C.E.M.TUITION

Student Name :_______________________________

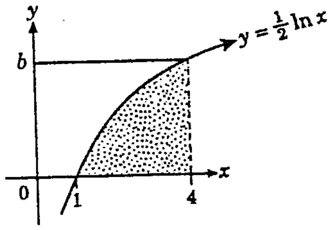
Review Topic : Logarithms and Exponentials

(HSC - PAPER 3)

2 Unit

19. Find the area enclosed between the curve $y = 2 - \frac{1}{x-1}$, the positive section of the x axis, and the line $x = 2$.
(*Hint* You will need to calculate the point where the curve cuts the x axis. A sketch would be useful.)

20.



(a) (i) Express the equation $y = \frac{1}{2}\log_e x$, with x as the subject.

(ii) Calculate the value of b, when $x = 4$.

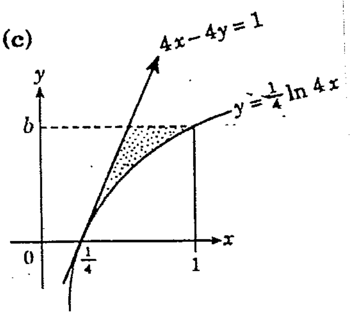
(b) By first calculating the area between the curve $y = \frac{1}{2}\log_e x$ and the y axis between the lines $y = 0$ and $y = b$, calculate the shaded area, leaving your answer in exact form.

21. By differentiating $y = e^{-2x} \sin x$, show that $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = 0$.

22. **(a)** Prove that equation of the tangent to $y = \frac{1}{4}\log_e 4x$ at the point where $x = \frac{1}{4}$ is $4x - 4y = 1$.

(b) Show that the curve $y = \frac{1}{4}\log_e 4x$ intersects the x axis at $\left(\frac{1}{4}, 0\right)$.

(c)



The shaded area is the region between the curve $y = \frac{1}{4}\log_e 4x$, the tangent at $x = \frac{1}{4}$ and the line $y = b$.

Note the line $y = b$ intersects $y = \frac{1}{4}\log_e 4x$ at $(1, b)$.

By firstly rewriting $y = \frac{1}{4}\log_e 4x$ with x as the subject, and using the y axis as a reference, calculate the shaded area.

23. **(a)** Show that

$$\frac{d}{dx}\left\{\log_e\left(x+\sqrt{1+x^2}\right)\right\} = \frac{1}{\sqrt{1+x^2}}.$$

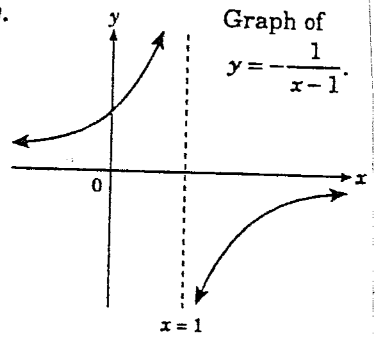
(b) Hence, or otherwise, evaluate $\displaystyle\int_0^2 \frac{dx}{\sqrt{1+x^2}}$, correct to 3 decimal places.

24. Consider the curve given by $y = xe^{2x}$.

(a) Find any turning points of this curve, and hence sketch the curve.

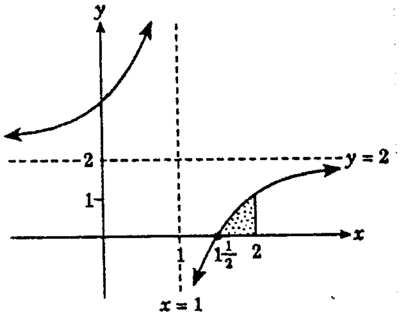
(b) Find the equation of the *normal* to the curve at the point where the curve cuts the x axis.

19. Graph of $y = -\frac{1}{x-1}$.



Now add 2 to this graph, giving $y = 2 - \frac{1}{x-1}$.

[Raise graph upward 2 units.]



Graph of $y = 2 - \frac{1}{x-1}$.

