Area & Volume

THE CIRCLE AND SECTOR

Questions relating to the **area** and **circumference** of a circle are frequently in tests. π (pronounced pie) is an **irrational** number which means it does not have an exact fractional or decimal equivalent. In circle problems involving π use the value on your calculator, unless told otherwise. It is important to know the formulae for both the area and the circumference of a circle, as they are not listed on the formula sheet.

$$\text{Circumference} = 2 \times \pi \times r = 2\pi r$$
$$\text{Area} = \pi \times r \times r = \pi r^2$$

where r = the radius of the circle.

Example: Find the area of a circle of diameter 20 cm. Use $\pi = 3.142$

Solution: Radius = diameter $\div$ 2 = 10 cm

$$\therefore \quad \text{Area} = \pi r^2 = 3.142 \times 10^2 = 314.2 \text{ cm}^2$$

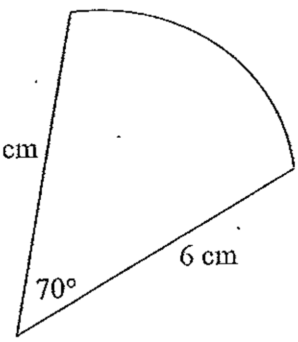
The formulae involving sectors appear on the formula sheet, so it is necessary to know how to apply them in a variety of situations.

$$\text{Area of a Sector:} \quad A = \frac{\theta}{360}\pi r^2$$

$$\text{Arc Length of a Sector:} \quad l = \frac{\theta}{360} 2\pi r$$

where θ = degrees in central angle

Example: Calculate the area and perimeter of a sector with a radius of 6 cm and a central angle of 70°.

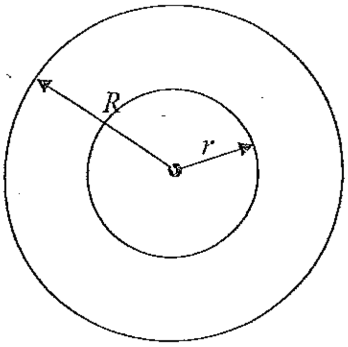


Solution:

$$\text{Area} = \frac{\theta}{360}\pi r^2$$
$$= \frac{70}{360} \times \pi \times 6^2 \text{ cm}^2$$
$$= 21.99 \text{ cm}^2$$

$$\text{Perimeter} = \text{arc length} + 2 \times \text{radius}$$
$$= \frac{70}{360} \times 2 \times \pi \times 6 \; + \; 2 \times 6 \text{ cm}$$
$$= 19.33 \text{ cm}$$

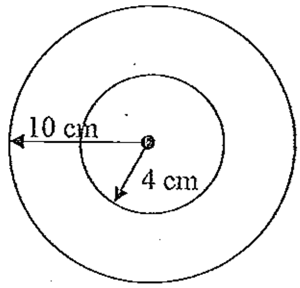
THE ANNULUS



An **annulus** is the area between two concentric circles. Concentric circles have the same centre.

> Area of an annulus:
>
> $$A = \pi (R^2 - r^2)$$
>
> where R = radius of outer circle
> r = radius of inner circle

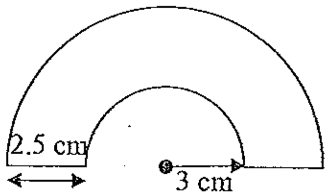
Example (i): Calculate the shaded area:



Solution:

$$\begin{aligned} A &= \pi (R^2 - r^2) \\ &= \pi (10^2 - 4^2)\ \text{cm}^2 \\ &= \pi \times 84\ \text{cm}^2 \\ &= 263.89\ \text{cm}^2 \text{ (to 2 d.p)} \end{aligned}$$

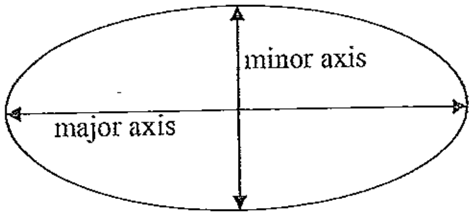
Example (ii): Calculate the shaded area correct to 3 significant figures.



