PROBABILITY

A. The Meaning of Probability

a. Definitions

- An ***outcome*** is the result of an experiment or game
- The ***sample space*** is the set of all possible outcomes
- An ***event*** is a group of one or more outcomes
- The ***theoretical probability*** an event occurring is calculated using the formula:

$$P(E) = \frac{number\ of\ favourable\ outcomes}{total\ number\ of\ outcomes} = \frac{n(E)}{n(S)}$$

EXAMPLE 1:

One card is selected at random from a normal deck of playing cards. What is the probability that the card is:

a. an ace?
b. a red ace?
c. a picture card?
d. a club or a red ace?

b. The Range of Probabilities

- A probability value can be expressed as a fraction or decimal ranging from 0 to 1
- or as a percentage ranging from 0% to 100%
- $0 \le P(E) \le 1$
- if $P(E) = 0$ the event is impossible
- if $P(E) = 1$ the event is certain

c. Complementary Events

- If $P(E)$ is the probability that an event will occur then $P(\tilde{E})$ is the probability that the event ***will not*** occur
- $\tilde{E}$ is called the ***complementary*** event and $P(E) + P(\tilde{E}) = 1$

$$P(\tilde{E}) = 1 - P(E)$$

P(the event does not occur) = 1 – P(the event occurs)

EXAMPLE 2:

A jar contains 12 red, 7 yellow, 8 white and 13 black jellybeans. Express as a decimal the probability that a jellybean randomly selected from the jar is:

a. green
b. black or white
c. not red
d. black, yellow or red
e. not blue

d. Experimental Probability

- The experimental probability of an event occurring is its relative frequency.

$$\text{Relative frequency of an event} = \frac{\text{frequency of the event}}{\text{total frequency}}$$

EXAMPLE 3:

A roulette wheel at a casino has 37 numbers, 0 to 36. The results of 250 spins of the wheel are shown in the table.

Outcome	Frequency
0	5
1 – 9	60
10 – 18	62
19 – 27	64
28 - 36	59

Express your answers to the following questions as percentages (to 1 decimal place where necessary).

a. What is the experimental probability (relative frequency) of spinning a number from 19 to 27?
b. What is the calculated probability (theoretical probability) of spinning a number from 19 to 27?
c. What is the calculated probability of spinning zero?
d. What is the experimental probability of spinning zero?
e. What is the experimental probability of spinning a number less than 10?

- New Century General Maths Ex 6.01 Page 197

B. Tree Diagrams & Tables

a. Using a Tree Diagram

- A ***tree diagram*** allows all the possible outcomes of a multistage event to be listed ***systematically***
- It ensures all possible arrangements have been covered
- Branches are used to illustrate the possibilities at every stage or level
- A ***multistage*** event consists of two or more events occurring together

EXAMPLE 4:

A coin is tossed three times. Find all possible outcomes (the sample space) by listing them and by drawing a tree diagram.

EXAMPLE 5:

The digits 7, 2, 3 & 6 are written on separate cards and two of them are selected at random to form a two digit number.

a. Use a tree diagram to list all possible outcomes
b. What is the probability that the number formed is divisible by 3?

b. Using a Table

- The outcomes of a two stage event can also be listed systematically in a table

EXAMPLE 6:

A pair of dice are rolled and the sum of the numbers calculated. Use a table to list all possible outcomes and hence find the probability of rolling a sum of ten.

- New Century General Maths Ex 6.02 Page 200

C. The Multiplication Principle for Counting

- More advanced counting techniques helps calculate probabilities when the total number of possibilities is very large
- Probability trees and tables are not practical in these cases
- If event *A* has *m* outcomes and event *B* has *n* outcomes then events *A* & *B* together have *m x n* possible arrangements
- Similarly If event *A* has *a* outcomes, event *B* has *b* outcomes, event *C* has *c* outcomes etc then events *A, B, C,* ..... together have *a x b x c x* .... possible arrangements

EXAMPLE 7:

From these lists of given names and surnames determine the number of possible given name – surname combinations.

Given Name	**Surname**
Alex	Garrett
Brionne	Hijazi
Cate	Iacono
Daniel	Johnson
Erin	Kee
Fiona	

EXAMPLE 8:

From this menu calculate the number of different 3-course meals possible.

