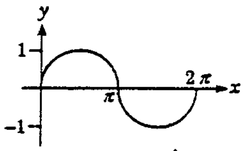
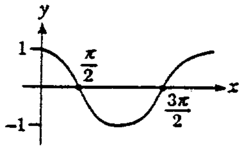
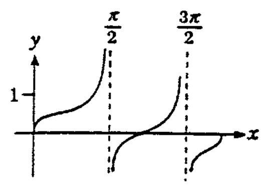
1. Arc length: $\ell = r\theta$
   Area of sector $= \frac{1}{2}r^2\theta$
   } θ in radians

2. 

$y = \sin x$
Period $= 2\pi$
Amplitude $= 1$



$y = \cos x$
Period $= 2\pi$
Amplitude $= 1$



$y = \tan x$
Period $= \pi$

3. When x is small, $\lim\limits_{x \to 0} \frac{\sin x}{x} = 1$

4. **Derivatives:**

$\frac{d}{dx}(\sin x) = \cos x$ $\qquad$ $\frac{d}{dx}[\sin f(x)] = f'(x)\cos f(x)$

$\frac{d}{dx}(\cos x) = -\sin x$ $\qquad$ $\frac{d}{dx}[\cos f(x)] = -f'(x)\sin f(x)$

$\frac{d}{dx}(\tan x) = \sec^2 x$ $\qquad$ $\frac{d}{dx}[\tan f(x)] = f'(x)\sec^2 f(x)$

5. **Integrals:**

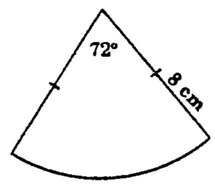
$\int \sin x\, dx = -\cos x + C$ $\qquad$ $\int \sin ax\, dx = -\frac{1}{a}\cos ax + C$

$\int \cos x\, dx = \sin x + C$ $\qquad$ $\int \cos ax\, dx = \frac{1}{a}\sin ax + C$

$\int \sec^2 x\, dx = \tan x + C$ $\qquad$ $\int \sec^2 ax\, dx = \frac{1}{a}\tan ax + C$

a and C constants

EXERCISE 9: EXAMINATION-TYPE QUESTIONS

1. (a) 

The figure shows a sector of a circle with radius 8 cm. Find the area of this sector correct to 2 decimal places.

   (b) Differentiate: (i) $\log_e(\sin x)$ (ii) $\cos\left(x + \frac{\pi}{2}\right)$

   (c) Evaluate: (i) $\int_0^{\frac{\pi}{4}} \cos 2x\, dx$ (ii) $\int_0^1 \sin\frac{\pi}{3}x\, dx$

   (d) Find the equation of the tangent to $y = 2\sin 2x$ at the point $x = \frac{\pi}{12}$.

2. (a) Solve, where $0 \le x \le 2\pi$, $2\sin x = \sqrt{3}$.

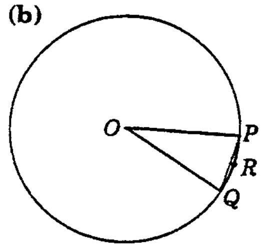
   (b) Evaluate $\lim\limits_{\theta \to 0} \frac{\sin 3\theta}{\theta}$.

   (c) (i) Sketch the graph of $y = \cos x$, for $0 \le x \le \pi$.
   (ii) Where does the curve cross the x axis?
   (iii) Show that the area between the curve and the x axis between $x = 0$ and $x = \pi$ is 2 units^2.

   (d) Find $\frac{dy}{dx}$ if: (i) $y = \tan\frac{x}{3}$ (ii) $y = \sin^2 x$

3. **(a)** Find the gradient of the normal to $y = \cos 3x$ at the point $x = \frac{\pi}{6}$.

(b)



The figure shows a circle centre O with triangle OPQ and R is a point on arc PQ. $OP = 10$ cm and $\angle POQ = 30°$.

(i) Find the area of ΔOPQ.

