = 2\sin\theta$

$\frac{dx}{d\theta} = 2\cos\theta$.

$x=\sqrt3$ $\sqrt3 = 2\sin\theta$

$= \int_{\pi/6}^{\pi/3} \frac{4\sin^2\theta}{\cancel{2\cos\theta}} \times \cancel{2\cos\theta}\, d\theta$ $\sin\theta = \frac{\sqrt3}{2}$

$\theta = \frac{\pi}{3}$.

$= \int_{\pi/6}^{\pi/3} 4\sin^2\theta\, d\theta$ $=1$ $1 = 2\sin\theta$

$\theta = \frac{\pi}{6}$

$= 4\int_{\pi/6}^{\pi/3} \frac12 - \frac12\cos2\theta\, d\theta$

$= 2\int_{\pi/6}^{\pi/3} 1-\cos2\theta\, d\theta$

$= 2\left[\theta - \frac12\sin2\theta\right]_{\pi/6}^{\pi/3}$

$= 2\left[\frac{\pi}{3} - \frac12\sin\frac{2\pi}{3} - \frac{\pi}{6} + \frac12\sin\frac{\pi}{3}\right]$.

$= 2\left[\frac{\pi}{6} - \frac12\times\frac{\sqrt3}{2} + \frac12\times\frac{\sqrt3}{2}\right]$

$= \frac{\pi}{3}$

3/3

~~d). $\int x^2\sqrt{3-x}\,dx$~~ Let $u^2 = 3-x$.

$x = 3-u^2$.

$dx = -2u\,du$.

~~$= \int (3-u^2)^2 u^2 \cdot -2u\,du$.~~

~~$= \int (9-6u^2+u^4)(-2u^3)\,du$.~~

~~$= \int -18u^3+12u^5-2u^7\,du$~~

~~$= \frac{-18u^4}{4} + 2u^6 - \frac{2u^8}{8} + C$.~~

~~$= -\frac{9(3-x)^2}{2} + u^4\left(\frac92 + 2u^2 - \frac14 u^4\right) + C$~~

~~$= (3-x)^2\left(\frac92 + 2(3-x) - \frac14(3-x)^2\right) + C$.~~

e) $\int_0^1 \tan^{-1}\theta\, d\theta$

$= \int_0^1 u\, dv$.

Let $u = \tan^{-1}\theta$ $v = \theta$

$du = \frac{1}{1+\theta^2}d\theta$ $dv = d\theta$

$\int_0^1 \tan^{-1}\theta\, d\theta = \left[\theta\tan^{-1}\theta\right]_0^1 - \int_0^1 \frac{\theta}{1+\theta^2}\,d\theta$

$= \theta\tan^{-1}\theta - \frac12\ln|1+\theta^2| + C$.

$= \tan^{-1}1 - \frac12\ln2$

1/3

d) $\int x^2\sqrt{3-x}\,dx$

Let $u = \sqrt{3-x}$

$u^2 = 3-x$

$x = 3-u^2$

$dx = -2u\,du$

$\int x^2\sqrt{3-x}\,dx = \int (3-u^2)^2 \cdot u \cdot -2u\,du$

$= -2\int (9+u^4-6u^2)u^2\,du$

$= -2\int 9u^2+u^6-6u^4\,du$

$= -2\left(3u^3+\frac{u^7}{7}-\frac{6u^5}{5}\right)+C$

$= -2\left(3(\sqrt{3-x})^3+\frac{(\sqrt{3-x})^7}{7}-\frac{6(\sqrt{3-x})^5}{5}\right)+C$

$= -2(\sqrt{3-x})^3\left(3+\frac{(3-x)^2}{7}-\frac{6(3-x)}{5}\right)+C$.

3/3

ASCHAM SCHOOL

MATHEMATICS WRITING BOOKLET

Name:	
Form Class:	Teacher's Initials

Question Number 2

- Enter the information requested in each of the boxes above
- Take a new writing booklet for each question
- You may ask for an extra writing booklet if you need one. Label it as above and put it inside the first booklet for the question.
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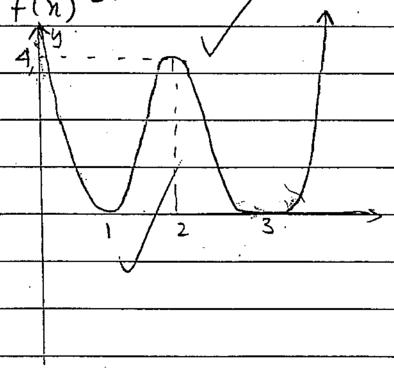
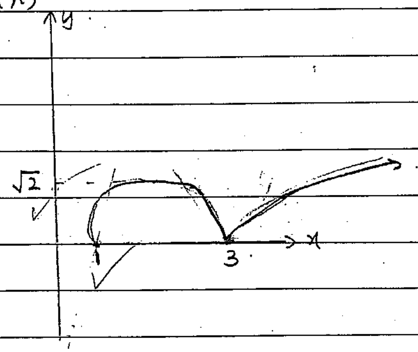
14

a) i) $y = f(x)^2$.



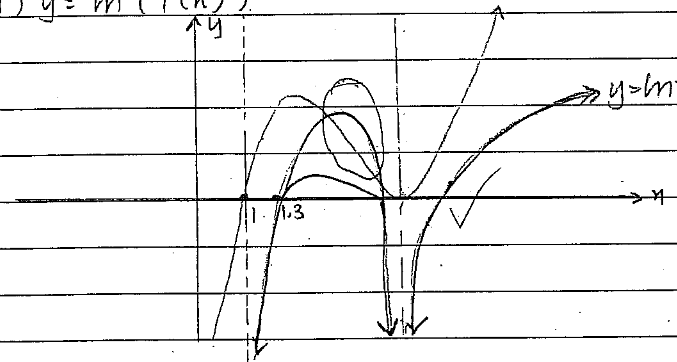
2/2

ii) $y = \sqrt{f(x)}$



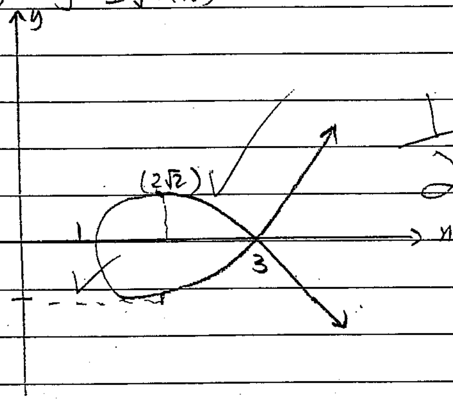
2/2

iii) $y = \ln(f(x))$.



