C.E.M. TUITION

Student Name : ______________________________

Review Topic : Logarithms and Exponentials

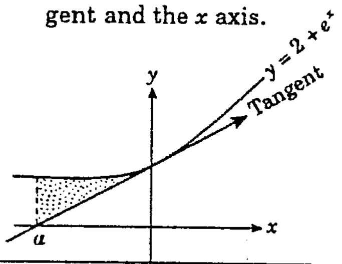
(HSC - PAPER 2)

2 Unit

9. **(a)** Find the equation of the tangent to $y = 2 + e^x$ at the point where $x = 0$.

(b) Find the point of intersection of this tangent and the x axis.

(c) The shaded area is the region between the tangent, the curve $y = 2 + e^x$ and the line $x = a$ where a is the point of intersection of the tangent and the x axis.



Calculate the shaded area correct to one decimal place.

10. **(a)** Show that

$$\frac{e^x - e^{-x}}{e^x} = 1 - e^{-2x}.$$

Hence, or otherwise, differentiate $y = \dfrac{e^x - e^{-x}}{e^x}$.

(b) Evaluate $\displaystyle\int_0^1 \frac{e^x - e^{-x}}{e^x}$ correct to 2 decimal places.

11. Show that $y = x \ln x$ has a minimum value of $\frac{-1}{e}$ when $x = \frac{1}{e}$.

12. **(a)** Find $\frac{d}{dx}(\log_e \sin x)$.

(b) Hence, or otherwise, show that $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot x \, dx = \frac{1}{2}\log_e 2$

13. Find the volume of the solid generated by rotating the area beneath the curve $y = \dfrac{2}{\sqrt{x}}$ between $x = 1$ and $x = 3$.

14. The curve $y = \dfrac{6}{\sqrt{2x-1}}$ in the first quadrant between $x = 1$ and $x = 3$ is rotated about the x axis. Find the exact volume generated.

15. Find the area of the region enclosed by the curve $y = \frac{x}{x^2+1}$, the x axis, and the lines $x = 0$ and $x = 1$.

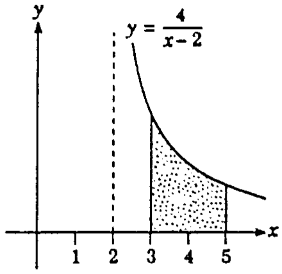
16. **(a)** Show that

$$\frac{d}{dx}\left[\log_e(\log_e x)\right] = \frac{1}{x\log_e x}.$$

(b) Hence, or otherwise, evaluate $\int_e^{e^2} \frac{dx}{x\log_e x}$.

17. Show that $y = \frac{\ln x}{x}$ has a maximum value of $\frac{1}{e}$ when $x = e$.

18. Calculate the exact area under the curve $y = \frac{4}{x-2}$ between $x = 3$ and $x = 5$.



9. (a) $y = 2 + e^x$

$\frac{dy}{dx} = e^x = e^0 = 1$ at $x = 0$.

At $x = 0$, $y = 2 + e^0 = 2 + 1 = 3$ Point is (0, 3).

Equn. of tangent at (0, 3) is of form $y - y_1 = m(x - x_1)$

i.e. $y - 3 = 1(x - 0)$

$y - 3 = x$

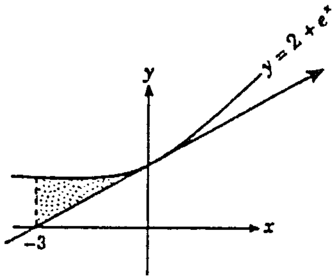
$\therefore$ $x - y + 3 = 0$ is equn.

(b) Tangent cuts x axis when $y = 0$. $x + 3 = 0$ $\therefore$ $x = -3$

Pt. of intersection is –3, 0).

(c) $a = -3$

We require area between -3 and 0 below $y = 2 + e^x$ but above $x - y + 3 = 0$, $[y = x + 3]$.



Area

$= \int_{-3}^{0} (2 + e^x)\, dx - \int_{-3}^{0} (x + 3)\, dx$

$= \left[2x + e^x\right]_{-3}^{0} - \left[\frac{1}{2}x^2 + 3x\right]_{-3}^{0}$

$e^0 = 1$

$= \left[(e^0) - (-6 + e^{-3})\right]$

$-\left[0 - \left(\frac{9}{2} - 9\right)\right]$

$= \left[1 + 6 - 4{\cdot}5 - e^{-3}\right]$

$= 2{\cdot}450\,212\,9 \approx 2{\cdot}5$ (1dp).

Area is $2{\cdot}5$ units^2.

10. (a) Put $y = \frac{e^x - e^{-x}}{e^x}$

$= \frac{e^x}{e^x} - \frac{e^{-x}}{e^x}$

$e^{-x} + e^x = e^{-2x}$

$= 1 - e^{-2x}$

$\therefore \frac{dy}{dx} = 2e^{-2x}$.

(b) $\int_0^1 (1 - e^{-2x}\, dx$

$= \left[x + \frac{1}{2}e^{-2x}\right]_0^1$

$= \left[\left(1 + \frac{1}{2}e^{-2}\right) - \left(0 + \frac{1}{2}e^0\right)\right]$

$= 1 - \frac{1}{2} + \frac{1}{2}e^{-2}$

$= \frac{1}{2}(1 + e^{-2})$

$= 0{\cdot}067\,667\,641$

$\approx 0{\cdot}07$ (2dp).

11. $y = x\ln x$

$y = uv$ $\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$

$\frac{dy}{dx} = 1.\ln x + x.\frac{1}{x}$

$= 1 + \ln x$.