(ii) Express 30° in radians and show that the area of sector OPQ is $\frac{25\pi}{3}$ cm^2.

(iii) Show that the area of the segment PRQ is $25\left(\frac{\pi - 3}{3}\right)$ cm^2.

(iv) Use the Cosine Rule to find the length of PQ.

(v) Show that the length PQ is shorter than the arc PRQ by 0·06 cm, correct to 2 dec. places.

(c) Find the nature of the stationary points on $y = \sin 2x$ if $0 < x < \pi$ (using calculus).

(d) Find c, if $\int_0^c \cos x \, dx = \frac{\sqrt{3}}{2}$, where $0 < c < \frac{\pi}{2}$.

4. **(a)** Differentiate: **(i)** $\frac{\sin x}{x}$ **(ii)** $e^{2\cos x}$

(b) Solve, for $0° \le x \le 360°$, $2\sin x - 1 = 0$

(c) Evaluate: **(i)** $\int_0^{\frac{\pi}{8}} \sec^2 2x \, dx$ **(ii)** $\int_0^{\frac{\pi}{2}} \sin x + \cos x \, dx$

(d) Prove that $\frac{d}{dx}\left[\frac{1 + \sin x}{\cos x}\right] = \frac{1}{1 - \sin x}$.

5. **(a)** Evaluate $\lim_{x \to 0} \frac{\sin 2x}{4x}$.

(b) **(i)** Sketch $y = \sin \pi x$, on a number plane, where $-2 \le x \le 2$.

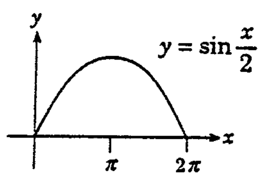
(ii) State the period and amplitude of $y = \sin \pi x$.

(iii) Use the sketch to solve the equation $\sin \pi x = 0$.

(c) If $y = 4\sin 3x$, prove that $y'' + 9y = 0$.

(d) **(i)** Show that $\int_0^{\pi} \sin \frac{x}{2} \, dx = 2$.

(ii)



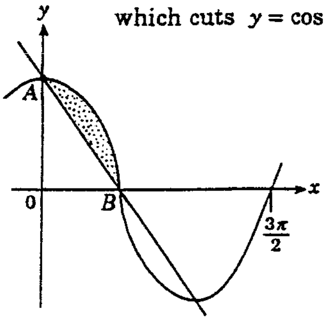
Find the area between the curve $y = \sin \frac{x}{2}$ and the x axis when $x = 0$ and $x = 2\pi$.

6. **(a)** The minute hand of a clock is 4 cm in length. What area is swept by the hand in an interval of 40 minutes? Answer in terms of π.

(b) Find the derivative of: **(i)** $\sin 2x + \cos x$ **(ii)** $\dfrac{1}{\cos x}$

(c) Find: **(i)** $\int (\sin x - \cos x)\, dx$ **(ii)** $\int \dfrac{\cos x}{\sin x + 1}\, dx$

(d) The diagram shows the graph of $y = \cos x$ and a straight line which cuts $y = \cos x$ at the points A and B.

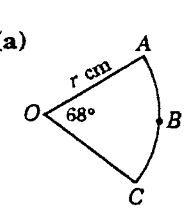


(i) Find the coordinates of the points A and B.

(ii) Show that the equation of the line passing through A and B is $y = \dfrac{\pi - 2x}{\pi}$.

(iii) Find the shaded area between $y = \cos x$ and $y = \dfrac{\pi - 2x}{\pi}$ (marked on the diagram).

7. **(a)**



The figure shows a sector of a circle with radius r cm. The length of the arc ABC is 7 cm.

(i) Find the value of r, to one decimal place.

(ii) Show that the area of the sector OAC is approximately 21 cm^2.

(b) Differentiate: **(i)** $\log(\sin x + \cos x)$ **(ii)** $\cos^2(3x - 1)$

(c) **(i)** Sketch the graph of $y = \cos x$, where $-\pi \le x \le \pi$.