~~2~~ 2

iv) $y^2 = f(x)$ $\quad y = \pm\sqrt{f(x)}$



1/2

b) i) $f'(x) = \frac{2-x}{x^2}$.

$$f''(x) = \frac{-1(x^2) - 2x(2-x)}{x^4}$$

$$= \frac{-x^2 - 4x + 2x^2}{x^4}$$

$$= \frac{x^2 - 4x}{x^4}$$

$$= \frac{\cancel{x}(x-4)}{\cancel{x^4}\,x^3}$$

$$= \frac{x-4}{x^3}.$$

$$f(x) = \int \frac{2-x}{x^2}\,dx.$$

$$= \int \frac{2}{x^2} - \frac{1}{x}\,dx.$$

$$= -\frac{2}{x} - \ln x + C.$$

3/3

when $f(1) = 0$

$0 = -2 - \ln 1 + C$

$C = 2$.

$\therefore f(x) = -\frac{2}{x} - \ln x + 2$.

ii) Turning pt $f'(x) = 0$

$\frac{2-x}{x^2} = 0 \qquad x = 2$.

$\therefore$ there is only 1 possible turning point.

value $f(2) = -\frac{2}{2} - \ln 2 + 2$

$= -1 - \ln 2 + 2$

$= 1 - \ln 2$.

$\therefore (2, 1-\ln 2)$.

$f''(x) = \frac{2-\cancel{4}}{2^3} < 0 \quad \therefore$ concave down.

$\therefore$ it is a max turning point

2/2

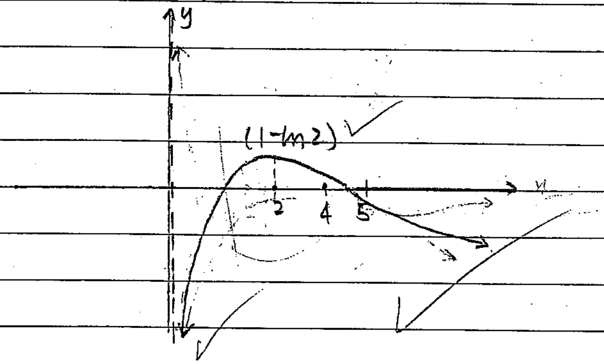
2/2

iii) $f(4) = -\frac{2}{4} - \ln 4 + 2$

$= 0.11\ldots > 0$

$f(5) = -\frac{2}{5} - \ln 5 + 2$

$= -0.0094\ldots < 0$



2/2

2

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MATHEMATICS WRITING BOOKLET

Name:	Question Number 3
Form Class:	Teacher's Initials

- Enter the information requested in each of the boxes above
- Take a new writing booklet for each question
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11

a) $(\sqrt{3}+i)^8 = (2\,\text{cis}\frac{\pi}{6})^8$

$= 2^8 \text{cis}\frac{4\pi}{3}$

$= 2^8\cos\frac{4\pi}{3} + 2^8 i\sin\frac{4\pi}{3}$

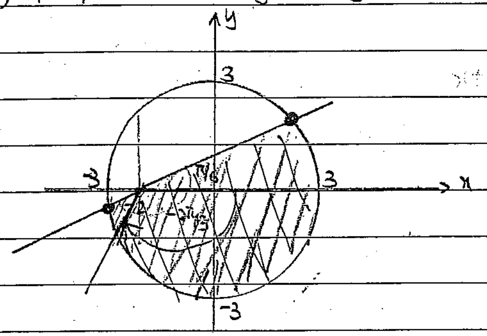
$= 2^8(-\frac{1}{2}) + 2^8 i(-\frac{\sqrt{3}}{2})$

$= -2^7 - 2^7\sqrt{3}\,i$

$= -128 - 128\sqrt{3}\,i$

3/3

b) $|z| \leq 3 \qquad -\frac{2\pi}{3} \leq \arg(z+2) \leq \frac{\pi}{6}$.



2/3

c) $\frac{\cancel{1+i}\,1+\sin\theta + i\cos\theta}{1+\sin\theta - i\cos\theta}$

$= \frac{1+\sin\theta + i\cos\theta}{1+\sin\theta - i\cos\theta} \times \frac{1+\sin\theta + i\cos\theta}{1+\sin\theta + i\cos\theta}$

$= \frac{(1+\sin\theta)^2 + 2(1+\sin\theta)i\cos\theta - \cos^2\theta}{(1+\sin\theta)^2 + \cos^2\theta}$.

$= \frac{1+2\sin\theta+\sin^2\theta + 2i\cos\theta + 2\sin\theta i\cos\theta - \cos^2\theta}{1+2\sin\theta+\sin^2\theta+\cos^2\theta}$

$= \frac{2(1+\sin\theta)i\cos\theta + (1+2\sin\theta+\sin^2\theta-\cos^2\theta)}{2(1+\sin\theta)}$.

$\frac{1+\sin\theta+i\cos\theta}{1+\sin\theta - i\cos\theta} = \frac{1+2\sin\theta+\sin^2\theta + 2\cos\theta(1+\sin\theta)i - \cos^2\theta}{1+2\sin\theta+\sin^2\theta+\cos^2\theta}$

$= \frac{\cancel{1+2\sin\theta} - \cancel{1}+\sin^2\theta + \sin^2\theta + \cos\theta i(2+2\sin\theta)}{2+2\sin\theta}$.

