Sydney Girls' High School

2009 ASSESSMENT TASK 2

March 3rd 2009

MATHEMATICS

YEAR 12

Time Allowed: 90 minutes + 5 minutes reading time

Topic: Quadratics, Series & Sequences.

Directions to Candidates:

- There are five (5) questions.
- Attempt ALL questions.
- Questions are of EQUAL value.
- Start each question on a new page.
- Write on one side of the paper only.
- Show all necessary working.
- Marks will be deducted for careless or badly arranged work.
- Board-approved calculators may be used.

Total: 100 marks

QUESTION 1 (20 marks) — Marks

a) Given $y = x^2 + x - 6$

i) Find the axis of symmetry of the parabola — 2

ii) Find the co-ordinates of the vertex of the parabola — 2

iii) Sketch the parabola $y = x^2 + x - 6$ — 1

iv) For what values of x is the parabola positive? — 2

b) Consider the arithmetic progression 21, 25, 29, 33...

Find the i) n^{th} term — 2

ii) 50^{th} term — 2

iii) sum of the first 50 terms — 3

c) The first term of a geometric series is 16 and the fourth term is $\frac{1}{4}$.

i) Find the common ratio — 2

ii) Find the limiting sum of the series. — 2

d) How many terms are there in the sequence

21, 16, 11, ..., -134. — 2

QUESTION 2 (20 marks)

Marks

a) If the roots of $3x^2-6x+1=0$ are α and β.

Find: i) $\alpha+\beta$ 1

ii) $\alpha\beta$ 1

iii) $\frac{1}{\alpha}+\frac{1}{\beta}$ 2

iv) $\alpha^2+\beta^2$ 2

v) $(2-\alpha)(2-\beta)$ 3

b) Express $3x^2-5x+6=0$ in the form $A(x-2)^2+B(x-2)+C=0$ 4

c) If $(x-2)$, $(x+2)$, and $(x+8)$ are in a geometric progression, find the value of x. 3

d) The first term of an arithmetic progression is 2 and the product of its fourth and seventh terms is 4.

What are the possible value(s) for the common difference? 4

QUESTION 3 (20 marks)

Marks

a) Consider the equation $x^2+(k+2)x+4=0$.

For what values of k does the equation have:

i) Equal roots 3

ii) Distinct real roots. 3

b) Evaluate $\sum_{k=15}^{45} 124 - 7k$ 4

c) If one root of $3x^2-8x+k=0$ is three times the other root, find 3

d) How many terms of the series $9 + 18 + 36 + \ldots\ldots$ are needed to give a sum of 1143? 4

e) Bacteria in a culture are multiplying so that at the end of each hour there are 5% more than at the beginning of the hour.

If 200 000 bacteria are present now, how many would you expect to be present in 8 hours from now? 3

QUESTION 4 (20 marks) Marks

a) Solve: i) $9^x - 10(3^x) + 9 = 0$ 4

ii) $(x-3)^4 - 18(x-3)^2 + 32 = 0$ 5

b) Find the first negative term of the series $90 + 84 + 78 + \ldots\ldots$ 3

c) The positive multiples of 7 are 7, 14, 21, ...

i) What is the largest multiple of 7 less than 1000? 1

ii) What is the sum of the positive multiples of 7 which are less than 1000? 2

d) Anne invests \$2 000 per year in a superannuation fund which pays 10.5% p.a. interest compounded annually. Anne collects her investment at the end of 25 years immediately after the interest has been paid.

How much does she collect? 5

QUESTION 5 (20 marks) Marks

a) Given the sum of the series is given by $S_n = 5n^2 - 3n$.

Find the first three terms of the series. 3

b) If α and β are the roots of $x^2 + hx + k = 0$, find the relationship between h and k if: $\frac{1}{\alpha} + \frac{1}{\beta} = 3$. 3

c) Jane borrows \$10 000 at 18% p.a. compound interest over 4 years. Interest is compounded monthly and repayments are made monthly.