(ii) On the same number plane, graph $y = \dfrac{1}{2}$.

(iii) Using (i) and (ii), solve $\cos x > \dfrac{1}{2}$ for $-\pi \le x \le \pi$.

(d) The area bounded by the curve $y = \sec x$, the x axis and the lines $x = 0$ and $x = \frac{\pi}{3}$ is rotated about the x axis. Find the volume of the solid formed.

8. **(a)** If $y'' = -3\cos x - 2\sin x$ and when $x = 0$, $y' = 0$, $y = 5$, find y in terms of x.

(b) Find the equation of the tangent to the curve $y = x\cos x$ at the point $x = \pi$.

(c) Solve the equation $\cos\dfrac{x}{2} = \dfrac{1}{\sqrt{2}}$ where $0 \le x \le 2\pi$.

(d) **(i)** Show that if $f(x) = 2\sin 2x + 1$, then $f'(x) = 4\cos 2x$.

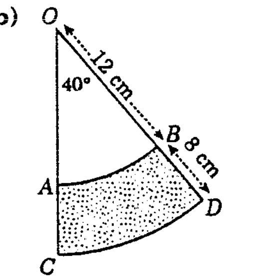
(ii) Hence, show that $\displaystyle\int_0^{\frac{\pi}{6}} \frac{4\cos 2x}{2\sin 2x + 1}\, dx = 1{\cdot}0051$, rounded off correct to four decimal places.

9. (a) (i) Show that the point $\left(\frac{\pi}{3}, \frac{\sqrt{3}}{3}\right)$ lies on the curve $y = \frac{\sin x}{1+\cos x}$.

(ii) Show that if $y = \frac{\sin x}{1+\cos x}$, then $\frac{dy}{dx} = \frac{1}{1+\cos x}$.

(iii) Find the equation of the tangent to the curve $y = \frac{\sin x}{1+\cos x}$ at the point $x = \frac{\pi}{3}$.

(b)



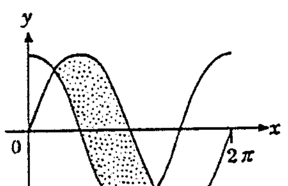
The diagram shows AB and CD as arcs of concentric circles, with centre O. It is known that $OB = 12$ cm and $BD = 8$ cm.

(i) Find the arc length CD, correct to 2 decimal places.

(ii) Show that the area of the shaded region is 17 868·8 cm^2.

10. (a) Show that $\frac{d}{dx}[x\sin x + \cos x] = x\cos x$, and hence evaluate $\int_0^{\frac{\pi}{2}} x\cos x \, dx$.

(b)



The diagram shows the graphs of $y = \sin x$ and $y = \cos x$ for the domain $0 \le x \le 2\pi$.

(i) Show that the points of intersection of $y = \sin x$ and $y = \cos x$ are $\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$ and $\left(\frac{5\pi}{4}, -\frac{1}{\sqrt{2}}\right)$.

(ii) Show that $y = \sin x$ cuts the x axis at $x = 0$, π and 2π, while $y = \cos x$ cuts the x axis at $x = \frac{\pi}{2}$ and $\frac{3\pi}{2}$.

(iii) Show that the tangent to $y = \sin x$ at $x = \pi$ is parallel to the tangent to $y = \cos x$ at $x = \frac{\pi}{2}$.

(iv) Find the exact area of the shaded region.

(c) Find the volume of the solid formed when the curve $y = \tan x$ is rotated about the x axis between $x = 0$ and $x = \frac{\pi}{4}$. Leave your answer in terms of π.

EXERCISE 9: WORKED SOLUTIONS

1. (a) $A=\frac{1}{2}r^2\theta$

Note θ must be in radians.

72 degrees $=72\times\frac{\pi}{180}$

$=1\cdot256\,6371$

(from calc.)