$= \frac{\sin\theta\cancel{(2+2\sin\theta)} + \cos\theta i\cancel{(2+2\sin\theta)}}{\cancel{2+2\sin\theta}}$

$= \sin\theta + \cos\theta i$ qed

2/3

3

d) i) $z = \dfrac{-1+i}{\sqrt{3}+i}$

$= \dfrac{(-1+i)(\sqrt{3}-i)}{(\sqrt{3}+i)(\sqrt{3}-i)}$

$= \dfrac{-\sqrt{3}+i+i\sqrt{3}+1}{3+1}$

$= \dfrac{1-\sqrt{3}+i(1+\sqrt{3})}{4}$ ①

$\cancel{z = \dfrac{-1}{\sqrt{3}+i} + \dfrac{i}{\sqrt{3}+i}}$

$= z = \dfrac{-1+i}{\sqrt{3}+i}$

d) i) $\cancel{z = \sqrt{2}\,cis(}$ $z = \dfrac{\sqrt{2}\,cis(\frac{3\pi}{4})}{2\,cis(\frac{\pi}{6})}$ ✗ $\frac{\pi}{6}$

$= \dfrac{1}{\sqrt{2}}\,cis\left(\dfrac{5\pi}{12}\right)$ ✗ $\dfrac{1}{\sqrt{2}}\,cis\left(\dfrac{7\pi}{12}\right)$ ✓ $\frac{1}{2}$

ii) $\cos\frac{7\pi}{12} = ?$

$z = \frac{1}{\sqrt{2}}\cos\frac{5\pi}{12} + \frac{1}{\sqrt{2}} i\sin\frac{5\pi}{12}$ $\quad$ $z = \frac{1}{\sqrt{2}}\cos\frac{7\pi}{12} + \frac{1}{\sqrt{2}} i\sin\frac{7\pi}{12}$

$\cos\frac{7\pi}{12} = \cos\left(-\frac{5\pi}{12}\right)$

$= \cos\frac{5\pi}{12}$.

$\cancel{(\sqrt{3}+i)z = -1+i}$ $\quad$ Take ①

$\cancel{(\sqrt{3}+i)z\ \sqrt{3}z + iz} = -1+i$

$\cancel{\sqrt{3}z + 1 + iz - 1 = 0}$

Taking real part $\frac{1}{\sqrt{2}}\cos\frac{5\pi}{12} = \frac{1-\sqrt{3}}{4}$. $\quad$ correct method

$\cancel{\therefore \cos\frac{7\pi}{12} = \cos\frac{5\pi}{12}}$ ✗

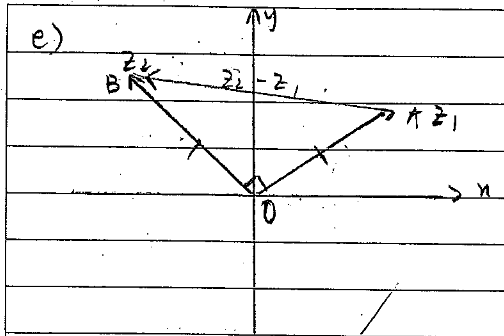
$= \dfrac{\sqrt{2}(1-\sqrt{3})}{4}$ ✓ $\quad \cos\frac{7\pi}{12} = \left(\frac{1-\sqrt{3}}{4}\right)\sqrt{2}$.

$\cancel{= \dfrac{1-\sqrt{3}}{2\sqrt{2}}}$ $\quad = \dfrac{\sqrt{2}-\sqrt{6}}{4}$ ✓

$\frac{2}{2}$

4

e)



$x^2 - y^2 +$

$\overrightarrow{AB} = (z_2 - z_1)$ ✓

Pythagoras $|(z_2 - z_1)^2| = |z_1^2| + |z_2^2|$

$z_2^2 - 2z_1z_2 + z_1^2 = z_1^2 + z_2^2$.

$(z_1^2 + 2z_1z_2 + z_2^2) = z_2^2 + z_1^2$

$(z_1 + z_2)^2 = \cancel{\ldots}$

$= z_1^2 + z_1^2$.

$= 2z_1^2$ $\quad$ since $|OA| = |OB|$

$= 2z_1z_2$ $\quad$ qed

From diagram $z_2 = iz_1$

since multiplying by i rotates it 90° in anti-c direction

$(z_1+z_2)^2 = (z_1 + iz_1)^2$

$= \cancel{z_1^2(1+i)^2} =$

$= z_1^2(1+i)^2$

$= z_1^2(1+2i-1)$

$= 2iz_1^2$

$= 2z_1 iz_1$

$= 2z_1z_2$ qed

$\frac{0}{2}$

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MATHEMATICS WRITING BOOKLET

Name:		Question Number
Form Class:	Teacher's Initials	4

- Enter the information requested in each of the boxes above
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13

a). $z = 1+i$ $\therefore \bar{z} = 1-i$ is also a root since real coefficients.

~~REDTS~~ $P(x) = ~~(1+i~~(x-(1+i))(x-(1-i))\,Q(x)$

Find 3rd root, α.

$z \cdot \bar{z} \cdot \alpha = -6$.

$(1+i)(1-i) \cdot \alpha = -6$.

$2\alpha = -6$

$\alpha = -3$. ✓

$\therefore$ roots are $1+i$, $1-i$ and -3.

$1+\cancel{i} + 1-\cancel{i} - 3 = -p$ (sum of roots).

$2-3 = -p$

$p = 1$. ✓

$(1+i)(1-i) + (1+i)(-3) + (1-i)(-3) = q$.

$2 \cdot \cancel{4} - 3 - \cancel{3i} - 3 + \cancel{3i} = q$

$2 - 6 = q$

$q = -4$. ✓

3/3

b). If a multiple root $f'(x) = 0$ $f(x) = x^3 + px + q$

~~$f'(x) = 3x^2 + p x = 0$.~~

~~$3x^2 = -p$~~

$f'(x) = 3x^2 + p = 0$. ✓

~~The~~ Let roots be α, α, β =

$\alpha^2\beta = -q$.

$f'(\alpha) = 3\alpha^2 + p = 0$

$p = -3\alpha^2$

$\alpha^2 = \frac{p}{-3}$

$f(x) = x^3 + px + q$.

$= x(x^2+p) + q$

$f(\alpha) = \alpha(\alpha^2+p) + q$

$= \alpha\left(\frac{p}{-3} + p\right) + q$

$= \left(\frac{2p}{3}\right)\alpha + q = 0$

1/3

$2p\alpha + 3q = 0$

$\alpha = \frac{-3q}{2p}$

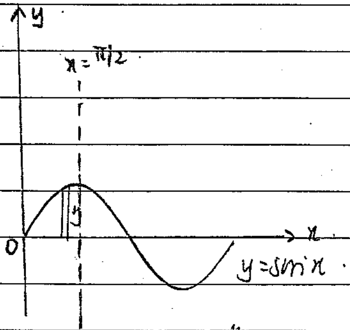
$\alpha^2 = \frac{9q^2}{4p^2}$

so $\left(\frac{-p}{3}\right) = \frac{9q^2}{4p^2}$.