i) Find the balance at the end of the first month after interest has been calculated and the first repayment (\$M) has been made. 2

ii) Calculate the value of \$M(the monthly repayment) so that the loan will be repaid at the end of four years. 5

iii) Calculate the balance owing at the end of two years. 3

iv) Calculate the total interest paid. 2

v) Calculate the equivalent annual Simple Interest rate. 2

Year 12 - Assessment 2

- MATHEMATICS - SOLUTIONS
- MARCH 3rd 2009

Question 1 (20 marks)

a) $y = x^2 + x - 6$

i) Axis of symmetry

$x = \frac{-b}{2a}$

$x = -\frac{1}{2}$ (2)

ii) Vertex at $x = -\frac{1}{2}$

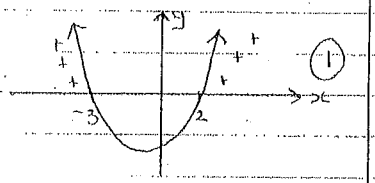
$y = (-\frac{1}{2})^2 + (-\frac{1}{2}) - 6$

$y = -6\frac{1}{4}$

$\therefore$ Vertex $(-\frac{1}{2}, -6\frac{1}{4})$ (2)

iii) $x^2 + x - 6 = 0$

$(x+3)(x-2) = 0$



(1)

iv) positive at

$(x+3)(x-2) > 0$ (2)

$x < -3$ and $x > 2$

b) $a = 21, \quad d = 4$

i) $T_n = a + (n-1)d$

$= 21 + (n-1)4$

$= 21 + 4n - 4$

$T_n = 4n + 17$ (2)

ii) $T_{50} = 4n + 17$

$= 4(50) + 17$

$= 217$ (2)

iii) $S_n = \frac{n}{2}(a + l)$

$S_{50} = \frac{50}{2}(21 + 217)$

$= 25 \times 238$

$S_{50} = 5950$ (3)

c) $a = 16 \quad T_4 = ar^3 = \frac{1}{4}$

i) $16r^3 = \frac{1}{4}$

$r^3 = \frac{1}{64}$

$\therefore r = \frac{1}{4}$ (2)

ii) since $|r| < 1$ ... limiting sum exists

$S = \frac{a}{1-r}$

$S = \frac{16}{1 - \frac{1}{4}} = 21\frac{1}{3}$ (2)

d) $T_n = a + (n-1)d$

$-134 = 21 + (n-1) - 5$

$-134 - 21 = -5n + 5$

$-160 = -5n$ (2)

$\therefore n = 32$

There are 32 terms in sequence

Question 2 (20 marks)

a) $3x^2 - 6x + 1 = 0$

i) $\alpha + \beta = \frac{-b}{a}$

$= \frac{6}{3}$

$\alpha + \beta = 2$ (1)

ii) $\alpha\beta = \frac{c}{a}$

$\alpha\beta = \frac{1}{3}$ (1)

iii) $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha + \beta}{\alpha\beta}$

$= 2 \div \frac{1}{3}$

$= 6$ (2)

iv) $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$= (2)^2 - 2(\frac{1}{3})$

$= 4 - \frac{2}{3}$

$= 3\frac{1}{3}$ (2)

v) $(2-\alpha)(2-\beta) = 4 - 2\alpha - 2\beta + \alpha\beta$

$= 4 - 2(\alpha + \beta) + \alpha\beta$

$= 4 - 2(2) + \frac{1}{3}$

$= \frac{1}{3}$ (3)

b) $3x^2 - 5x + 6 \equiv A(x-2)^2 + B(x-2) + C$

$A(x^2 - 4x + 4) + Bx - 2B + C$

$Ax^2 + x(B - 4A) + 4A - 2B + C$

$\therefore A = 3 \quad B - 4A = -5$

$B - 12 = -5$

$B = 7$

and, $4A + C - 2B = 6$

$12 + C - 14 = 6$

$C = 8$ (4)

$\therefore 3x^2 - 5x + 6 \equiv 3(x-2)^2 + 7(x-2) + 8$

c) $r = \frac{T_2}{T_1} = \frac{T_3}{T_2}$

$\frac{(x+2)}{(x-1)} = \frac{(x+8)}{(x+2)}$

$(x+2)^2 = (x+8)(x-2)$

$x^2 + 4x + 4 = x^2 + 8x - 2x - 16$

$4x + 4 = 6x - 16$

$-2x = -20$

$x = 10$ (3)

d) $T_1 = a = 2$

$T_4 = a + 3d$

$T_7 = a + 6d$

$(2 + 3d)(2 + 6d) = 4$

$4 + 12d + 6d + 18d^2 = 4$

$18d^2 + 18d = 0$

$18d(d + 1) = 0$

$\therefore d = 0$ or $d = -1$ (4)