$\therefore A=\frac{1}{2}r^2\theta$

$=\frac{1}{2}\times 8^2\times 1\cdot256\,637\,1$

$=40\cdot212\,386$

$=40\cdot21$ (2 dec. places)

$\therefore$ area is $40\cdot21\text{ cm}^2$.

(b) (i) $\frac{d}{dx}\left[\log_e(\sin x)\right]=\frac{\cos x}{\sin x}$

$\frac{d}{dx}\left[\log f(x)\right]=\frac{f'(x)}{f(x)}$

$=\cot x.$

Note $\frac{\cos x}{\sin x}=\cot x$

(ii) $\frac{d}{dx}\left[\cos\left(x+\frac{\pi}{2}\right)\right]$

$=-\sin\left(x+\frac{\pi}{2}\right)$

(c) (i) $\int_0^{\frac{\pi}{4}}\cos 2x\,dx$

$=\left[\frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{4}}$

$=\frac{1}{2}\left[\sin 2\left(\frac{\pi}{4}\right)-\sin 2(0)\right]$

$=\frac{1}{2}\left[\sin\frac{\pi}{2}-\sin 0\right]$

$=\frac{1}{2}[1-0]$

$=\frac{1}{2}.$

(ii) $\int_0^1\sin\frac{\pi}{3}x\,dx=\left[-\frac{3}{\pi}\cos\frac{\pi}{3}x\right]_0^1$

$=-\frac{3}{\pi}\left[\cos\frac{\pi}{3}(1)-\cos\frac{\pi}{3}(0)\right]$

$=-\frac{3}{\pi}\left[\cos\frac{\pi}{3}-\cos 0\right]$

$=-\frac{3}{\pi}\left[\frac{1}{2}-1\right]$

$=-\frac{3}{\pi}\left[-\frac{1}{2}\right]$

$=\frac{3}{2\pi}.$

(d) $y=2\sin 2x$

Subs. $x=\frac{\pi}{12}$ in $y=2\sin 2x$

$=2\sin 2\left(\frac{\pi}{12}\right)$

$=2\sin\frac{\pi}{6}$

$=2\times\frac{1}{2}$

$=1$

$\therefore$ the point $\left(\frac{\pi}{12},1\right)$.

For gradient, $\frac{dy}{dx}=4\cos 2x$

Subs. in $x=\frac{\pi}{12}$

$\therefore \frac{dy}{dx}=4\cos 2\left(\frac{\pi}{12}\right)$

$=4\cos\frac{\pi}{6}$

$=4\left(\frac{\sqrt{3}}{2}\right)$

$=2\sqrt{3}.$

$\therefore$ Equation of tangent:

$y-y_1=m(x-x_1)$

$y-1=2\sqrt{3}\left(x-\frac{\pi}{12}\right)$

$y-1=2\sqrt{3}x-\frac{\sqrt{3}\pi}{6}$

2. (a) $2\sin x=\sqrt{3}$

$\therefore \sin x=\frac{\sqrt{3}}{2}$

$x=\frac{\pi}{3},\frac{2\pi}{3}$

Pos. in 1st and in 2nd quad.

(b) $\lim_{\theta\to 0}\frac{\sin 3\theta}{\theta}$

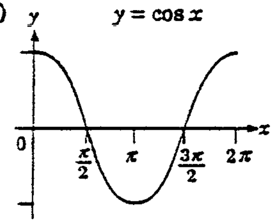
$=\lim_{\theta\to 0}\frac{\sin 3\theta}{3\theta}.3$

$=3\lim_{\theta\to 0}\frac{\sin 3\theta}{3\theta}$

$=3.1$ $\quad\lim_{x\to 0}\frac{\sin x}{x}=1$

$=3$

(c) (i)



(ii) Cut x axis when

$x=\frac{\pi}{2},\frac{3\pi}{2}$

(as seen from the diagram).