Solution: This is half an annulus with $r = 3$ cm and $R = (2.5 + 3)$ cm $= 5.5$ cm

$$\begin{aligned} \text{Area} &= \frac{1}{2} \times \pi (R^2 - r^2) \\ &= \frac{1}{2} \times \pi (5.5^2 - 3^2)\ \text{cm}^2 \\ &= \frac{1}{2} \times \pi \times 21.25\ \text{cm}^2 \\ &= 33.4\ \text{cm}^2 \text{ (to 3 sign. fig)} \end{aligned}$$

THE ELLIPSE

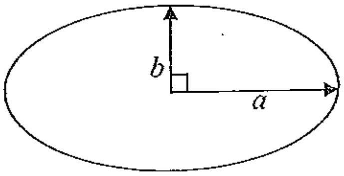


Ellipses have two axes of symmetry. The **major** axis is the longer axis and the shorter axis is called the **minor** axis.

The **semi-major** axis is $\frac{1}{2}$ the major axis.

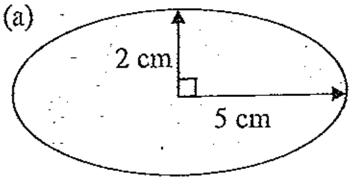
The **semi-minor** axis is $\frac{1}{2}$ the minor axis.

> Area of an ellipse = πab
>
> where a = length of semi-major axis
> b = length of semi-minor axis

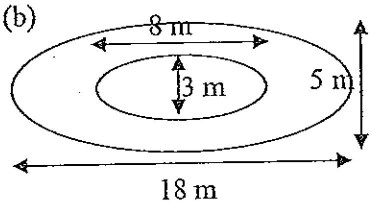


Example: Calculate, correct to 2 decimal places, the shaded areas:

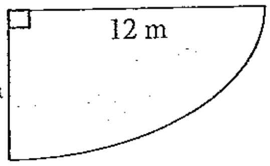
(a)



(b)



(c)



Solution:

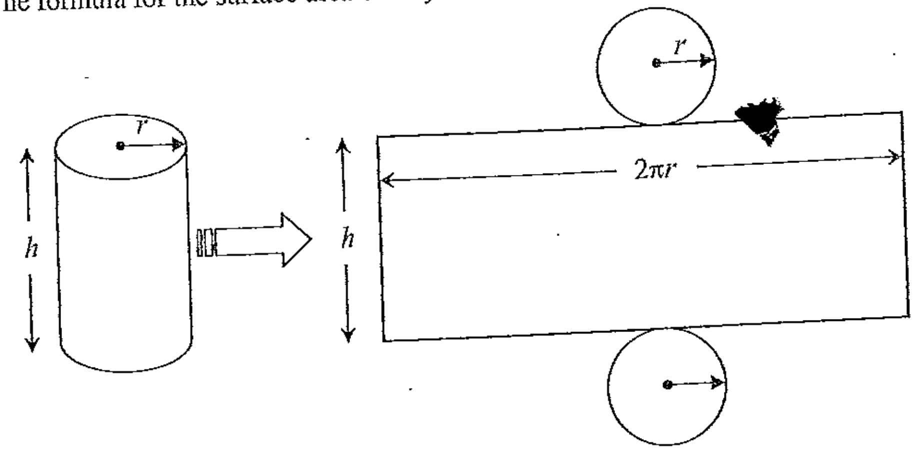
(a) $A = \pi ab$
$= \pi \times 5 \times 2 \text{ cm}^2$
$= 31.42 \text{ cm}^2$

(b) A = outside ellipse – inside ellipse
$= \pi a_1 b_1 - \pi a_2 b_2$
$= \pi \times 9 \times 2.5 - \pi \times 4 \times 1.5 \text{ m}^2$
$= 51.84 \text{ m}^2$

(c) $A = \frac{1}{4} \times \pi ab$
$= 0.25 \times \pi \times 12 \times 8 \text{ m}^2$
$= 75.40 \text{ m}^2$

SURFACE AREA OF A CYLINDER

The formula for the surface area of a cylinder can be deducted from its net.