When $y = 0$, $\frac{1}{x-1} = 2$

$\therefore \quad 1 = 2x - 2$

$\therefore \quad 2x = 3$

$x = 1\frac{1}{2}$.

Curve cuts x axis at $\left(1\frac{1}{2}, 0\right)$.

Area required is shaded.

$$A = \int_{1\frac{1}{2}}^{2} \left(2 - \frac{1}{x-1}\right) dx$$

$$= \int_{1\frac{1}{2}}^{2} 2\, dx - \int_{1\frac{1}{2}}^{2} \frac{1}{x-1}\, dx$$

$$= \left[2x\right]_{1\frac{1}{2}}^{2} - \left[\log(x-1)\right]_{1\frac{1}{2}}^{2}$$

$$= [4-3] - \left[\log 1 - \log\left(\frac{1}{2}\right)\right]$$

$$= 1 + \log 2^{-1}$$

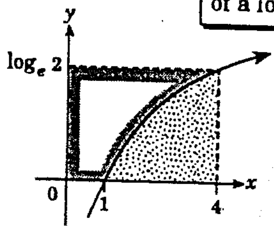
$$= 1 - \log 2.$$

Area enclosed is $(1 - \log 2)$ units^2.

20. (a) (i) $y = \frac{1}{2}\log_e x$

$\Rightarrow 2y = \log_e x$

$\Rightarrow \quad x = e^{2y}$ — Definition of a log



(ii) When $x = 4$,

$y = \frac{1}{2}\log_e 4$

$= \log_e 4^{\frac{1}{2}} = \log_e 2$

$\therefore \ b = \log_e 2$.

(b) Area between curve and y axis is given by

$$A = \int_0^{\ln 2} x\, dy$$

$$= \int_0^{\ln 2} e^{2y}\, dy$$

$$= \left[\frac{1}{2}e^{2y}\right]_0^{\log_e 2}$$

$$= \left[\frac{1}{2}e^{2\log_e 2} - \frac{1}{2}e^0\right]$$

$$= \frac{1}{2}.4 - \frac{1}{2}.1$$

$$= 1\frac{1}{2}.$$

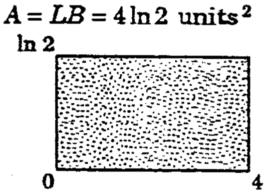
Note (i) $2\log_e 2 = \log_e 2^2 = \log_e 4$

(ii) $e^{\log_e 4} = 4$

(iii) $e^0 = 1$

$\therefore$ area between curve and y axis is 1·5 units^2.

Now area of rectangle 4 units long and (ln 2) units wide is given by $A = LB = 4\ln 2$ units^2



Then shaded area (area between curve, x axis and $x = 4$) is the difference between the calculated areas.

$A = 4\ln 2 - 1{\cdot}5$

Area is $(4\ln 2 - 1{\cdot}5)$ units^2.

21. $y = e^{-2x}\sin x$

$$\frac{dy}{dx} = -2e^{-2x}\sin x + e^{-2x}\cos x$$
$$= e^{-2x}(\cos x - 2\sin x)$$

$$\frac{d^2y}{dx^2} = -2e^{-2x}(\cos x - 2\sin x) + e^{-2x}(-\sin x - 2\cos x)$$
$$= e^{-2x}(-2\cos x + 4\sin x - \sin x - 2\cos x)$$
$$= e^{-2x}(3\sin x - 4\cos x).$$

Then $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y$

$$= e^{-2x}(3\sin x - 4\cos x) + 4e^{-2x}(\cos x - 2\sin x) + 5e^{-2x}\sin x$$
$$= e^{-2x}(3\sin x - 4\cos x + 4\cos x - 8\sin x + 5\sin x)$$
$$= e^{-2x}(0)$$
$$= 0,$$

i.e. $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = 0$.