(iii) Area $=\int_0^{\frac{\pi}{2}}\cos x\,dx+\left|\int_{\frac{\pi}{2}}^{\pi}\cos x\,dx\right|$

$=\left[\sin x\right]_0^{\frac{\pi}{2}}+\left|\left[\sin x\right]_{\frac{\pi}{2}}^{\pi}\right|$

$=\left(\sin\frac{\pi}{2}-\sin 0\right)+\left|\sin\pi-\sin\frac{\pi}{2}\right|$

$=(1-0)+\left|-1-0\right|$

$=1+\left|-1\right|$

$=1+1$

$=2.$

$\therefore$ area is 2 units^2.

(d) (i) $y=\tan\frac{x}{3}$ $\quad\frac{x}{3}=\frac{1}{3}x$

$\therefore y=\tan\frac{1}{3}x$

$\therefore \frac{dy}{dx}=\frac{1}{3}\sec^2\frac{1}{3}x$

$=\frac{1}{3}\sec^2\frac{x}{3}.$

(ii) $y = \sin^2 x$

$\therefore \quad y = (\sin x)^2$

$\therefore \quad \frac{dy}{dx} = 2(\sin x)'.\cos x$

$= 2\sin x \cos x.$

3. (a) $y = \cos 3x$

$\frac{dy}{dx} = -3\sin 3x$

Subs. $x = \frac{\pi}{6}$ in $\frac{dy}{dx}$

$\therefore \quad \frac{dy}{dx} = -3\sin 3\left(\frac{\pi}{6}\right)$

$= -3\sin\frac{\pi}{2}$

$= -3.1$

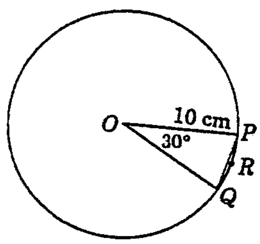
$= -3$

$\therefore$ grad. of tangent $= -3,$

$\therefore$ grad. of normal $= \frac{1}{3}$

$\therefore$ gradient of normal to $y = \cos 3x$ at $x = \frac{\pi}{6}$ is $\frac{1}{3}$.

(b)



(i) $A = \frac{1}{2}ab\sin C$

$= \frac{1}{2} \times 10 \times 10 \times \sin 30$

$= \frac{1}{2} \times 10 \times 10 \times \frac{1}{2}$

$= 25$

$\therefore$ area of $\Delta\ OPQ$ is $25\ \text{cm}^2$.

(ii) $30° = 30 \times \frac{\pi}{180} = \frac{\pi}{6}$

Deg. → Rad: $\times \frac{\pi}{180}$

$A = \frac{1}{2}r^2\theta$

$= \frac{1}{2}.10^2.\frac{\pi}{6}$

$= \frac{25\pi}{3}$

$\therefore$ area of sector OPQ is $\frac{25\pi}{3}\ \text{cm}^2$.

(iii) Area of segment

= area of sector

− area of triangle

$= \frac{25\pi}{3} - 25$

$= 25\left(\frac{\pi}{3} - 1\right)$

$= 25\left(\frac{\pi - 3}{3}\right)$

$\therefore$ area of segment is $25\left(\frac{\pi - 3}{3}\right)\ \text{cm}^2$.

(iv)

$PQ^2 = 10^2 + 10^2 - 2(10)(10)\cos 30°$

$a^2 = b^2 + c^2 - 2bc\cos A$

$= 100 + 100 - 200.\frac{\sqrt{3}}{2}$

$= 200 - 100\sqrt{3}$

$= 26{\cdot}794\,919$ (from calc.)

$\therefore\ PQ = \sqrt{26{\cdot}794\,919}$

$= 5{\cdot}176\,380\,9$ (from calc.)

(v)

Now arc length $PRQ = r\theta$ | $\ell = r\theta$ |

$= 10.\frac{\pi}{6}$

$= \frac{5\pi}{3}$

$= 5{\cdot}235\,987\,8$ (from calc.)