The curved surface of the cylinder opens up to form a rectangle. The length of the rectangle is equal to the circumference of the circular base. The area of the curved surface is given by:

Length × height = circumference of base × height

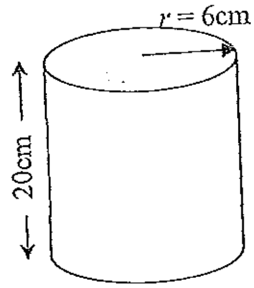
$$\text{Curved surface area} = 2\pi rh$$

Total surface area = area of curved surface + area of 2 circular ends

$$\text{Total surface area} = 2\pi rh + 2\pi r^2$$

Example: Calculate the surface area of a closed cylinder having a diameter of 12 cm and a height of 20 cm. Answer correct to 4 significant figures.

Solution: Diameter is 12 cm, ∴ radius is 6 cm



Area of base $= \pi \times 6^2$
$= 36\pi \text{ cm}^2$

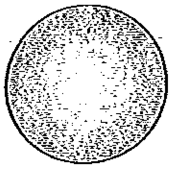
Curved surface area $= 2 \times \pi \times 6 \times 20$
$= 240\pi \text{ cm}^2$

Total surface area $= 2 \times 36\pi + 240\pi$
$\approx 980.2 \text{ cm}^2$ correct to 4 sign. fig.

Note: The formula for calculating the surface area of a cylinder does not appear on the examination formula sheet. It is not necessary to learn the formula given here as it is very unlikely that you will be asked to calculate the total surface area of a cylinder. You may however be required to find the base area or substitute values into a formula given as part of a question.

SURFACE AREA OF A SPHERE

The formula for the surface area of a sphere appears on the formula sheet so it is important that you learn how to use the formula in a variety of situations.



Surface area of a sphere: $A = 4\pi r^2$

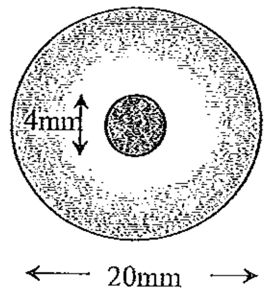
where r = radius

Example (i): Calculate the surface area of a sphere with diameter 7 cm.

Solution: Diameter is 7 cm, so the radius is 3.5 cm

$$\text{Surface area} = 4 \times \pi \times 3.5^2$$
$$= 153.94 \text{ cm}^2 \text{ correct to 2 decimal places}$$

Example (ii): Calculate the surface area of a spherical bead of diameter 20 mm which has a cylindrical hole of diameter 4 mm drilled through the centre.



Solution: sphere r = 10 mm

$$\text{Surface area of sphere} = 4\pi r^2$$
$$= 4 \times \pi \times 10^2$$
$$= 400\pi \text{ mm}^2$$

We need to determine the surface area of the hole which is a cylinder with r = 2 mm, h = 20 mm

$$\text{Curved surface area of cylinder} = 2\pi r h$$
$$= 2 \times \pi \times 2 \times 20$$
$$= 80\pi \text{ mm}^2$$

$$\text{Circular ends of cylinder} = 2\pi r^2$$
$$= 2 \times \pi \times 2^2$$
$$= 8\pi \text{ mm}^2$$

Surface Area of bead = surface area of sphere + curved surface area of cylinder – circular ends of cylinder

$$= 400\pi + 80\pi - 8\pi$$
$$= 472\pi$$
$$\approx 1482.83 \text{ mm}^2 \text{ correct to 2 decimal places}$$

VOLUME OF A PRISM

Any solid with a **uniform** or **constant** 'cross sectional area' is called a PRISM. Therefore there are many different types of prisms, so it is necessary for you to have a good understanding of the area formulae in order to be able to calculate volumes.