Entree	**Main Course**	**Dessert**
Pumpkin Soup	Cajun Prawns	Pavlova
Calamari Rings	Steak Diane	Black Forest Cake
Potato Wedges	Roast Lamb	Chocolate Mousse
	Chicken Dijon	Mangoes and Ice Cream
	Grilled Perch	

EXAMPLE 9:

An internet user password is made up of 6 characters, alphabetic or numeric. How many different passwords are possible?

- New Century General Maths Ex 6.03 Page 203

D. Counting Arrangements

EXAMPLE 10:

Six friends – Rachel, Ross, Chandler, Monica, Pheobe and Joey – stand in line for a group photo. How many possible arrangements of position are there?

EXAMPLE 11:

Harry's office has 10 carspaces for employees. In how many different ways can 10 cars be parked in 10 spaces?

a. Factorial Notation

- In mathematics factorial notation represents a type of multiplication
- The symbols 4! and 8! are read "4 factorial" and "8 factorial"
- $4! = 4 \times 3 \times 2 \times 1$
- $8! = 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$
- $x!$ means the product of all the numbers from x down to 1
- The number of ways n different items can be arranged is $n!$
- $n! = n \times (n-1) \times (n-2) \times \dots\dots\dots \times 3 \times 2 \times 1$

EXAMPLE 12:

Suppose the 6 friends from example 10 want to have smaller group photos on a couch that sits 4. How many different photos are possible? This means in how many ways can you arrange 6 people into 4 positions?

EXAMPLE 13:

A girl's school is electing a captain and a vice-captain. There are 5 candidates, Ang, Beth, Cassie, Dasha and Elena.

a. How many possible pairings of captain and vice-captain are possible?
b. List the combinations.

- New Century Maths General Ex 6.04 Page 207

E. Counting Unordered Selections

- The number of unordered selections that can be made from n different items when there are r positions is:

$$No\ of\ arrangements = \frac{n\times(n-1)\times(n-2)\times\ldots\ldots}{r!}[r\ terms]$$

EXAMPLE 14:

Six friends visit a tennis court. How many different doubles teams are possible?
Let the players be denoted by the letters A, B, C, D, E & F.
When the teams are formed remember that AB and BA are the same arrangement, order doesn't matter.

EXAMPLE 15:

Three students are to be selected from a group of 8 to represent the school. How many combinations of 3 students are possible?

EXAMPLE 16:

In lotto 6 balls are randomly selected from 44 numbered balls. How many different selections of 6 balls are possible?

EXAMPLE 17:

A student council must select 8 junior members and 4 senior members. There are 25 junior candidates and 11 senior candidates.

a. How many different ways can the 8 junior representatives be chosen?
b. How many different arrangements of 4 senior representatives exist?
c. How many different student councils are possible?

- New Century General Maths Ex 6.05 Page 210

F. Ordered and Unordered Selections

- You need to be able to distinguish between these two types of selections
- ***Ordered selections,*** called ***permutations,*** are arrangements where order within the group is ***important***
- There are ***more*** possible ordered arrangements because ABC, BAC, CBA etc are all counted as ***different***
- Examples of ordered selections include:
  - first three places in a race
  - captain and vice-captain in a team
  - photos arranged on a page in an album

- ***Unordered selections,*** called ***combinations***, are arrangements where order within the group is ***not important***
- There are ***fewer*** possible ordered arrangements because ABC, BAC, CBA etc are all counted as ***the same***
- Examples of ordered selections include:
  - choosing 5 players for a basketball team
  - selecting 6 numbers for lotto
  - choosing 20 items to be tested

Arrangements ***Arranging in Order*** ***Permutations***

Ordered Selections

Selections ***Groupings*** ***Combinations***

Unordered Selections

EXAMPLE 18:

Poker is a card game in which each player is dealt a hand of 5 cards from a normal deck of 52 cards.

a. How many different hands of 5 cards are possible?
b. In how many ways can 3 aces be selected from the 4 aces in the deck?
c. In how many ways can 2 queens be selected from the 4 queens in the deck?
d. Hence, what is the probability of being dealt a hand with 3 aces and 2 queens?

EXAMPLE 19:

Dan believes that in a 14-horse race 3 of his 5 favourites must come 1^{st}, 2^{nd} and 3^{rd}, in any order. A trifecta is a bet on the first 3 places of a horse race in the correct order.

a. How many trifectas are possible from 14 horses?
b. If Dan wants to cover all trifectas of his 5 favourites how many trifecta bets is this?
c. If all 14 horses are equally likely to win the race what is the probability that Dan will win from one of his bets?