22. (a) $y = \frac{1}{4}\log_e 4x$

$$\therefore \ \frac{dy}{dx} = \frac{1}{4}.\frac{1}{4x}.4 = \frac{1}{4x}$$
$$= \frac{1}{4 \times \frac{1}{4}} \text{ at } x = \frac{1}{4}$$
$$= 1.$$

When $x = \frac{1}{4}$, $y = \frac{1}{4}\log_e 1 = 0$.

Equation of tangent is of form $y - 0 = 1\left(x - \frac{1}{4}\right)$

$y = x - \frac{1}{4}$

$\therefore \ 4x - 4y = 1$.

(b) When $y = 0$, $0 = \frac{1}{4}\log_e 4x$

$\therefore \ 4x = e^0$

$4x = 1$

$x = \frac{1}{4}$.

Curve cuts x axis at $\left(\frac{1}{4}, 0\right)$.

(c) When $x = 1$, $y = \frac{1}{4}\ln 4$

i.e. $b = \frac{1}{4}\ln 4$

$= \ln 4^{\frac{1}{4}}$

$= \ln(2^2)^{\frac{1}{4}}$

$= \ln\sqrt{2}$ $\boxed{2^{\frac{1}{2}} = \sqrt{2}}$

Area required is area between curve and y axis less area between tangent and y axis.

Now, if $y = \frac{1}{4}\ln 4x$

then $4y = \ln 4x$

$\therefore$ $4x = e^{4y}$

$x = \frac{1}{4}e^{4y}$

$$\boxed{\begin{aligned} 4x - 4y &= 1 \\ 4x &= 4y + 1 \\ x &= y + \frac{1}{4} \end{aligned}}$$

Also, tangent is $x = y + \frac{1}{4}$ [from (a)].

Area required

$$= \int_0^{\ln\sqrt{2}} \frac{1}{4}e^{4y}\,dy - \int_0^{\ln\sqrt{2}} \left(y + \frac{1}{4}\right) dy$$

$$= \int_0^{\ln\sqrt{2}} \left[\frac{1}{4}e^{4y} - \left(y + \frac{1}{4}\right)\right] dy$$

$$\boxed{\int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b f(x) - g(x)\,dx}$$

$$= \int_0^{\ln\sqrt{2}} \left[\frac{1}{4}e^{4y} - y - \frac{1}{4}\right] dy$$

$$= \frac{1}{4}\int_0^{\ln\sqrt{2}} \left[e^{4y} - 4y - 1\right] dy$$

$\boxed{\text{Take out factor of } \frac{1}{4}.}$

$$= \frac{1}{4}\left[\frac{1}{4}e^{4y} - 2y^2 - y\right]_0^{\ln\sqrt{2}}$$

$$= \frac{1}{4}\left\{\left[\frac{1}{4}e^{4\ln\sqrt{2}} - 2\left(\ln\sqrt{2}\right)^2 - \ln\sqrt{2}\right] - \left[\frac{1}{4}e^0 - 0 - 0\right]\right\}$$

$$= \frac{1}{4}\left\{\frac{1}{4}e^{\ln(\sqrt{2})^4} - 2\left(\ln\sqrt{2}\right)^2 - \ln\sqrt{2} - \frac{1}{4}\right\}$$ $\boxed{e^0 = 1}$

$\boxed{n\log_e a = \log_e a^n}$

$$= \frac{1}{4}\left\{\frac{1}{4}e^{\ln 4} - 2\left(\ln\sqrt{2}\right)^2 - \ln\sqrt{2} - \frac{1}{4}\right\}$$ $\boxed{(\sqrt{2})^4 = (2^{\frac{1}{2}})^4 = 2^2 = 4}$

$$= \frac{1}{4}\left[\frac{1}{4}.4 - 2\left(\ln\sqrt{2}\right)^2 - \ln\sqrt{2} - \frac{1}{4}\right]$$

$\boxed{e^{\log_e x} = x}$

$$= \frac{1}{4}\left[1 - \frac{1}{4} - \ln\sqrt{2}(2\ln\sqrt{2} + 1)\right]$$

$$= \frac{1}{4}\left[\frac{3}{4} - \frac{1}{2}\ln 2\left(2.\frac{1}{2}\ln 2 + 1\right)\right]$$