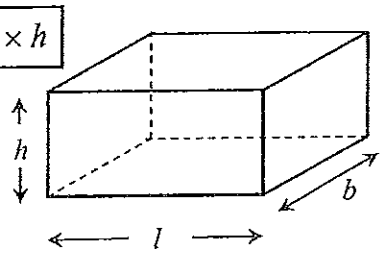
$$\boxed{\text{Volume of any prism} = A \times h}$$

where A = cross sectional area of prism
h = height or depth of the prism

IN PARTICULAR:

$$\boxed{\text{Volume of a rectangular prism} = l \times b \times h}$$

where l = length
b = breadth
h = height

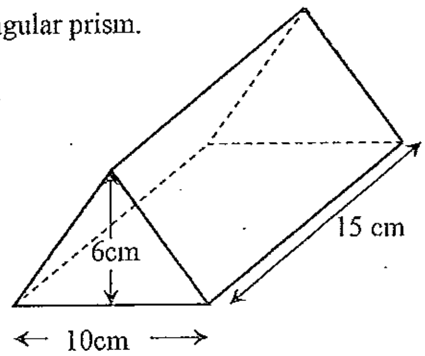


Example (i): Calculate the volume of a rectangular shipping container which is 8 metres in length, 3 metres wide and 4 metres high.

Solution:

$$\begin{aligned} V &= l \times b \times h \\ &= 8 \times 3 \times 4 \\ &= 96 \text{ m}^3 \end{aligned}$$

Example (ii): Calculate the volume of the triangular prism.



Solution: The uniform cross sectional area is a triangle.

$$\begin{aligned} \text{Area of triangle} &= \frac{1}{2}(b \times h) \\ &= \frac{1}{2}(10 \times 6) \\ &= 30 \text{ cm}^2 \end{aligned}$$

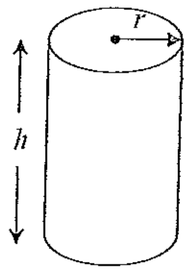
$$\begin{aligned} \therefore \quad \text{Volume} &= A \times h \\ &= 30 \times 15 \\ &= 450 \text{ cm}^3 \end{aligned}$$

NOTE: For volume, the answer must be expressed in cubic units, i.e. units^3

VOLUME OF COMMON SOLIDS

Formulae for several common solids appear on the examination formula sheet and it is important to be able to use the formula accurately.

