in terms of k and draw a sketch to show the positions of C, D when (a) k = 2 (b) k = −2.

USE OF COORDINATE GEOMETRY TO PROVE GEOMETRIC RESULTS

René Déscartes invented the method of Analytical Geometry and carried through the proof of every theorem in Euclid's "Elements" by his method. The following examples illustrate this method.

(1) FIRST STEP. *To take the simplest possible general figure.* For example, if we want to prove some facts about Δ ABC, then we cannot assume the triangle is isosceles, equilateral, etc. Instead of taking the general triangle ABC as one with vertices $A(x_1,y_1)$, $B(x_2,y_2)$, $C(x_3,y_3)$, we place the axes in certain positions to simplify the coordinates, but no *loss of generality* is allowed. Thus we can here take the x axis along BC, and the y axis, perpendicular to x axis, through A. We then rotate the axes around until the line BC i.e. the x axis is horizontal and the y axis is vertical. Then let C be (c,o), B be (−b,o) A be (o,a).

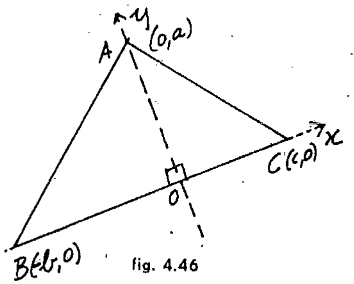

fig. 4.46

OTHERWISE, we could put the x axis along BC and put the y axis perpendicular to the x axis, through B.

Then let C be (c,o), A be (a,b).

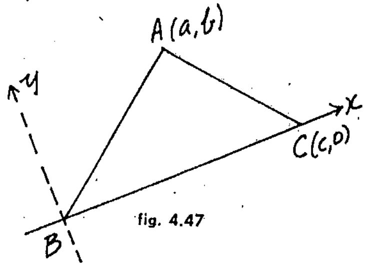

fig. 4.47

CONCURRENCE:

Medians − are intervals joining a vertex to the midpoint of the opposite side. These are concurrent at the *CENTROID* which is the point of trisection of each median.

Altitudes − are intervals drawn from a vertex perpendicular to the opposite side. These are concurrent at the *ORTHOCENTRE*.

AD, BE, CF *medians*
G is *centroid* of ΔABC
*AG :*GD =*BG :*GE
=*CG :*GF = 2 : 1.

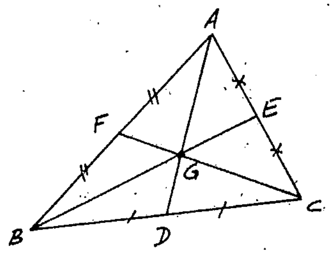

fig. 4.48

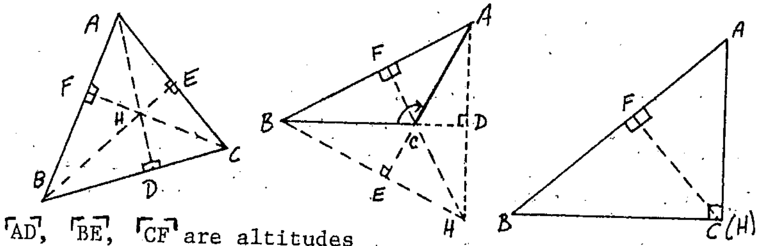

fig. 4.49

AD, BE, CF are altitudes
In these figures, H is the *ORTHOCENTRE* of ΔABC

Right Bisectors of Sides − these are lines bisecting the sides at right angles. They meet at the *CIRCUMCENTRE*.

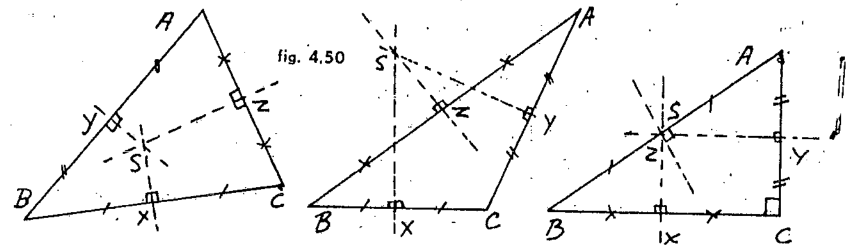

fig. 4.50

In these figures, SX, SY, SZ are right bisectors of the sides. S is the *CIRCUMCENTRE*.

SUMMARY:

MEDIANS	ALTITUDES	RIGHT BISECTORS OF SIDES
CENTROID	ORTHOCENTRE	CIRCUMCENTRE

EXAMPLE: Prove that in ΔABC, the altitudes are concurrent.
PROOF: Take *any* ΔABC so that the x axis lies along BC and the y axis passes through A. Then rotate the axes so that the x axis is horizontal. Take the co-ordinates of C as (c,o), B (−b,o), A (o,a)

Now *eqn. of altitude AO* is x=0 and for *eqn. of altitude BE* we find the gradient of

$$AC = \frac{y_2 - y_1}{x_2 - x_1} = \frac{a}{-c}$$

∴ Gradient of BE is $+\frac{c}{a}$

and eqn. of BE is given by $y - y_1 = m(x - x_1)$

as $y - o = \frac{c}{a}(x + b)$

i.e. $ay = cx + bc$

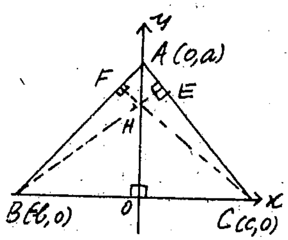

fig. 4.51

For eqn. of altitude CF, calculate gradient of $AB = \frac{a}{b}$, and hence of CF as $\frac{-b}{a}$

∴ Eqn. of CF is $y - o = \frac{-b}{a}(x - c)$

The *orthocentre* is the point of concurrence of the altitudes. To *prove* the altitudes are concurrent, we find the point of intersection of 2 altitudes, and then show that this point satisfies the third equation.