$-4p^3 = 27q^2$

$\therefore 4p^3 + 27q^2 = 0$ qed

c)



Volume of slice $= \cancel{x \cdot x}\ y_0\ y^{4} \cdot \delta x$

$= \cancel{x^2 \delta x}\ y^2 \delta x$

$y^2 = \sin^2 x$

Volume of slice $= \sin^2 x\, dx$

Volume $= \lim\limits_{\delta x \to 0} \sum\limits_{x=0}^{\pi/2} \sin^2 x\, \delta x$

$= \int_0^{\pi/2} \sin^2 x\, dx$

$= \int_0^{\pi/2} \frac{1}{2} - \frac{1}{2}\cos 2x\, dx$

$= \frac{1}{2}\left[x - \frac{1}{2}\sin 2x\right]_0^{\pi/2}$

$= \frac{1}{2}\left(\frac{\pi}{2} - 0 - 0 + 0\right)$

$= \frac{\pi}{4}\ u^3$

2/3

d) i) $z^5 = 1$

$= cis(0 + 2k\pi)$ where $k = 0, 1, 2 \ldots$

$z = cis\left(\frac{2k\pi}{5}\right)$

when $k = 0$ $\quad z_0 = cis\, 0$

$= 1$ $\quad z_1 = cis\frac{2\pi}{5}$

$= 2$ $\quad z_2 = cis\frac{4\pi}{5}$

$= 3$ $\quad z_3 = cis\frac{6\pi}{5}$

$= cis\frac{-4\pi}{5}$

$= 4$ $\quad z_4 = cis\frac{8\pi}{5}$

$= cis\frac{-2\pi}{5}$

2/2

2

$\therefore$ roots are $cis\,0,\ cis\frac{2\pi}{5},\ cis\frac{4\pi}{5},\ cis\frac{-4\pi}{5},\ cis\frac{-2\pi}{5}$.

ii) $z^5 - 1 = (z - cis\,0)(z - cis\frac{2\pi}{5})(z - cis\frac{-2\pi}{5})(z - cis\frac{4\pi}{5})(z - cis\frac{-4\pi}{5})$

$= (z-1)(z^2 - z(cis\frac{2\pi}{5} + cis\frac{-2\pi}{5}) + cis\frac{2\pi}{5} \cdot cis\frac{-2\pi}{5})(z^2 - z(cis\frac{4\pi}{5} + cis\frac{-4\pi}{5})$

$+ cis\frac{4\pi}{5} \cdot cis\frac{-4\pi}{5})$ $\quad z + \bar{z} = 2Re(z)$ $\quad$ 1

$= (z-1)(z^2 - 2z\cos\frac{2\pi}{5} + 1)(z^2 - 2z\cos\frac{4\pi}{5} + 1)$

2/2

iii) $\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} = -\frac{1}{2}$.

2

Sum roots $2\cos\frac{2\pi}{5} + 2\cos\frac{4\pi}{5} + 1 = 0$

$2\cos\frac{2\pi}{5} + 2\cos\frac{4\pi}{5} = -1$

knowing $\quad \cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} = -\frac{1}{2}$.

that $cis\frac{2\pi}{5} + cis\frac{-2\pi}{5} = 2Re(z)$

2/2

2

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MATHEMATICS WRITING BOOKLET

Name:		
Form Class:	Teacher's Initials	Question Number 5

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- If you do not attempt a question still hand in the booklet with "NOT ATTEMPTED" written on the front.
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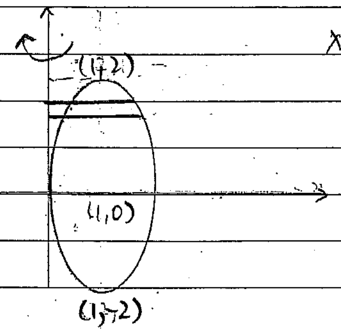
8

a)



x in terms of y.

$4(x-1)^2 = 4-y^2$.

$x-1 =$

Volume slice $= (\pi R^2 - \pi r^2)\delta y$

$R = x_2$

$r = x - x_1$

$x^2 - 2x + 1 + \frac{y^2}{4} = 1$.

$\frac{y^2}{4} = \cancel{1} - x^2 + 2x - \cancel{1}$

$= 2x - x^2$

$y^2 = 8x - 4x^2$

$4x^2 - 8x + y^2 = 0$

Find $x = \frac{8 \pm \sqrt{64 - 16y^2}}{8}$

$= \frac{8 \pm 4\sqrt{4-y^2}}{8}$

$= \frac{2 \pm \sqrt{4-y^2}}{2}$.

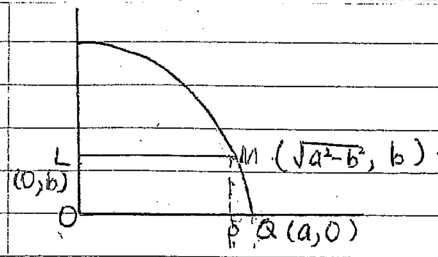
$x_1 = \frac{2 - \sqrt{4-y^2}}{2}$ ✓ / $x_2 = \frac{2 + \sqrt{4-y^2}}{2}$ ✓

Volume slice $= \cancel{(\pi R - \pi r)(\pi R + \pi r)}$

$= \cancel{\pi\left(x - x + \frac{2-\sqrt{4-y^2}}{2}\right)\left(x + x - \frac{2-\sqrt{4-y^2}}{2}\right)}$.

$= \pi\, \frac{2-\sqrt{4-y^2}}{2}\left(2x - \frac{2-\sqrt{4-y^2}}{2}\right).\delta y$.

Question 5c)



LM divides the area in half

i) $x^2+y^2=a$

when $y=b$

$x^2+b^2=a^2$

$x=\sqrt{a^2-b^2}$

Area LMQO $=\frac{1}{2}\left(\frac{\pi r^2}{4}\right)$

$=\frac{1}{2}\left(\frac{\pi a^2}{4}\right)$

Area of LMQO = area LMPO + area of $\frac{1}{2}$ segment MPQ.