- New Century Maths General Ex 6.06 Page 214

G. Probability Tree Diagrams

- The probabilities of outcomes are listed at every stage on the branches of probability tree diagrams
- To calculate the probability of a particular outcome ***multiply*** the probabilities along the branches
- To calculate the probability of an event with 2 or more outcomes ***add*** the calculated probabilities together
- for complementary events ***P(at least one) = 1 - P(none)***

EXAMPLE 20:

To drive to work Mr Katehos passes through 3 sets of traffic lights. The probability of a red light (including amber) on each light is 0.3. Construct a tree diagram showing all possible arrangements of red and green signals for the 3 sets of lights. Calculate the probability that he meets:

a. all green lights
b. one red then 2 green lights
c. 1 red and 2 green lights in any order
d. at least 1 red light

EXAMPLE 21:

A student council has 8 Year 10 students, 6 Year 11 and 4 Year 12 students. Two students are selected at random from the council to represent the school at the Lord Mayor's lunch. Construct a probability diagram to show all possible selections and use it to calculate the probability that:

a. both are from Year 10
b. there is one from each of Year 10 and Year 11
c. at least one of the representatives is from Year 12
d. each representative is from a different year group

- New Century General Maths Ex 6.07 Page 219

H. Expectation

- If the probability of an event *E* is *p* and the experiment is conducted *n* times then the expected number of times *E* will occur is *n* x *p*
- ***Financial expectation*** is calculated by ***multiplying*** every possible financial outcome by its probability and ***adding*** the results together

EXAMPLE 22:

In a lotto draw 6 numbers are selected from 44.

a. What is the probability that Carlos' luck number 34 is selected?
b. There are 104 Lotto draws in a year. How many times can Carlos expect his number to be selected over the year?

EXAMPLE 23:

A pair of coins is tossed 300 times. How often would you expect 2 heads to come up?

EXAMPLE 24:

Samantha plays a game involving the tossing of 2 coins. Each game costs 40c to play. She wins $5 if both coins show a head, $1 for a head and a tail and loses $6 if both are tails.

a. What is Samantha's final expectation for the game?
b. On average will she make a profit or a loss?

EXAMPLE 25:

Graham rolls a pair of dice in a game that costs $1 per bet. The table lists the financial outcomes for each event.

a. Calculate the financial expectations for this game.
b. Is this game fair? Justify your answer.

Event	**Financial Outcome**
Doubles	win $2
Sum of 7	win $3
Odd Sum (except 7)	win $1
Even Sum (except doubles)	lose $1

- New Century Maths General Ex 6.08 Page 222

I. Probability Simulations

- A probability simulation is the use of some method to model or simulate a real experiment or situation.
- Simulation can involve:
  - calculators and computers to generate random numbers
  - dice, coins, spinners and coloured counters
  - random number tables

EXAMPLE 26:

For families with 5 children what is the probability that boys outnumber girls? Devise a simulation of this situation and use it to determine whether the probability is more than, less than or equal to ½.

Assuming there is an equal chance of the child being a boy or a girl each time list some simulations that would be suitable using:

- Coins
- Cards
- Dice

EXAMPLE 27:

Four people are randomly selected. What is the probability that two or more of them have birthdays in the same month?

a. how could you simulate the problem?

b. run your simulation 100 times and approximate the probability

- New Century Maths General Ex 6.09 Page 229

J. Probability in Testing

- Doctors and scientists use diagnostic tests to determine the existence of a particular disease or condition.
- A ***positive*** result means the patient tested ***has*** the disease
- A ***negative*** result means the patient ***does not have*** the disease
- Tests are not 100% reliable so there is a chance the diagnosis is incorrect
- A ***false positive*** occurs when the patient receives a positive result but does not have the disease
- A ***false negative*** occurs when the patient receives a negative result but has the disease

	Test Result	
	Positive	Negative
Disease Present	***True Positive*** disease present and detected	***False Negative*** disease present but not detected
Disease Absent	***False Positive*** disease not present but wrongly detected	***True Negative*** disease not present and test indicates

EXAMPLE 28:

The following table shows the results of a medical test that determines the presence of coronary heart disease.

	Test Results		
	Positive	Negative	Total
Disease Present	231	15	246
Disease Absent	84	410	494
Total	315	425	

a. How many subjects were diagnosed as having the disease?

b. How many false positive results were there?

c. How many tests were done?

d. What percentage (to 2 dp) of:

i. positive results were correct?

ii. negative results were correct?

iii. all results were incorrect?