$\boxed{\ln\sqrt{2} = \ln 2^{\frac{1}{2}} = \frac{1}{2}\ln 2}$

$$= \frac{3}{16} - \frac{1}{8}\ln 2(\ln 2 + 1).$$

Area is $\frac{3}{16} - \frac{1}{8}\ln 2(\ln 2 + 1)$ units^2

23. (a) $\frac{d}{dx}\left\{\ln(x + \sqrt{1 + x^2})\right\}$

$$= \frac{1}{x + \sqrt{1+x^2}} \times \frac{d}{dx}(x + \sqrt{1 + x^2})$$

$$= \frac{1}{x + \sqrt{1+x^2}} \times 1 + \frac{1}{2}(1 + x^2)^{-\frac{1}{2}}.2x$$

$$= \frac{1}{x + \sqrt{1+x^2}} \times \left[1 + \frac{x}{\sqrt{1+x^2}}\right]$$

$$= \frac{1}{x + \sqrt{1+x^2}} \times \frac{\sqrt{1+x^2} + x}{\sqrt{1+x^2}}$$

$$= \frac{1}{\sqrt{1+x^2}}.$$

(b) $\int_0^2 \frac{dx}{\sqrt{1+x^2}}$

$$= \left[\ln(x + \sqrt{1+x^2})\right]_0^2$$

$$= \left[\ln(2 + \sqrt{1+4}) - \ln(\sqrt{1})\right]$$

$$= \ln(2 + \sqrt{5})$$

$=$

$\approx$ (3dp).

24. $y = xe^{2x}$

(a) $\frac{dy}{dx} = 1.e^{2x} + x.2e^{2x}$

$= e^{2x}(1 + 2x).$

For turning pts. $\frac{dy}{dx} = 0$

$\therefore$ $e^{2x}(1 + 2x) = 0.$

Now $e^{2x} \neq 0$

$\therefore$ $1 + 2x = 0$

i.e. $2x = -1$

$x = -\frac{1}{2}.$

$x = -\frac{1}{2}$, $y = -\frac{1}{2}e^{-1} = -\frac{1}{2e}$

Turning point at $\left(-\frac{1}{2}, -\frac{1}{2e}\right)$.

$$\frac{d^2y}{dx^2} = 2e^{2x}(1 + 2x) + e^{2x}.2$$

$= 2e^{2x}(2 + 2x)$

$\boxed{1 + 2x + 1}$

$= 4e^{2x}(1 + x)$

$= 4e^{-1}\left(1 - \frac{1}{2}\right)$ at $x = -\frac{1}{2}$

$= \frac{2}{e} > 0.$

$\therefore$ Minimum turning point at $x = -\frac{1}{2}$.

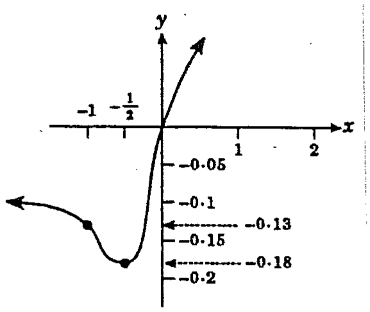
Inflexion point where $\frac{d^2y}{dx^2} = 0$

i.e. $1 + x = 0$, $e^{2x} \neq 0$

$x = -1$

$x = -1$, $y = -1e^{-2}$

$= \frac{-1}{e^2} = -0\cdot 14$



When $x = 0, y = 0$.

Also as $x \to -\infty$, $y \to 0$.
Curve is asymptotic to negative section of x axis for $x < -1$.

(b) Curve cuts x axis at (0, 0).

$$\frac{dy}{dx} = e^{2x}(1+2x)$$
$$= e^0 = 1.$$

Gradient of normal = −1 (negative reciprocal).

Equation of normal is of form $y - 0 = -1(x - 0)$
$$= -x$$
$$\therefore \; x + y = 0.$$

Equn. of normal is $x + y = 0$.