Question 3 (20 marks)

a) $x^2 + (k+2)x + 4 = 0$

i) Equal roots $\Delta = 0$

$\Delta = b^2 - 4ac$

$= (k+2)^2 - 4(1)(4)$

$0 = (k+2)^2 - 16$

$16 = (k+2)^2$

$k+2 = \pm 4$

$k = \pm 4 - 2$

$\therefore k = 4-2$ or $-4-2$

$k = 2$ or -6 (3)

ii) Distinct real roots

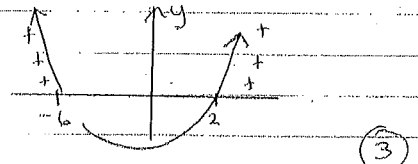
$\Delta > 0$

$(k+2)^2 - 16 > 0$

$k^2 + 4k + 4 - 16 > 0$

$k^2 + 4k - 12 > 0$

$(k+6)(k-2) > 0$



(3)

$\therefore k < -6$ or $k > 2$

b) Evaluate

$\sum_{k=15}^{45} 124 - 7k$

$k = 15 \quad T_{15} = 124 - 7(15)$

$= 19$

$T_{45} = 124 - 7(45)$

$= -191$

$S_{um} = \frac{n}{2}(a+l)$

$n = 45 - 15 + 1$

$n = 31$

$\therefore S = \frac{31}{2}(19 - 191)$

$= \frac{31}{2} \times -172$

$= -2666$ (4)

c) $\alpha + \beta = \frac{-b}{a}$

$= \frac{8}{3}$

$\alpha\beta = \frac{c}{a}$

$= \frac{k}{3}$

and $\alpha = 3\beta$

$4\beta = \frac{8}{3}$

$\therefore \beta = \frac{2}{3}$ and $\alpha = 3 \times \frac{2}{3}$

$\alpha = 2$

Now, $\frac{2}{3} \times 2 = \frac{k}{3}$

$\frac{4}{3} = \frac{k}{3}$

$\therefore k = 4$ (3)

d) $r = \frac{18}{9} = \frac{36}{18} = 2$

$S = \frac{a(r^n - 1)}{r-1}$

$1143 = \frac{9(2^n - 1)}{2-1}$

Question 3 (cont'd)

$1143 = 9(2^n - 1)$

$127 = 2^n - 1$

$128 = 2^n$

$2^7 = 2^n$ (4)

$\therefore n = 7$

$\therefore$ 7 terms are needed.

e) $A_n = P\left(1 + \frac{r}{100}\right)^n$

$P = 200\,000$

$r = 5\%$

$n = 8$ (3)

$A_8 = 200000\left(1 + \frac{5}{100}\right)^8$

$= 200000\left(\frac{1}{120}\right)^8$

$= \$295\,491.09$

Question 4 (20 marks)

a) i) $9^x - 10(3^x) + 9 = 0$

let $m = 3^x$

$m^2 - 10m + 9 = 0$

$(m-9)(m-1) = 0$

$m = 9$ or $m = 1$

$3^x = 9 \qquad 3^x = 1$

$3^x = 3^2 \qquad 3^x = 3^0$

$\therefore x = 2$ or $x = 0$ (4)

ii) $(x-3)^4 - 18(x-3)^2 + 32 = 0$

let $m = (x-3)^2$

$m^2 - 18m + 32 = 0$

$(m-16)(m-2) = 0$

$m - 16 = 0$ or $m - 2 = 0$

$m = 16 \qquad m = 2$

$(x-3)^2 = 16 \qquad (x-3)^2 = 2$

$x - 3 = \pm 4 \qquad x - 3 = \pm\sqrt{2}$

$x = \pm 4 + 3 \qquad x = \pm\sqrt{2} + 3$

$x_1 = 7$ (5) $x_3 = \sqrt{2} + 3$

$x_2 = -1 \qquad x_4 = -\sqrt{2} + 3$

b) $a = 90 \quad d = -6$

$T_n = a + (n-1)d$

$90 - 6n + 6 < 0$

$96 - 6n < 0$

$-6n < -96$

(3) $n > 16$

$\therefore n = 17$ would be the first negative term of series

$T_{17} = 90 + 16 \times -6 = -6$

c) $a = 7, \quad d = 7$

i) $n = 142$

$T_{142} = 994$ (1)

ii) $S_n = \frac{n}{2}(a+l)$

$S_{142} = \frac{142}{2}(7 + 994)$

$= 71 \times 1001$ (2)