Solving AO , BE i.e. $\begin{cases} x = o \\ cx - ay + bc = o \end{cases}$

we find that $-ay + bc = 0$ i.e. $y = \frac{bc}{a}$

$\therefore$ The orthocentre H is $(o, \frac{bc}{a})$

Now substituting in the eqn. of CF , when $x = o$, $y = \frac{bc}{a}$

$\therefore$ $ay + bx - bc = a(\frac{bc}{a}) + b(o) - bc = o$

and hence that H lies on CF. Thus the 3 altitudes pass through H, and the result that 'the altitudes of *a triangle are concurrent' is true* for *any* triangle.

SET 4K

1. ABC is a triangle, right-angled at A. [Let the co-ords. be A(o,o); B(2a,o); C(o,2b)].
   (i) If D is the midpoint of $\overline{BC}$, show that D is equidistant from A, B, C.
   (ii) If $\overline{BE}$, $\overline{CF}$ are medians, show that $*BE^2 + *CF^2 = 5*EF^2$.

2. (a) ABC is an equilateral triangle in which $\overline{BC}$ is produced to D, so that *BC = *CD . Prove that $*AB^2 = \frac{1}{3}*AD^2$.
   (b) In ΔABC, $\hat{A} = 90°$ and *AB=*AC. If X is any point on BC, such that *BX=*XC, show that $2*AX^2 = *XB^2 + *XC^2$
   (c) ABC is a triangle in which AD is an altitude. Show that $*AB^2 - *AC^2 = *BD^2 - *DC^2 = *XB^2 - *XC^2$ where X is any point on $\overline{AD}$.

3. (a) PQR is a triangle, right angled at Q. If S is the midpoint of $\overline{PQ}$ and T a point on $\overline{QR}$ so that *QR =*RT , prove that $*PR^2 - *ST^2 = 3(*PS^2 + *RT^2)$
   (b) Prove that P(a cos θ, a sin θ) lies on the circle $x^2+y^2 = a^2$. If $\overline{AB}$ is a diameter, prove that $*A\hat{P}B = 90°$. [*Hint*, let A, B be (-a, o), (a,o) respectively.]

4. Show that
   (a) The interval joining the midpoints of two sides of a triangle is parallel to and half the third side.
   (b) The diagonals of a parallelogram bisect each other. [Take vertices at (o,o), (a,o), (b,c), (b-a,c)].
   (c) The sum of the squares on any two sides of a triangle is equal to twice the square on half the third side together with twice the square on the median drawn to that side (Apollonius' Theorem).

5. (a) ABCD is a square in which E, F are the midpoints of $\overline{BC}$, $\overline{CD}$ respectively. Prove that $*AE^2 + *AF^2 = 3*AB^2 - *EF^2$.
   (b) The diagonals of a quadrilateral ABCD meet at right angles. Show that $*AB^2 + *CD^2 = *AD^2 + *BC^2$. [Let vertices be (-a,o), (o,b), (c,o), (o,-d)].

6. Show that the sum of the squares on the sides of a rhombus are equal to the sum of the squares on the diagonals. [Take vertices as (-a,o), (a,o), (o,b), (o,-b)]. Repeat for a parallelogram.

7. ABC is a triangle right angled at A. P, Q are points on $\overline{AC}$ and $\overline{AB}$ produced respectively. Show that $*BC^2 - *BP^2 = *CQ^2 - *PQ^2$.

8. The line segments joining the midpoints of the adjacent sides of a quadrilateral form a parallelogram. [Let the quadrilateral have vertices (o,o), (2a,o), (2b,2c), (2d,2e)].

9. Prove that the altitudes of a triangle are concurrent, (for the two cases below),
   Case 1: Take the vertices as A(a,o), B(o,b), C(o,-c)
   Case 2: Take the vertices as A(a,b), B(o,o), C(c,o)

10. Prove that the medians of a triangle are concurrent. [Take the vertices at A(2a,2b), B(-2c,o), C(2c,o). Show the centroid is $(\frac{2a}{3}, \frac{2b}{3})$]

11. Prove that the right bisectors of the sides of a triangle are concurrent. [Take the vertices as (-2c,o), (2c,o), (2a,2b)]

12. Draw any triangle ABC on your page. Show how you would choose axes of co-ordinates and scale so that the co-ordinates of A, B, C are respectively (2a, 2b), (-2c,o), (2c,o).

    Write down the co-ordinates of the midpoints D, E, F of the sides BC, CA, AB respectively. Hence show that AD, BE, CF are concurrent.

    What theorem have you proven for *any* triangle?

13. For a trapezium show that
    (i) the distance between the midpoints of the non-parallel sides is one-half the sum of the parallel sides.
    (ii) the diagonals are equal if the trapezium is isosceles. (An isosceles trapezium is one in which the non-parallel sides are equal)

14. ABCD is a rectangle in which X is any point in $\overline{AB}$. Show that $*AX^2 + *CX^2 - *BX^2 = *DX^2$. Further, show this result is also true if X is outside the rectangle.