$\frac{1}{2}\left(\frac{\pi a^2}{4}\right) = b\sqrt{a^2-b^2} + \frac{1}{2}\left(\frac{1}{2}r^2(2\theta-\sin 2\theta)\right)$

$a^2\frac{\pi}{4} = 2b\sqrt{a^2-b^2} + \frac{1}{2}r^2(2\theta-\sin 2\theta)$

$= 2b\sqrt{a^2-b^2} + \frac{1}{2}a^2\left(2\sin^{-1}\left(\frac{b}{a}\right) - 2\sin\left(\sin^{-1}\frac{b}{a}\right)\times\cos\sin\right.$

$\frac{\pi}{4} = \frac{2b\sqrt{a^2+b^2}}{a^2} + \sin^{-1}\left(\frac{b}{a}\right) - \frac{b}{a}\left(\frac{\sqrt{a^2-b^2}}{a}\right)$

Pythagoras.

$= \frac{b\sqrt{a^2-b^2}}{a^2} + \sin^{-1}\left(\frac{b}{a}\right)$ as required.

ii) If radius of circle is 1 unit show b can be found.

$a=1$ ① $\theta=\sin^{-1}b$ ② $b=\sin\theta$ ③

sub ①,②,③ into part i)

$\sin^{-1}b + \frac{b\sqrt{1-b^2}}{1^2} = \frac{\pi}{4}$

$\theta + \sin\theta\sqrt{1-\sin^2\theta} = \frac{\pi}{4}$

$\theta + \sin\theta\cos\theta = \frac{\pi}{4}$

$2\theta + 2\sin\theta\cos\theta = \frac{\pi}{2}$

$2\theta + \sin 2\theta = \frac{\pi}{2}$ qed

iii) using Newtons method.

Geometry Qu5

b) $\angle AFE = 90°$ ($\angle$ in a semicircle).

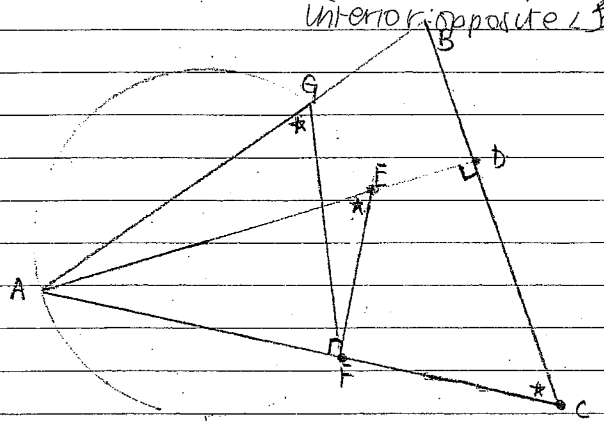
$\angle AFE = \angle EDC$ hence EDFC is a cyclic quadrilateral since the external $\angle$ = internal opposite $\angle$ B

In EDFC, $\angle AEF = \angle DCF$ (external $\angle$ = opposite internal $\angle$).

$= \angle BCF$.

Also $\angle AGF = \angle AEF$ ($\angle$s in the same segment)

So $\angle BCF = \angle AGF$ and GBFC is concyclic since the exterior $\angle$ = interior opposite $\angle$ B.



c) i)

volume of slice $= \pi(R+r)(R-r)$

$= \pi \left(\cancel{\frac{2+\sqrt{4-y^2}}{2} + x - \frac{2+\sqrt{4-y^2}}{2}} \right)\left(\cancel{\frac{2+\sqrt{4-y^2}}{2} - x + \frac{2+\sqrt{4-y^2}}{2}} \right)$

$= \cancel{\pi x}\ \pi (x_2 + x_1)(x_2 - x_1)\ \delta y$

$= \pi \left(\frac{2+\sqrt{4-y^2}}{2} + \frac{2-\sqrt{4-y^2}}{2} \right)\left(\frac{2+\sqrt{4-y^2}}{2} - \frac{2-\sqrt{4-y^2}}{2} \right)\delta y$

$= \pi \left(\frac{4}{2} \right)\left(\sqrt{4-y^2} \right) \delta y$

$= 2\pi \sqrt{4-y^2} \cdot \delta y$

$$\text{Volume} = \lim_{\delta y \to 0} \sum_{y=-2}^{2} 2\pi\sqrt{4-y^2}\,\delta y$$

$= \int_{-2}^{2} 2\pi\sqrt{4-y^2}\, dy$

$= \cancel{4}\pi \int_{-2}^{2} \sqrt{4-y^2}\, dy \quad 2\pi \times \frac{1}{2}\pi \times 4$

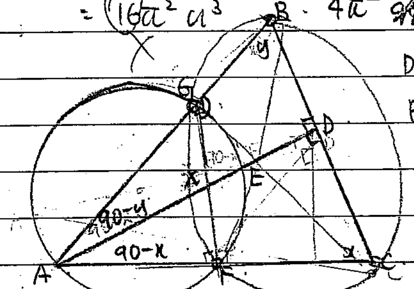
$= \cancel{4}\pi \times \frac{1}{2}\pi r^2$!! silly

$= \cancel{4}\pi \times \frac{1}{2}\pi(\cancel{4})$

$= 16\pi^2 a^3 \quad 4\pi^2$

3/4

b)



D C A are cyclic.

BDA are cyclic.

Prove

- GBED and EDFC are cyclic (exterior $\angle$ of a cyclic quadrilateral is equal to interior opposite $\angle$)

~~$\angle GFA = 90°$ ($\angle$ in a semicircle).~~

~~$\angle XDC = 90°$ (given).~~

~~$\therefore$ XDFC are concyclic (opposite $\angle$s interior $\angle$s of cyclic quadrilateral are supplementary.~~ $\therefore$ Combining the cyclic quadrilateral BGFC are cyclic.

$\angle ACD = 90° - \angle DAC$.