EXAMPLE 29:

A lie detector was tested for its reliability over 250 trials and the results presented in a table:

	Test Results		
	Accurate	Not Accurate	Total
True Statements	116	9	
False Statements	108	17	
Total			

a. Complete the table
b. How many false statements were accurately detected?
c. What percentage of test results was accurate?
d. Is the lie detector better at detecting true or false statements?
e. How many statements were judged as being false, rightly or wrongly?
f. What is the percentage probability (to 2 dp) that a statement judged to be true actually was true?

- New Century Maths General Ex 6.10 Page 232

GENERAL MATHS: PROBABILITY

EXAMPLE 1

a) $1/13$

b) $\frac{1}{26}$

c) $3/13$

d) $\frac{1}{4} + \frac{1}{26} = \frac{15}{52}$

c) $1/37$

d) $\frac{5}{250} = \frac{1}{50}$

e) $\frac{65}{250} = \frac{13}{50}$

EXAMPLE 2

a) 0

b) 0.525

c) 0.7

d) 0.8

e) 1

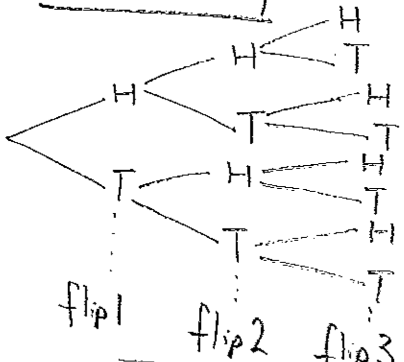
EXAMPLE 3

a) $\frac{64}{250} = \frac{32}{125}$

b) 9 numbers out of a possible 37

ie. $9/37$

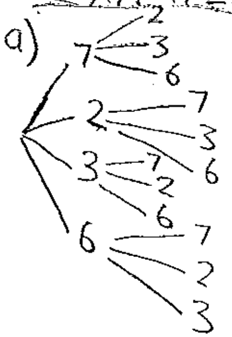
EXAMPLE 4



HHH	TTT
HHT	
HTH	
HTT	
THH	
THT	
TTH	

EXAMPLE 5

a)



b) Numbers divisible by 3: 72, 27, 36, 63

ie $\frac{4}{12} \Rightarrow \frac{1}{3}$

EXAMPLE 6

\	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

PROBABILITY OF ROLLING A SUM OF 10: $\frac{3}{36} = \frac{1}{12}$

EXAMPLE 7

TOTAL COMBINATIONS = N.O. OF NAMES × NO. OF SURNAMES

= 6 × 5 = 30 COMBINATIONS

EXAMPLE 8

TOTAL COMBINATIONS = N.O OF ENTREES × N.O OF MAIN COURSES × N.O OF DESSERTS

= 3 × 5 × 4

= 60 possible combinations

EXAMPLE 9

26 LETTERS

10 NUMBERS

So 36 POSSIBLE CHARACTERS FOR EACH SLOT

(36)(36)(36)(36)(36)(36

$\Rightarrow 36^6$

$\Rightarrow$ 2176782336 POSSIBLE COMBINATION

EXAMPLE 10

6 people

6 × 5 × 4 × 3 × 2 × 1

= 720 possible positi

EXAMPLE 11

10 CARS, 10 POSITIONS

10 × 9 × 8 × 7 × 6 × 5 × 4 × 3 × 2 × 1

= 3628800 possible positions.

EXAMPLE 12

6 PEOPLE INTO 4 POSITIONS

$$\frac{6!}{(6-4)!} = 360$$ possible ways.

EXAMPLE 13

a) $$\frac{5!}{(5-2)!} = 20$$ possible ways.

b)

Capt. + Vice capt.	Capt. + Vice capt.
Ang + Beth	Beth + Ang
Ang + Cassie	Cassie + Ang
Ang + Dasha	Dasha + Ang
Ang + Elena	Elena + Ang
Beth + Cassie	Cassie + Beth
Beth + Dasha	Dasha + Beth.
Beth + Elena	Elena + Beth.
Cassie + Dasha	Dasha + Cassie
Cassie + Elena	Elena + Cassie
Dasha + Elena.	Elena + Dasha

20 possible combinations.

EXAMPLE 14

6 Friends

Doubles pair of 2.

NO ORDER.