CYLINDER

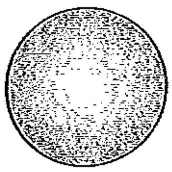


Volume = $\pi r^2 h$

where r = radius of base

h = height

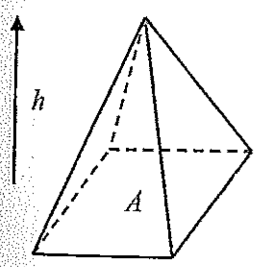
SPHERE



Volume = $\frac{4}{3} \pi r^3$

where r = radius of sphere

PYRAMID

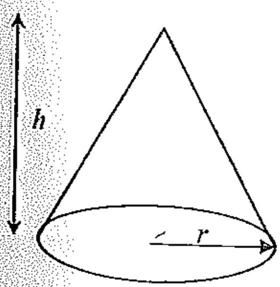


Volume = $\frac{1}{3} \times A \times h$

where A = area of base

h = perpendicular height

CONE



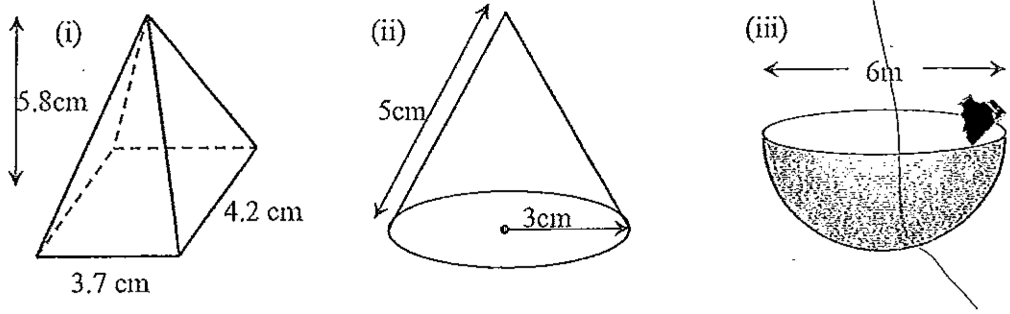
Volume = $\frac{1}{3} \pi r^2 h$

where r = radius of circular base

h = perpendicular height

VOLUME EXAMPLES

Find the volumes of the 3 solids shown correct to 2 decimal places:



Solutions: (i) Volume of pyramid $= \frac{1}{3} \times A \times h$

$= \frac{1}{3} \times (3.7 \times 4.2) \times 5.8$

$= 30.04 \text{ cm}^3$

(ii) Must first calculate the vertical height, h, of the cone using Pythagoras' Theorem.

$5^2 = h^2 + 3^2$

$\therefore \quad h^2 = 25 - 9$

$\therefore \quad h = \sqrt{16}$

$\therefore \quad h = 4$

Volume of cone $= \frac{1}{3} \pi r^2 h$

$= \frac{1}{3} \times \pi \times 3^2 \times 4$

$= 37.70 \text{ cm}^3$

(iii) Volume of sphere $= \frac{4}{3} \pi r^3$

Volume of hemisphere $= \frac{1}{2} \times \left(\frac{4}{3} \pi r^3 \right)$

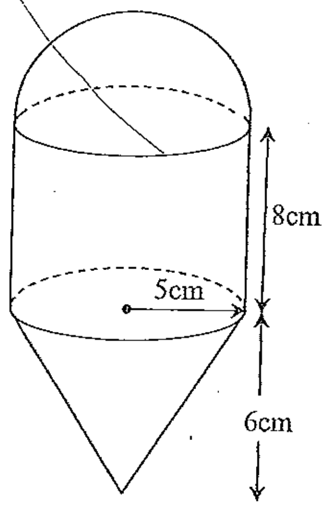
$= \frac{1}{2} \times \left(\frac{4}{3} \times \pi \times 3^3 \right)$

$= 56.55 \text{ m}^3$

COMPOSITE VOLUMES

The volumes of more complicated solids can be calculated by dividing the solid up into simpler solids. The volume of each of these simpler solids is calculated separately before finally adding together to give the total volume.

Example (i): Calculate the volume of the solid shown correct to 1 decimal place.



Solution: The solid is made up of a hemisphere, cylinder and cone.

$$\text{Volume of hemisphere} = \frac{1}{2}\times\left(\frac{4}{3}\pi r^3\right)$$

$$= \frac{1}{2}\times\left(\frac{4}{3}\pi\times 5^3\right)$$

$$= 83\frac{1}{3}\pi \text{ cm}^3$$

$$\text{Volume of cylinder} = \pi r^2 h$$

$$= \pi\times 5^2\times 8$$

$$= 200\pi \text{ cm}^3$$

$$\text{Volume of cone} = \frac{1}{3}\pi r^2 h$$

$$= \frac{1}{3}\pi\times 5^2\times 6$$

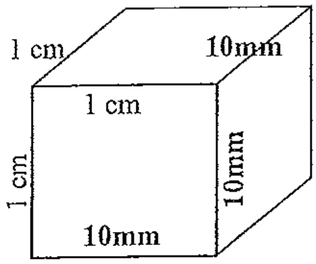
$$= 50\pi \text{ cm}^3$$

$$\text{Total volume} = 83\frac{1}{3}\pi + 200\pi + 50\pi$$

$$= 333\frac{1}{3}\times\pi$$

$$= 1047.2 \text{ cm}^3$$

VOLUME AND CAPACITY UNITS



$1\text{cm} \times 1\text{cm} \times 1\text{cm} = 10\text{mm} \times 10\text{mm} \times 10\text{mm}$

