C.E.M.TUITION

Student Name : ______________________________

Review : Logarithms & Exponentials

(HSC - PAPER 1)

Year 12 - 2 Unit

1. Find derivatives of:

(a) e^{5x}

(b) $\log_e(3x-1)$

2. (a) Find $\int \frac{dx}{2x-5}$

(b) Evaluate $\int_0^1 e^{3x}\,dx$

(Leave answer in terms of e.)

3. Differentiate:

(a) $4x\log_e x$

(b) xe^{2x}

4. **(a)** Evaluate $\int_1^2 \frac{dx}{2x}$ correct to 2 decimal places.

(b) Find $\int e^{\frac{1}{2}x}\,dx$

5. **(a)** Find **(i)** $\int \frac{x}{x^2+5}\,dx$

(ii) $\int \frac{x^2+5}{x}\,dx$

(b) Differentiate $\frac{\ln x}{4x}$

Note $\ln x = \log_e x$

6. Use log laws to aid in differentiating

$$y = \log_e \sqrt{\frac{1+2x}{1-2x}}\,.$$

7. Differentiate $\dfrac{e^{x}+e^{-x}}{e^{x}-e^{-x}}$.

8. If $y=e^{-kx}$ is a solution of

$$\frac{d^2y}{dx^2}+3\frac{dy}{dx}-4y=0,$$

find any valid values of k

(k is a constant).

1. (a) $\frac{d}{dx}(e^{5x}) = e^{5x} \times \frac{d}{dx}(5x)$
$= 5e^{5x}$.

(b) $\frac{d}{dx}[\log_e(3x-1)]$
$= \frac{1}{3x-1} \times \frac{d}{dx}(3x-1)$
$= \frac{3}{3x-1}$.

2. (a) $\int \frac{dx}{2x-5} = \frac{1}{2}\int \frac{2dx}{2x-5}\,dx$

> Numerator must be differential of denominator. $\frac{1}{2}$ in front compensates for multiplying by 2.

$= \frac{1}{2}\log_e(2x-5) + k$.

(b) $\int_0^1 e^{3x}\,dx$
$= \left[\frac{1}{3}e^{3x}\right]_0^1$
$= \frac{1}{3}[e^3 - e^0]$ $\boxed{e^0 = 1}$
$= \frac{1}{3}(e^3 - 1)$.

3. (a) $\frac{d}{dx}(4x\log_e x)$

> Using $\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$ where $u = 4x$, $v = \log_e x$.

$= (\log_e x)4 + 4x\left(\frac{1}{x}\right)$
$= 4\log_e x + 4$
$= 4(\log_e x + 1)$.

(b) $\frac{d}{dx}(xe^{2x})$

> Using $\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$ where $u = x$, $v = e^{2x}$.

$= e^{2x}(1) + x(2e^{2x})$
$= e^{2x} + 2xe^{2x}$
$= e^{2x}(1+2x)$.

4. (a) $\int_1^2 \frac{dx}{2x} = \frac{1}{2}\int_1^2 \frac{dx}{x}$
$= \frac{1}{2}[\log_e x]_1^2$
$= \frac{1}{2}[\log_e 2 - \log_e 1]$
$\boxed{\log_e 1 = 0}$
$= \frac{1}{2}\log_e 2$
$= 0{\cdot}35$ (correct to 2 dec. pl.)

(b) $\int e^{\frac{1}{2}x}\,dx = 2e^{\frac{1}{2}x} + c$

> Check: $\frac{d}{dx}(2e^{\frac{1}{2}x})$
> $= 2\left(\frac{1}{2}e^{\frac{1}{2}x}\right) = e^{\frac{1}{2}x}$.

5. (a) (i) $\int \frac{x}{x^2+5}\,dx$
$= \frac{1}{2}\int \frac{2x}{x^2+5}\,dx$
$\boxed{\frac{d}{dx}(x^2+5) = 2x}$
$= \frac{1}{2}\log_e(x^2+5) + k$.

(ii) $\int \frac{x^2+5}{x}\,dx$

> Split the numerator: $\frac{x^2}{x} + \frac{5}{x} = \frac{x^2+5}{x}$

$= \int \frac{x^2}{x} + \frac{5}{x}\,dx$
$= \int x + \frac{5}{x}\,dx$
$= \frac{1}{2}x^2 + 5\log_e x + c$.

(b) $\frac{d}{dx}\left(\frac{\ln x}{4x}\right)$

> Using $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ where $u = \ln x$, $v = 4x$.

$= \frac{4x\left(\frac{1}{x}\right) - (\ln x)4}{(4x)^2}$
$= \frac{4 - 4\ln x}{16x^2}$ $\boxed{\frac{4(1-\ln x)}{16x^2}}$
$= \frac{1-\ln x}{4x^2}$.

6. $y = \log_e\sqrt{\frac{1+2x}{1-2x}}$
$= \log_e\left(\frac{1+2x}{1-2x}\right)^{\frac{1}{2}}$ $\boxed{\sqrt{a} = a^{\frac{1}{2}}}$
$= \frac{1}{2}\log_e\left(\frac{1+2x}{1-2x}\right)$
$\boxed{\log_e a^2 = 2\log_e a}$
$= \frac{1}{2}[\log_e(1+2x) - \log_e(1-2x)]$
$\boxed{\log_e\left(\frac{a}{b}\right) = \log_e a - \log_e b}$

$\therefore \frac{dy}{dx} = \frac{1}{2}\left[\frac{1}{1+2x} \times \frac{d}{dx}(1+2x) - \frac{1}{1-2x} \times \frac{d}{dx}(1-2x)\right]$
$= \frac{1}{2}\left[\frac{2}{1+2x} + \frac{2}{1-2x}\right]$
$\boxed{-(-2) = 2}$
$= \frac{1}{2}\left[\frac{2(1-2x) + 2(1+2x)}{(1+2x)(1-2x)}\right]$
Add the fractions
$= \frac{1}{2}\left[\frac{2(1+1-2x+2x)}{1-4x^2}\right]$
$= \frac{2}{1-4x^2}$.

7. $y = \frac{e^x + e^{-x}}{e^x - e^{-x}} = \frac{u}{v}$

> *Note* $\frac{d}{dx}(e^{-x}) = -e^{-x}$

$u = e^x + e^{-x}$, $\frac{du}{dx} = e^x - e^{-x}$

$v = e^x - e^{-x}$, $\frac{dv}{dx} = e^x + e^{-x}$

$\therefore \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

> $e^x \times e^x = e^{2x}$
> $e^x - e^{-x} = e^0 = 1$

$= \frac{(e^x - e^{-x})(e^x - e^{-x}) - (e^x + e^{-x})(e^x + e^{-x})}{(e^x - e^{-x})^2}$
$= \frac{e^{2x} - 2 + e^{-2x} - (e^{2x} + 2 + e^{-2x})}{(e^x - e^{-x})^2}$
$= \frac{-4}{(e^x - e^{-x})^2}$.

8. $y = e^{-kx}$
$\therefore \frac{dy}{dx} = -ke^{-kx}$
$\frac{d^2y}{dx^2} = -k(-k)e^{-kx}$
$= k^2e^{-kx}$.

If $y = e^{-kx}$ is a solution, then it must satisfy equation.
$\therefore k^2e^{-kx} - 3ke^{-kx} - 4e^{-kx} = 0$
$(k^2 - 3k - 4)e^{-kx} = 0$
$\therefore k^2 - 3k - 4 = 0$ $\boxed{(e^{-kx} \neq 0)}$
$(k-4)(k+1) = 0$
$\therefore k = 4$ or -1.
Valid solutions are -1 or 4.