$r = 2$, $n = 6$.

$$\Rightarrow \frac{6 \times 5}{2 \times 1} \Rightarrow 15$$ possible doubles formations

EXAMPLE 15

8 people

Combination of 3 students

$$\frac{8 \times 7 \times 6}{3 \times 2 \times 1} = 56$$ possible combinations

EXAMPLE 16

$$\frac{44 \times 43 \times 42 \times 41 \times 40 \times 39}{6 \times 5 \times 4 \times 3 \times 2 \times 1}$$

This can be rewritten in combinatronic form.

$$\frac{44!}{6!(44-6)!} = 7059052$$ possible combinations

EXAMPLE 17

a) $$\frac{25!}{8!(25-8)!} = 1081575$$

b) $$\frac{11!}{4!(11-4)!} = 330$$

c) 330×1081575

$= 356919750$ possible student councils.

EXAMPLE 18

a) $\dfrac{52!}{5!\,(52-5)!}$

$= 2598960$ possible hands

b) $\dfrac{4!}{3!\,(1!)} = 4$

c) $\dfrac{4!}{2!\,(2!)} = 6$

d) $\dfrac{6 \times 4}{2598960} = \dfrac{1}{108290}$

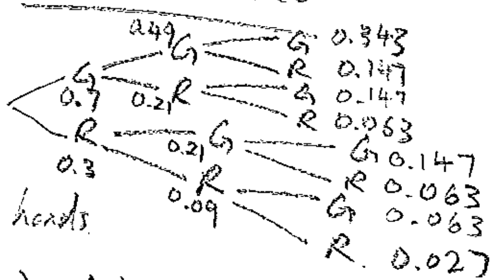
EXAMPLE 19

a) $\dfrac{14!}{3!\,(14-3)!} = 364$ possible trifectas

b) $\dfrac{5!}{3!\,(5-3)!} = 10$ possible trifectas

c) $\dfrac{10}{364} = \dfrac{5}{182}$ he will win one of his bets

EXAMPLE 20



a) All green $= 0.343 = \dfrac{343}{1000}$

b) 1 Red then 2 Green $= 0.147 = \dfrac{147}{1000}$

c) 1 Red, 2 Green, any order $= 0.441 = \dfrac{441}{1000}$

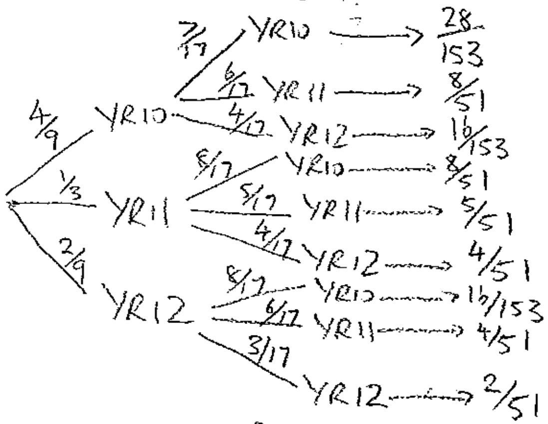
d) At least 1 Red means NOT ALL GREEN

$1 - 0.343 = 0.657 = \dfrac{657}{1000}$

EXAMPLE 21

8 Year 10
6 Year 11
4 Year 12

Total 18 students



a) Both from Year 10

$= \dfrac{28}{153}$

b) 1 Yr 10 and 1 Yr 11

$\dfrac{8}{51} \times 2 = \dfrac{16}{51}$

c) NOT Yr 10 + Yr 10, Yr 11 + Yr 11 or Yr 12 + Yr 12

$= 1 - \left[\left(\dfrac{28}{153}\right) + \left(\dfrac{5}{51}\right) + \left(\dfrac{2}{51}\right)\right]$

$= \dfrac{104}{153}$

EXAMPLE 22

Probability of a certain number in a Lotto Draw.

AS FIRST

$\dfrac{1}{44} \times \dfrac{42}{43} \cdots \times \dfrac{38}{39}$

AS SECOND

$\dfrac{43}{44} \times \dfrac{1}{43} \cdots \times \dfrac{38}{39}$

⋮

AS SIXTH

$\dfrac{43}{44} \times \dfrac{42}{43} \cdots \times \dfrac{1}{39}$

$= \dfrac{19}{946} + \dfrac{19}{924} + \dfrac{19}{902} + \dfrac{19}{880} + \dfrac{19}{858} + \dfrac{1}{44}$

$= \dfrac{18095787}{141181040}$

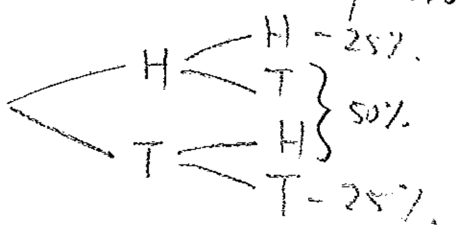
≈ 0.128

b) 0.128×104

Around 13 times.