$\therefore$ Difference

$= 5{\cdot}235\,987\,8 - 5{\cdot}176\,380\,9$

$= 0{\cdot}059\,606\,8$ (from calc.)

$= 0{\cdot}06$ (to 2 dec. pl.)

$\therefore\ PQ$ is shorter by 0·06 cm.

(c) $y = \sin 2x$

$\frac{dy}{dx} = 2\cos 2x = 0$

Stat. pts. when $\frac{dy}{dx} = 0$

$\therefore \quad \cos 2x = 0$

$\therefore \quad 2x = \frac{\pi}{2}, \frac{3\pi}{2}$

$\therefore \quad x = \frac{\pi}{4}, \frac{3\pi}{4}.$

Subs. in $y = \sin 2x$

$\therefore x = \frac{\pi}{4} \quad \therefore\ y = \sin 2\left(\frac{\pi}{4}\right)$

$= \sin\frac{\pi}{2}$

$= 1$

$\therefore x = \frac{3\pi}{4} \quad \therefore\ y = \sin 2\left(\frac{3\pi}{4}\right)$

$= \sin\frac{3\pi}{2}$

$= -1$

$\therefore$ stat. points at $\left(\frac{\pi}{4}, 1\right), \left(\frac{3\pi}{4}, -1\right)$.

$\frac{d^2y}{dx^2} = -4\sin 2x$

Now subs. $x = \frac{\pi}{4}$ in $\frac{d^2y}{dx^2}$

$\therefore \quad \frac{d^2y}{dx^2} = -4\sin 2\left(\frac{\pi}{4}\right)$

$= -4\sin\frac{\pi}{2}$

$= -4.1$

$= -4 < 0$

$\therefore$ max.

Also, subs. $x = \frac{3\pi}{4}$ in $\frac{d^2y}{dx^2}$

$\therefore \quad \frac{d^2y}{dx^2} = -4\sin 2\left(\frac{3\pi}{4}\right)$

$= -4\sin\frac{3\pi}{2}$

$= -4(-1)$

$= 4 > 0$

$\therefore$ min.

$\therefore$ maximum at $\left(\frac{\pi}{4}, 1\right)$ and minimum at $\left(\frac{3\pi}{4}, -1\right)$.

(d) $\int_0^c \cos x\, dx = \frac{\sqrt{3}}{2}$

$\therefore \int_0^c \cos x\, dx = \left[\sin x\right]_0^c$

$= \sin c - \sin 0$
$= \sin c$

$\therefore \quad \sin c = \frac{\sqrt{3}}{2}$

$\therefore \quad c = \frac{\pi}{6}.$

4. (a) (i) $\frac{d}{dx}\left[\frac{\sin x}{x}\right]$

$u = \sin x$	$v = x$
$\frac{du}{dx} = \cos x$	$\frac{dv}{dx} = 1$

$= \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$= \frac{x.\cos x - \sin x.1}{x^2}$

$= \frac{x\cos x - \sin x}{x^2}.$

(ii) $\frac{d}{dx}\left[e^{2\cos x}\right]$

$\boxed{\frac{d}{dx}e^{f(x)} = f'(x)e^{f(x)}}$

$= -2\sin x e^{2\cos x}$

(b)

$2\sin x - 1 = 0$
$2\sin x = 1$
$\sin x = \frac{1}{2}$

$\therefore \quad x = \frac{\pi}{6}, \frac{5\pi}{6}.$

(c) (i) $\int_0^{\frac{\pi}{8}} \sec^2 2x \, dx$

$= \left[\frac{1}{2}\tan 2x\right]_0^{\frac{\pi}{8}}$

$= \frac{1}{2}\left[\tan 2\left(\frac{\pi}{8}\right) - \tan 2(0)\right]$

$= \frac{1}{2}\left[\tan\frac{\pi}{4} - \tan 0\right]$

$= \frac{1}{2}[1 - 0]$

$= \frac{1}{2}$

(ii) $\int_0^{\frac{\pi}{2}} \sin x + \cos x \, dx$

$= \left[-\cos x + \sin x\right]_0^{\frac{\pi}{2}}$

$= \left(-\cos\frac{\pi}{2} + \sin\frac{\pi}{2}\right) - \left(-\cos 0 + \sin 0\right)$