$= 71071$

Question 4 (con't)

d) The first $2000 = 2000×1.105^{25}

The second $2000 = 2000×1.105^{24}

The third $2000 = 2000×1.105^{23}

⋮

The last $2000 = 2000×1.105

∴ At the end of 25 years

$= \$2000 (1.105 + 1.105^2 + \cdots + 1.105^{23} + 1.105^{24} + 1.105^{25})$

$S_n = \frac{ar(r^n - 1)}{r - 1}$

$S_{25} = \frac{2000 \times 1.105 (1.105^{25} - 1)}{1.105 - 1}$ (5)

$= \$234\ 375.34$

Question 5

a) $S_n = 5n^2 - 3n$

$S_1 = 5(1)^2 - 3(1)$

$a = S_1 = 2$

$S_2 = 5(2)^2 - 3(2)$

$= 20 - 6$

$= 14$

$\therefore T_2 = S_2 - S_1$

$= 14 - 2$

$= 12$

$S_3 = 5(3)^2 - 3(3)$

$= 45 - 9$

$= 36$

$T_3 = 36 - 14$

$= 22$

$\therefore T_1 = 2,$

$T_2 = 12$ and

$T_3 = 22$

(3)

Question 5 (con't)

b) $x^2 + bx + k = 0$

$\alpha + \beta = \frac{-b}{a}$ $\quad \alpha\beta = \frac{c}{a}$

$= \frac{-h}{1}$ $\quad = \frac{k}{1}$

$\therefore \alpha + \beta = -h$ $\quad \alpha\beta = k$

Now, $\frac{1}{\alpha} + \frac{1}{\beta} = 3$

$\frac{\beta + \alpha}{\alpha\beta} = 3$

$\frac{-h}{k} = 3$ (3)

$-h = 3k$

$\therefore h = -3k$

c) Loan = $10 000, 4 years, 18% p.a.

i) $\therefore n = 4 \times 12$ $\quad r = 18 \div 12$

$= 48$ months $\quad = 1.5\%$ per month

$A_1 = 10\,000 \times 1.015 - \$M$ (2)

ii) $A_2 = 10\,000 \times 1.015^2 - \$M(1 + 1.015)$

$A_3 = 10\,000 \times 1.015^3 - \$M(1 + 1.015 + 1.015^2)$

⋮

$A_n = 10\,000 \times 1.015^n - \$M(1 + 1.015 + \cdots + 1.015^{n-1})$

At the end of 4 years $A_{48} = 0$

$0 = 10\,000 \times 1.015^{48} - \$M(1 + 1.015 + \cdots + 1.015^{47})$

Question 5 (cont)

$$\therefore \$M = \frac{10000 \times 1.015^{48}}{(1 + 1.015 + 1.015^2 + \cdots + 1.015^{47})}$$

(denominator: a.s)

$$S_{48} = \frac{a(r^n - 1)}{r - 1}$$

$$= \frac{1(1.015^{48} - 1)}{1.015 - 1}$$

$$\therefore \$M = \frac{10000 \times 1.015^{48} \times 0.015}{1.015^{48} - 1} \quad (5)$$

$\therefore$ Monthly Repayment = $293.75

iii) End of two years $n = 24$

$$\therefore A_{24} = 10000 \times 1.015^{24} - \$293.75(1 + 1.015 + \cdots + 1.015^{23})$$

$$= 10000 \times 1.015^{24} - 293.75 \frac{(1.015^{24} - 1)}{0.015} \quad (3)$$

Balance after 2 years = $5883.93

iv) Total interest = $293.75 \times 48 - 10,000$

= $4100 (2)

v) Equivalent Simple Interest Rate

$I = PRT$

$$4100 = 10000 \times \frac{R}{100} \times 4 \quad (2)$$

$$4100 = 400R$$

$$\therefore R = \frac{4100}{400} = 10.25\% \text{ p.a.}$$