Put $\frac{dy}{dx} = 0$ for stationary values

$\therefore$ $1 + \ln x = 0$

$\therefore$ $\ln x = -1$,

then $x = e^{-1}$

$\therefore$ $x = \frac{1}{e}$.

$\log_a b = c \Rightarrow a^c = b$

When $x = \frac{1}{e}$,

$y = \frac{1}{e}\ln\left(\frac{1}{e}\right)$

$= \frac{1}{e}\ln e^{-1}$

$= \frac{1}{e}.-1\ln e$

$= -\frac{1}{e}$.

$\log_e e = 1$

Now $\frac{d^2y}{dx^2} = \frac{1}{x}$

$= e$ at $x = \frac{1}{e} > 0$,

$\therefore$ minimum value of $-\frac{1}{e}$ occurs at $x = \frac{1}{e}$.

12. (a) $\frac{d}{dx}(\log_e \sin x)$

$= \frac{1}{\sin x} \times \frac{d}{dx}(\sin x)$

$= \frac{1}{\sin x} \times \cos x = \frac{\cos x}{\sin x}$

$= \cot x$.

(b) $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot x\, dx$

$= \left[\log_e \sin x\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$

$= \left[\log_e \sin\frac{\pi}{2} - \log_e \sin\frac{\pi}{4}\right]$

$= \left[\log_e 1 - \log_e \frac{1}{\sqrt{2}}\right]$

$= 0 - \log_e 2^{-\frac{1}{2}}$

$= \frac{1}{2}\log_e 2$.

13. $V = \pi\int y^2\, dx$

$y = \frac{2}{\sqrt{x}}$ $\therefore$ $y^2 = \frac{4}{x}$

$V = \pi\int_1^3 \frac{4}{x}\, dx$

$= 4\pi\int_1^3 \frac{dx}{x}$

$\log_e 1 = 0$

$= 4\pi\left[\log_e x\right]_1^3$

$= 4\pi\left[\log_e 3 - \log_e 1\right]$

$= 4\pi\log_e 3$.

Volume is $4\pi\ln 3$ units^3.

14. $V = \pi\int y^2\, dx$

$y = \frac{6}{\sqrt{2x-1}}$
$y^2 = \frac{36}{2x-1}$

$V = \pi\int_1^3 \frac{36}{2x-1}\, dx$

$= 18\pi\int_1^3 \frac{2}{2x-1}\, dx$

$= 18\pi\left[\log_e (2x-1)\right]_1^3$

$= 18\pi\left[\log_e 5 - \log_e 1\right]$

$= 18\pi\ln 5$.

$\log_e 1 = 0$

Volume is $18\pi\ln 5$ units^3.

15. Area $= \int_0^1 \frac{x}{x^2+1}\,dx$

$= \frac{1}{2}\int_0^1 \frac{2x}{x^2+1}\,dx$

$= \frac{1}{2}\left[\log_e(x^2+1)\right]_0^1$

$= \frac{1}{2}\left[\log_e 2 - \log_e 1\right]$

$= \frac{1}{2}\log_e 2.$ $\boxed{\log_e 1 = 0}$

Area is $\frac{1}{2}\ln 2$ units^2.

16. (a) $\frac{d}{dx}\left[\log_e(\log_e x)\right]$

$= \frac{1}{\log_e x} \times \frac{d}{dx}(\log_e x)$

$= \frac{1}{\log_e x} \times \frac{1}{x}$

$= \frac{1}{x\log_e x}.$

(b) $\int_e^{e^2} \frac{dx}{x\log_e x}$

$= \left[\log_e(\log_e x)\right]_e^{e^2}$

$= \left[\log_e(\log_e e^2) - \log_e(\log_e e)\right]$

$= \left[\log_e 2 - \log_e 1\right]$

$= \log_e 2.$

Note $\log_e e^2 = 2\log_e = 2$ as $\log_e e = 1$

17. $y = \frac{\ln x}{x}$

$y = \frac{u}{v}$ $\qquad \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$\frac{dy}{dx} = \frac{x.\frac{1}{x} - 1.\ln x}{x^2}$

$= \frac{1-\ln x}{x^2}$

$= 0$ for stationary values,

$\therefore \quad 1 - \ln x = 0$

$\therefore \quad \ln x = 1$

$\therefore \quad x = e^1 = e.$

At $x = e$,

$y = \frac{\ln e}{e} = \frac{1}{e}$ $\boxed{\log_e e = 1}$

$\frac{d^2y}{dx^2} = \frac{x^2\left(-\frac{1}{x}\right) - (1-\ln x)2x}{x^4}$

$\boxed{\text{By Quotient Rule}}$

$= \frac{-x - 2x + 2x\ln x}{x^4}$

$= \frac{-3x + 2x\ln x}{x^4} = \frac{\cancel{x}(-3+2\ln x)}{\cancel{x^4}\,x^3}$

$= \frac{-3+2\ln x}{x^3}$

$= \frac{-3+2\ln e}{e^3}$ at $x = e$

$= \frac{-3+2}{e^3}$ $\boxed{\log_e e = 1}$

$= \frac{-1}{e^3} < 0$

$\therefore$ maximum value of $\frac{1}{e}$ occurs when $x = e$.

18. $A = \int_3^5 \frac{4}{x-2}\,dx$

$= 4\int_3^5 \frac{1}{x-2}\,dx$

$= 4\left[\log(x-2)\right]_3^5$

$= 4\left[\log 3 - \log 1\right]$

$= 4\log 3.$

Area is $4\log 3$ units^2.