$= (0 + 1) - (-1 + 0)$

$= 1 + 1$

$= 2$

(d) $\frac{d}{dx}\left[\frac{1+\sin x}{\cos x}\right]$

$u = 1 + \sin x$	$v = \cos x$
$\frac{du}{dx} = \cos x$	$\frac{dv}{dx} = -\sin x$

$= \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$= \frac{\cos x.\cos x - (1+\sin x).-\sin x}{(\cos x)^2}$

$= \frac{\cos^2 x + \sin x + \sin^2 x}{\cos^2 x}$

$\boxed{(\cos x)^2 = \cos^2 x}$

$= \frac{1+\sin x}{1-\sin^2 x}$ $\quad \boxed{\cos^2 x = 1 - \sin^2 x}$

$= \frac{\cancel{1+\sin x}}{(1-\sin x)(\cancel{1+\sin x})}$

$\boxed{a^2 - b^2 = (a-b)(a+b)}$

$= \frac{1}{1-\sin x}$

$\therefore \quad \frac{d}{dx}\left[\frac{1+\sin x}{\cos x}\right] = \frac{1}{1-\sin x}.$

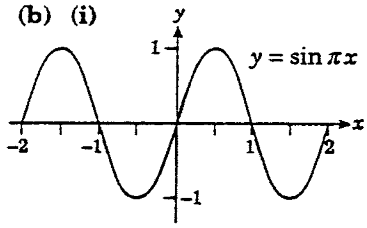
5. (a) $\lim_{x\to 0}\frac{\sin 2x}{4x}$

$= \frac{1}{2}\lim_{x\to 0}\frac{\sin 2x}{2x}$

$= \frac{1}{2}.1$ $\quad \boxed{\lim_{X\to 0}\frac{\sin X}{X}}$

$= \frac{1}{2}$

(b) (i)



(ii) $y = \sin \pi x$

$\therefore$ period: 2

Note
Period $= \frac{2\pi}{n} = \frac{2\pi}{\pi} = 2$

amplitude: 1

(iii) For $\sin \pi x = 0$,
$\therefore$ intersection of
$y = \sin \pi x$ and $y = 0$
[i.e. x axis],
$\therefore x = -2, -1, 0, 1, 2.$

(c) $y = 4\sin 3x$
$y' = 12\cos 3x$
$y'' = -36\sin 3x$

For $y'' + 9y = 0$

$\therefore$ LHS $= y'' + 9y$
$= -36\sin 3x + 9(4\sin 3x)$
$= -36\sin 3x + 36\sin 3x$
$= 0 \qquad$ = RHS

$\therefore \quad y'' + 9y = 0$

(d) (i) $\int_0^{\pi} \sin\frac{x}{2} \, dx$

$= \int_0^{\pi} \sin\frac{1}{2}x \, dx$

$= \left[-2\cos\frac{x}{2}\right]_0^{\pi}$

$= -2\left[\cos\frac{\pi}{2} - \cos 0\right]$

$= -2(0 - 1)$

$= 2$

$\therefore \quad \int_0^{\pi} \sin\frac{x}{2} \, dx = 2.$

(ii) $A=\int_0^{2\pi}\sin\frac{x}{2}\,dx$

$=2\int_0^{\pi}\sin\frac{x}{2}\,dx$

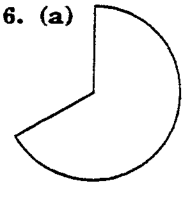
symmetrical about $x=\pi$

$=2(2)$ [from (i)]

$=4$

$\therefore$ area is 4 units^2.