$\angle AFE = 90°$ ($\angle$ in a semicircle)

Similarly $\angle AGE = 90°$

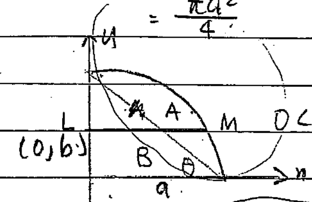
$\angle AGE = \angle GED = 90°$ and $\angle EFA = \angle EDC = 90°$

1/4

c) $x^2 + y^2 = a^2$.

i) Area $= \frac{1}{4}\pi r^2$

$= \frac{\pi a^2}{4}$



Area A + Area B $= \frac{\pi a^2}{4}$

$0 < b < a$.

~~Area A $= \frac{1}{2}r^2(\theta - \sin\theta)$~~

Area B $= \frac{ab}{2}$.

$\theta = \sin^{-1}\frac{b}{a}$.

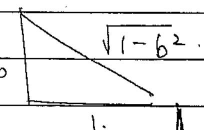
1/3

ii) $\sin^{-1}\frac{b}{a} + \frac{b\sqrt{a^2-b^2}}{a^2} = \frac{\pi}{4}$

$r = 1 = a$

$\sin^{-1} b + \frac{b\sqrt{1-b^2}}{1} = \frac{\pi}{4}$

$\sin^{-1} b = \frac{\pi}{4} - b\sqrt{1-b^2}$



0/3

iii)

0/1

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MATHEMATICS WRITING BOOKLET

Name:	
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Question Number 6

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a) $\frac{x^2}{4} + \frac{y^2}{3} = 1$

i) $b^2 = a^2(1-e^2)$

$3 = 4(1-e^2)$

$= 4 - 4e^2$

$-1 = -4e^2$

$e^2 = \frac{1}{4}$

$e = \frac{1}{2}$

Foci $(\pm ae, 0) \rightarrow (\pm 2 \times \frac{1}{2}, 0) \therefore (\pm 1, 0)$

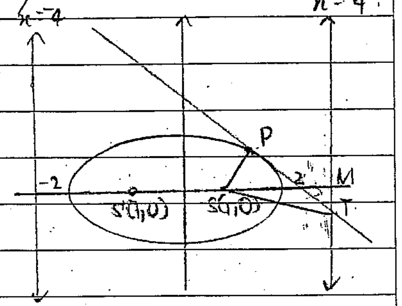
ii) ~~SP + S'P~~ $\frac{SP}{P'}$

Directrix: $x = \pm \frac{a}{e}$

$= \pm \frac{2}{1/2}$

$x = \pm 4$

3/3



ii) $\frac{SP}{PM} = e$ $\quad \frac{S'P}{PM'} = e(PM')$

$SP = e\,PM$

$SP + S'P = e(PM + PM')$

$= e(2OM)$

$= 2e(4)$

$= 8e$

$\therefore$ independent of point $P(x_1, y_1)$

3/3

iii) Tangent

$m_{tangent} = \frac{dy}{dx}$

$\frac{x^2}{4} + \frac{y^2}{3} = 1$ Differentiate implicitly

$\frac{2x}{4} + \frac{2y}{3}\frac{dy}{dx} = 0$

$\frac{2y}{3}\frac{dy}{dx} = -\frac{x}{2}$

$\frac{dy}{dx} = -\frac{x}{2} \times \frac{3}{2y}$

$= \frac{-3x_1}{4y_1}$

$y - y_1 = \frac{-3x_1}{4y_1}(x - x_1)$

$4y_1 y - 4y_1^2 = -3x_1 x + 3x_1^2$

$4y_1 y + 3x_1 x = 3x_1^2 + 4y_1^2$

~~$y_1 y$~~ $\frac{x_1 x}{4} + \frac{y_1 y}{3} = \frac{x_1^2}{4} + \frac{y_1^2}{4}$

$\therefore \frac{x_1 x}{4} + \frac{y_1 y}{3} = 1$ From eqn of ellipse

qed

3/3

(v) $m_{PS} = \frac{y_1}{x_1 - 1}$

Find T when $x = 4 \longrightarrow$ Tangent.

$\frac{4x_1}{4} + \frac{yy_1}{3} = 1$

$x_1 + \frac{yy_1}{3} = 1$

$yy_1 = 3 - 3x_1$

$y = \frac{3-3x_1}{y_1}$

$m_{ST} = \dfrac{\frac{3-3x_1}{y_1} - 0}{4-1}$

$= \dfrac{\frac{\cancel{3}(1-x_1)}{y_1}}{\cancel{3}}$

$= \frac{1-x_1}{y_1}$

$= \frac{-(x_1-1)}{y_1}$

$m_{ST} \times m_{PS} = \frac{-(x_1-1)}{y_1} \times \frac{y_1}{x_1-1}$

$= -1$

$\therefore \angle PST$ is a right $\angle$

3/3

b) $a+b+c=1$

i) $a^2+b^2 \geq 2ab$

Prove $a^2+b^2-2ab \geq 0$

$a^2+b^2-2ab = (a-b)^2 \geq 0$

$\therefore a^2+b^2 \geq 2ab$

1

ii) $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq 9$

LHS $= \frac{bc+ac+ab}{abc}$ ~~[illegible]~~ $= \frac{(bc+ac+ab)(a+b+c)}{abc}$

~~An~~ Consider $a^2+b^2 \geq 2ab$ $= \frac{(a^2b+ab^2+abc+abc+b^2c+bc^2 + a^2c+abc+ac^2)}{abc}$

~~$\therefore a^2+c^2 \geq 2ac$~~

~~$b^2+c^2 \geq 2bc$~~

~~LHS $= \frac{2(ab+bc+ac)}{2abc}$~~

~~$\geq 2($~~

$= \frac{3abc+a(b^2+c^2)+b(a^2+c^2)+c(b^2+a^2)}{abc}$

$\geq \frac{3abc+a(2bc)+b(2ac)+c(2ba)}{abc}$

$\geq \frac{9abc}{abc} \geq 9$ qed

2

$bc+ac+ab \geq 9abc$.

$2(bc+ac+ab) \geq 9abc$.

$a^2+b^2+c^2+a^2+b^2+c^2 \geq 9abc$.

Prove. $2(a^2+b^2+c^2) \geq 9abc$

Consider ~~[illegible]~~ $(a+b+c)^2 = a^2+b^2+2ab+2bc+2ac+c^2$.

$= (a^2+b^2+c^2) + 2(ab+bc+ac)$.