$\therefore \boxed{1\text{cm}^3 = 1\,000\text{ mm}^3}$

Similarly,

$1\text{ m} \times 1\text{ m} \times 1\text{ m} = 100\text{cm} \times 100\text{cm} \times 100\text{cm}$

$\therefore \boxed{1\text{ m}^3 = 1\,000\,000\text{ cm}^3}$

Capacity is a measure of the volume of fluids. The capacity of a container is the amount of liquid it can hold. The base unit is the **litre**.

1000 millilitres (mL)	=	1 litre (L)
1000 litres	=	1 kilolitre (kL)

For the metric system of units:

$1\text{ cm}^3 = 1\text{ mL}$

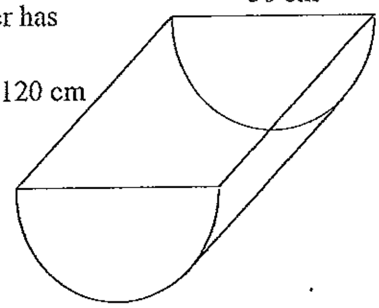
$1000\text{ cm}^3 = 1\text{ L}$

$1\text{ m}^3 = 1\text{ kL}$

1 cm^3 or 1 mL water weighs 1 g

1000 cm^3 or 1 L water weighs 1 kg

Example: A trough in the shape of a half cylinder has dimensions as shown in the diagram. How many litres of water does the trough hold when it is filled to the top?



Solution: The trough is a half a cylinder,

$$\therefore V = \frac{1}{2}\pi r^2 h \quad \text{where } r = 28 \text{ and } h = 120$$

$$= \frac{1}{2} \times \pi \times 28^2 \times 120$$

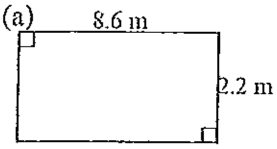
$$= 147\,780.52\text{ cm}^3$$

$$= 147\,780.52\text{ mL} \quad \approx 148\text{ L (to the nearest litre)}$$

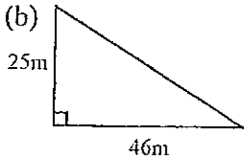
REVIEW EXERCISE – LEVEL 1

1. Find the area of the following:

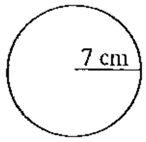
(a)



(b)



(c)



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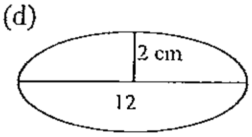

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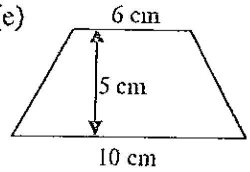

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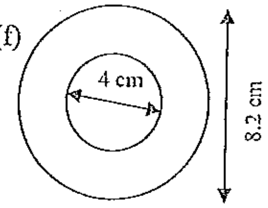
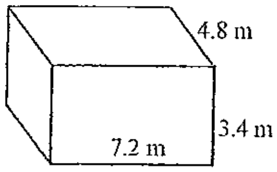


2. Calculate the surface area and volume of these solids:

(a)



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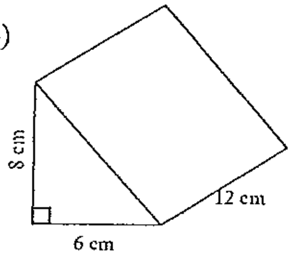

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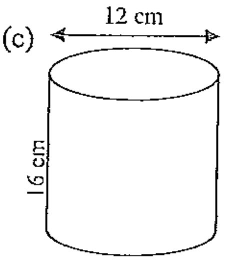