EXAMPLE 23

THE PROBABILITY OF 2 HEADS TO APPEAR WHEN 2 COINS ARE TOSSED IS 0.25 / 25%



So the expected probability is 75 times for 2 heads to appear

Drawing a probability table

a)	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Chance of rolling doubles = $\frac{1}{6}$
Chance of sum 7 = $\frac{1}{6}$
Chance of odd sum (not 7) = $\frac{1}{3}$
Chance of even sum (no double) = $\frac{1}{3}$

EXAMPLE 24

1)

SAMANTHA HAS A

25% chance of winning $4.60*
50% chance of winning $0.60*
25% chance of losing $6.40*

* Taken into consideration 40¢ fee.

b) CONSIDER THE EVENT WHERE 100 GAMES ARE PLAYED

Scientific probability dictates that Samantha will win $(25 \times 4.6) + (50 \times 0.6)$ but lose $-(25 \times 6.4)$

This gives a loss of $15 over 100 games, or 15c over 1 game.

So on average, she will make a loss

If we consider the $1 bet to be a loss, this gives us

Doubles - win $1
Sum 7 - win $2
Odd sum - win nothing
Even sum - lose $2

a) CONSIDER THE EVENT WHERE 60 GAMES ARE PLAYED

Scientific probability dictates that there will be 10 doubles, 10 sum 7, 20 odd sum and 20 even sum

giving $10 + 20 + 0 + 20(-2) = -10$, So Financial expectations are at a loss

b) A game is fair if Graham and the house both have an equal chance to win. As Graham LOSES money

EXAMPLE 26

WITH COINS
FLIP A COIN, SET EACH FACE OF THE COIN TO A GENDER.
WITH CARDS
SHUFFLE A DECK OF CARDS AND PICK 5 FROM RANDOM POSITIONS WITHIN THE DECK, SET A PARTICULAR CARD COLOUR TO A GENDER
WITH DICE
TOSS A DICE, SET A NUMBER THRESHOLD ie. 1-3 FOR BOY 4-6 FOR GIRLS [THE SPLIT MUST BE EQUAL]

EXAMPLE 27

THIS IS AN EXAMPLE OF THE BIRTHDAY PARADOX.

WITH 4 PEOPLE, WE HAVE

$$\frac{4 \times 3}{2} = 6 \text{ PAIRS}$$

{2 OR MORE => AT LEAST 2.}

THE PROBABILITY THAT ANY GIVEN PAIR DOES NOT HAVE THE SAME BIRTHDAY IS

$$1 - \frac{1}{365} = \frac{364}{365} \approx 99.7\%$$

SO WITH 6 PAIRS, THE PROBABILITY THAT NO PAIRS SHARE THE SAME BIRTHDAY.

THE SAME CONCEPT APPLIES TO MONTH. $1 - \frac{1}{12} = \frac{11}{12} \Rightarrow \approx 91.67\%$

SO THE PROBABILITIES THAT NO 6 PAIRS SHARE THE SAME MONTH IS

$$\left(\frac{11}{12}\right)^6 \approx 59.3\%$$

SO AT LEAST 2 IS

$$1 - 59.3\% = \underline{40.67\%}$$

which is the scientific probability that at least 2 will share the same birthday.

a) RANDOM NUMBER GENERATOR FROM 1-12, Repeat ~~6~~ times and note down any repetitions

b) TRY THIS YOURSELF!

EXAMPLE 28

a) 246

b) 84

c) 740 Tests

d) i) 73.3%

ii) 96.5%

iii) 13.4%

EXAMPLE 29

a)

	TEST RESULTS		
	ACCURATE	NOT ACCURATE	TOTAL
TRUE STATEMENTS	116	9	125
FALSE STATEMENTS	108	17	125
TOTAL	224	26	

b) 108

c) 89.6%

d) For True statements

success rate = $\frac{116}{125}$ = 92.8%

For False statements

success rate = $\frac{108}{125}$ = 86.4%

Better at detecting true statements

e) 108+9 = 117

f) Total True Judgements

= 116+17 = 133

success rate = $\frac{116}{133}$ = 87.22%