$\therefore$ LHS $= 2(a+b+c)^2 - 4(ab+bc+ac)$

$= 2 - 4(ab+bc+ac)$

$\frac{1}{a^2} + \frac{1}{b^2} \geq 2$

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MATHEMATICS WRITING BOOKLET

Name:

Form Class:

Teacher's Initials

Question Number 7

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12

a) i) P: $y=0$

$x+0=2ct$

$x=2ct$ ✓

Q: $x=0$

$t^2y=2ct$

$y=\frac{2ct}{t^2}$

$=\frac{2c}{t}$ ✓

2

ii) $m_{\text{Tangent}}=-\frac{1}{t^2}$ $t^2y=-x+2ct$

$m_{\text{normal}}=t^2$ ✓ $y=-\frac{1}{t^2}+2ct$

$y-\frac{c}{t}=t^2(x-ct)$ ✓

$y-\frac{c}{t}=t^2x-ct^3$

$y=t^2x+\frac{c}{t}-ct^3$ ✓

2

iii) ~~When~~ At R At R, $x=y$

$x=t^2x+\frac{c}{t}-ct^3$

$x(1-t^2)=\frac{c}{t}-ct^3$

$x=\frac{\frac{c}{t}-ct^3}{1-t^2}$

$=\frac{c}{t}(1-t^4)\times\frac{1}{1-t^2}$

$=\frac{c}{t}\,\frac{\cancel{(1-t^2)}(1+t^2)}{\cancel{1-t^2}}$ ✓

$=\frac{c}{t}(1+t^2)$ qed ✓

$\therefore R\left(\frac{c}{t}(1+t^2),\frac{c}{t}(1+t^2)\right)$ ✓

2

Question 7 b(i)

$I_n = \int \frac{1\,dx}{(x^2+1)^n} = \int u\,dv$

Let $u = (x^2+1)^{-n}$ $\quad v = x$

$\frac{du}{dx} = -2nx(x^2+1)^{-n-1}$ $\quad \frac{dv}{dx} = 1$

$I_n = uv - \int v\,du$

$= x(x^2+1)^{-n} - \int x(-2nx)(x^2+1)^{-(n+1)}\,dx$

$= \frac{x}{(x^2+1)^n} + 2n\int \frac{x^2}{(x^2+1)^{n+1}}\,dx$

$= \frac{x}{(x^2+1)^n} + 2n\int \frac{x^2+1}{(x^2+1)^{n+1}} - \frac{1}{(x^2+1)^{n+1}}\,dx$

$I_n = \frac{x}{(x^2+1)^n} + 2n\int \frac{dx}{(x^2+1)^n} - 2n\int \frac{dx}{(x^2+1)^{n+1}}$

$= \frac{x}{(x^2+1)^n} + 2nI_n - 2nI_{n+1}$

$2nI_{n+1} = \frac{x}{(x^2+1)^n} + I_n(2n-1)$

replace n with $n-1$

$2(n-1)I_n = \frac{x}{(x^2+1)^{n-1}} + I_{n-1}(2(n-1)-1)$

$I_n = \frac{1}{2(n-1)}\left(\frac{x}{(x^2+1)^{n-1}} + (2n-3)I_{n-1}\right)$ qed

iv) $PR^2 = \cancel{4c^2t^2 \ldots 4ct}$

$PR^2 = \left(2ct - \frac{c}{t}(1+t^2)\right)^2 + \left(0 - \frac{c}{t}(1+t^2)\right)^2$

$= \left(2ct - \frac{c}{t} - ct\right)^2 + \left(-\frac{c}{t}(1+t^2)\right)^2 = \left(ct - \frac{c}{t}\right)^2 + \left(-\frac{c}{t}(1+t^2)\right)^2$ ✓

$PQ^2 = \left(0 - \frac{c}{t}(1+t^2)\right)^2 + \left(\frac{2c}{t} - \frac{c}{t}(1+t^2)\right)^2$

$= \left(-\frac{c}{t}(1+t^2)\right)^2 + \left(\frac{c}{t} - ct\right)^2$ ✓

$= \left(ct - \frac{c}{t}\right)^2 + \left(-\frac{c}{t}(1+t^2)\right)^2$

$= PR^2$

$\therefore PQ = PR$

$\therefore$ PRQ is isosceles

3/3

b) $I_n = \int \frac{dx}{(x^2+1)^n}$

$= \int \frac{1}{(x^2+1)} \cdot \frac{1}{(x^2+1)^{n-1}}\,dx$

Let $u = \frac{1}{x^2+1}$ $\quad v = I_{n-1}$

$du = -2x(x^2+1)^{-2}$ $\quad dv = \int \frac{1}{(x^2+1)^{n-1}}\,dx$

$\frac{-2x}{(x^2+1)^2}\,dx$

$I_n = \left(\frac{1}{x^2+1}\right)I_{n-1}$

b) i) $I_n = \int \frac{dx}{(x^2+1)^n}$ integrate by parts

$= \int \frac{1}{(x^2+1)^{n-1}} \cdot \frac{1}{(x^2+1)}\, dx$

Let $u = \frac{1}{x^2+1}$

$\frac{0}{4}$

ii) $\int_0^1 \frac{dx}{(x^2+1)^2} = \frac{1}{2(2-1)}\left(\frac{x}{x^2+1} + (2n-3)I_1\right)$

$= \left[\frac{1}{2}\left(\frac{x}{x^2+1} + (2n-3)\tan^{-1}x\right)\right]_0^1$

$= \frac{1}{2}\left(\frac{1}{2} + \tan^{-1}1 - 0 - 0\right)$

$= \frac{1}{2}\left(\frac{1}{2} + \frac{\pi}{4}\right)$

$= \frac{1}{4} + \frac{\pi}{8}$

$\frac{2}{2}$

ASCHAM SCHOOL

MATHEMATICS WRITING BOOKLET

Name:

Form Class:

Teacher's Initials

Question Number 8

- Enter the information requested in each of the boxes above
- Take a new writing booklet for each question
- You may ask for an extra writing booklet if you need one. Label it as above and put it inside the first booklet for the question.
- If you do not attempt a question still hand in the booklet with "NOT ATTEMPTED" written on the front.
- Write the question part in the margin
- Write on the ruled lines in black or blue ink

Do not write in this box

8

a) BV^2N

$M\ddot{x} = -BV^2$

i) $\ddot{x} = -\frac{BV^2}{M}$

$v\frac{dv}{dx} = \frac{-Bv^2}{m}$

$\frac{dv}{dx} = \frac{-Bv}{m}$

$\frac{dx}{dv} = -\frac{m}{Bv}$

$x = -\frac{m}{B}\int \frac{1}{v}\,dv$

$= -\frac{m}{B}\ln v + C$

when $x = 0$ $v = V$

$0 = -\frac{m}{B}\ln V + C$

$C = \frac{m}{B}\ln V$

$x = -\frac{m}{B}\ln v + \frac{m}{B}\ln V$

$= \frac{m}{B}\ln\frac{V}{v}$ when $x = D_1$ $v = u$

$\therefore D_1 = \frac{m}{B}\ln\frac{V}{u}$ qed.