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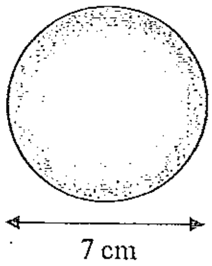

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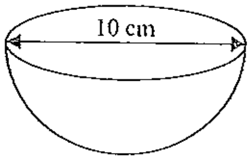

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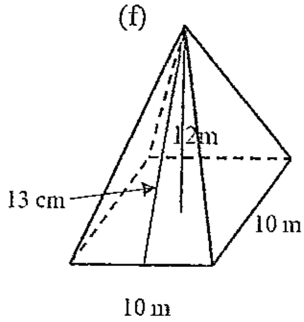
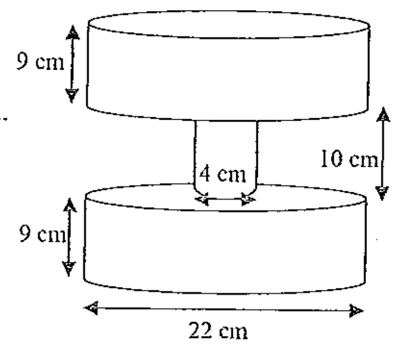


3. Calculate the capacity of a cylindrical glass of diameter 7 cm which is filled to a height of 9 cm. How many such glasses could be filled from a 2 litre bottle of lemonade?

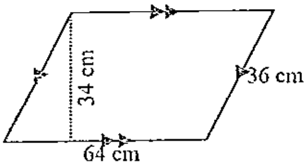
4. Calculate the volume and surface area of the following composite solid.



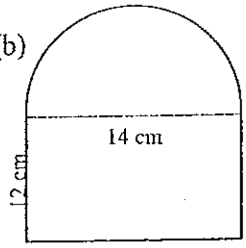
REVIEW EXERCISE – LEVEL 2

1. Calculate the area of these shapes:

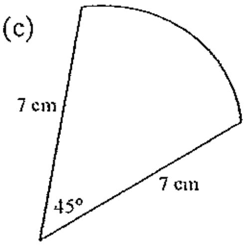
(a)



(b)

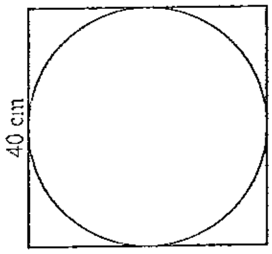


(c)

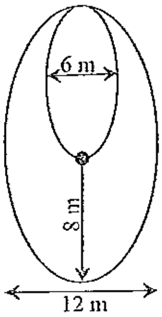


2. Find the shaded area

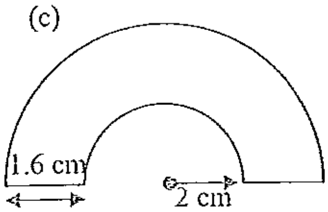
(a)



(b)



(c)

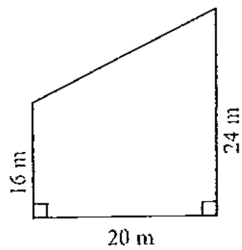


3. A cylindrical vase, diameter 10 cm, is filled with 200 mL of water. What is the depth of the water?

4. A bedroom is 4.2 metres long, 3.4 metres wide and 2.2 metres high. Doors and windows amount to 10.6 m^2 of the area of the walls. A litre of paint covers 4.4 m^2. How many litres are required to paint the walls and ceiling of the room?

5. Jessica wants to put decking around her circular pool, which has a diameter of 12.4 metres. Decking costs \$23 per square metre. What is the cost of putting decking, 2 metres wide, around the pool?

6. The diagram shows John's backyard. John has to apply fertiliser at 250 grams per square metre. How many kilograms of fertiliser, at \$3.20 per kg, will he need and what will it cost?



7. An ingot of gold, 30 cm long, 12 cm wide and 8 cm high is melted down and reshaped into cubes, each with sides 4 cm. How many cubes are obtained?