4

4

ii) ~~Bx~~ $M\ddot{x} = -(A+Bv^2)$.

$\ddot{x} = -(A+Bv^2) \div M$.

$v\frac{dv}{dx} = \frac{-(A+Bv^2)}{M}$

$\frac{dx}{dv} = \frac{Mv}{-(A+Bv^2)}$

$x = \int \frac{Mv}{-(A+Bv^2)}\,dv$

$= -\cancel{\frac{A}{2MB}} - \frac{M}{2B}\ln(A+Bv^2) + C_1$

When $x = 0$ $v = u$

$0 = -\frac{M}{2B}\ln(A+Bu^2) + C_1$

$C_1 = \frac{M}{2B}\ln(A+Bu^2)$

~~Cop~~ Question 8b)

b) i)

x	1	2	3	$n-1$
$y = n\sin\frac{\pi x}{2n}$	$n\sin\frac{\pi}{2n}$	$n\sin\frac{2\pi}{2n}$	$n\sin\frac{3\pi}{2n}$	$n\sin\frac{(n-1)\pi}{2n}$

Areas of rectangles $= 1\times n\sin(\frac{\pi}{2n}) + 1\times n\sin(\frac{2\pi}{2n}) + 1\times n\sin(\frac{3\pi}{2n}) + \dots + 1\times n\sin(\frac{(n-1)\pi}{2n})$

Area under curve $= \int_0^n n\sin(\frac{\pi x}{2n})\,dx$

$= -\left[n\left(\frac{2n}{\pi}\cos(\frac{\pi x}{2n}\right)\right]_0^n$

$= -\frac{2n^2}{\pi}\left(\cos\frac{\pi}{2} - \cos 0\right)$

$= \frac{2n^2}{\pi}$

area of rectangles < area under curve etc.

ii) from i) $\sum_{r=1}^{n-1} \sin(\frac{\pi r}{2n}) < \frac{2n}{\pi}$

$2n\sum_{r=1}^{n-1} \sin\frac{\pi r}{2n} < \frac{4n^2}{\pi} = \frac{4\pi n^2}{\pi^2}$

but $\frac{4}{\pi} < \frac{\pi}{2}$ since $(8 < \pi^2)$

so $\frac{4\pi n^2}{\pi^2} < \frac{\pi^2 n^2}{2\pi} = \frac{\pi n^2}{2}$

$\therefore 2n\sum_{r=1}^{n-1} \sin\frac{\pi r}{2n} < \frac{4\pi n^2}{\pi^2} < \frac{\pi n^2}{2}$ qed

$x = -\frac{M}{2B}\ln(A+Bv^2) + \frac{M}{2B}\ln(A+Bu^2)$.

$= \frac{M}{2B}\ln\frac{A+Bu^2}{\cancel{A}\,\cancel{BA}+Bv^2}$

When $x = D_2$ $v = 0$

$D_2 = \frac{M}{2B}\ln\frac{A+Bu^2}{\cancel{B}A+0}$

$= \frac{M}{2B}\ln\left(\frac{A+Bu^2}{\cancel{B}A}\right)$

$\therefore D_2 = \frac{M}{2B}\ln(1+\frac{B}{A}u^2)$ qed

4/4

iii) 0/2

b) $y = n\sin\frac{\pi x}{2n}$ $0 \le x \le 2n$

i) Area ~~a~~ when $x=1$ $y = n\sin\frac{\pi}{2n}$

Area $= (\sin\frac{\pi}{2n})\,1\times n$

when $x = 2$ Area $= (\sin\frac{2\pi}{2n})\times n$ etc

$\therefore$ total area $= \sin\frac{\pi}{2n} + \sin\frac{2\pi}{2n} + \sin\frac{3\pi}{2n} + \ldots + \sin\frac{\pi(n-1)}{2n} + \sin\frac{\pi n}{2n}$

$= n(\sin\frac{\pi}{2n} + \sin\frac{2\pi}{2n} + \sin\frac{3\pi}{2n} + \ldots + \sin\frac{\pi(n-1)}{2n})$ since $\sin\frac{\pi n}{2n} = 0$

0/3

$\sin\frac{\pi}{2n} + \sin\frac{2\pi}{2n} + \sin\frac{3\pi}{2n} + \ldots + \sin\frac{4\pi(n-1)}{2n} < \cancel{\int_0^n n\sin\frac{\pi x}{2n}\,dx}\cdot 2n\times n$

$< \cancel{2n^2\cos\frac{\pi x}{2n}}\cdot 2n^2$

ii) $2n\sum_{r=1}^{n-1}\sin\frac{\pi r}{2n} < \frac{\pi n^2}{2}$

$2n\left(\sin\frac{\pi}{2n} + \sin\frac{2\pi}{2n} + \ldots + \frac{\sin\pi(n-1)}{2n}\right) < \frac{2n}{\pi}\times$

0/2

$M = 100$ tonnes $= 100\,000$ kg

$V = 90$

$u = 60$

$BV^2 = 125V^2$ so $B = 125$

$A = 75\,000$ N

$D = D_1 + D_2$

$= \frac{M}{B}\ln\left(\frac{V}{u}\right) + \frac{M}{2B}\ln\left(1+\frac{B}{A}u^2\right)$

$= \frac{100000}{125}\ln\left(\frac{90}{60}\right) + \frac{100000}{2\times125}\ln\left(1+\frac{125}{75000}\times60^2\right)$

$= 800\ln(\frac{3}{2}) + 400\ln 7$

$= 1102.74$ m (6 sf)