6. (a)



40 minutes

$=\frac{40\text{ min}}{1\text{ h}}\times 2\pi$

$=\frac{40}{60}\times 2\pi$

$=\frac{2}{3}\times 2\pi$

$=\frac{4\pi}{3}$ radians

$\therefore\quad A=\frac{1}{2}r^2\theta$

$=\frac{1}{2}.4^2.\frac{4\pi}{3}$

$=\frac{32\pi}{3}$

$\therefore$ area is $\frac{32\pi}{3}$ cm^2.

(b) (i) $\frac{d}{dx}\left[\sin 2x+\cos x\right]$

$=2\cos 2x-\sin x.$

(ii) $\frac{d}{dx}\left[\frac{1}{\cos x}\right]$

$=\frac{d}{dx}\left[(\cos x)^{-1}\right]$

$=-1(\cos x)^{-2}.-\sin x.$

Chain Rule

$=\frac{\sin x}{\cos^2 x}$

$=\frac{\sin x}{\cos x}.\frac{1}{\cos x}$

$=\tan x.\sec x.$

(c) (i) $\int(\sin x-\cos x)\,dx$

$=-\cos x-\sin x+c$

(ii) $\int\frac{\cos x}{\sin x+1}\,dx$

$=-\int\frac{-\cos x}{\sin x+1}\,dx$

$=-\log_e(\sin x+1)+c.$

(d) (i) For A:

subs. $x=0$ in $y=\cos x$

$\therefore\quad y=\cos 0$

$\therefore\quad y=1$

$\therefore A(0,1).$

For B:

subs. $y=0$ in $y=\cos x$

$0=\cos x$

$\therefore\quad \cos x=0$

$x=\frac{\pi}{2},\frac{3\pi}{2},\ldots$

$\therefore\ B\left(\frac{\pi}{2},0\right)$

$\therefore A(0,1)$ and $B\left(\frac{\pi}{2},0\right)$.

(ii) Eqn. of AB:

$\frac{y-y_1}{x-x_1}=\frac{y_2-y_1}{x_2-x_1}$

$\frac{y-1}{x-0}=\frac{0-1}{\frac{\pi}{2}-0}$

$\therefore\quad \frac{y-1}{x}=\frac{-1}{\frac{\pi}{2}}$

$\therefore\quad \frac{y-1}{x}=\frac{-2}{\pi}$

$\therefore\ \pi y-\pi=-2x$

$\therefore\quad \pi y=\pi-2x$

$\therefore\ y=\frac{\pi-2x}{\pi}.$

(iii) Area $=\int_0^{\frac{\pi}{2}}\cos x\,dx$

$-\int_0^{\frac{\pi}{2}}\frac{\pi-2x}{\pi}\,dx$

$=\int_0^{\frac{\pi}{2}}\cos x\,dx$

$-\frac{1}{\pi}\int_0^{\frac{\pi}{2}}(\pi-2x)\,dx$

$=\left[\sin x\right]_0^{\frac{\pi}{2}}-\frac{1}{\pi}\left[\pi x-x^2\right]_0^{\frac{\pi}{2}}$

$=\left(\sin\frac{\pi}{2}-\sin 0\right)$

$-\frac{1}{\pi}\left[\left(\frac{\pi^2}{2}-\frac{\pi^2}{4}\right)-(0)\right]$

$=(1-0)-\frac{1}{\pi}\left(\frac{\pi^2}{4}\right)$

$=1-\frac{\pi}{4}$

$=\frac{4-\pi}{4}$

$\therefore$ area is $\frac{4-\pi}{4}$ units^2.

7. (a) (i) $68^\circ\to$ radians

$\therefore\ 68\times\frac{\pi}{180}$

$=1\cdot 186\,823\,9$

$=\theta$ (from calc.)

Now, $\ell=r\theta$

$\therefore\ 7=r(1\cdot 186\,823\,9)$

$r=\frac{7}{1\cdot 186\,823\,9}$

$=5\cdot 898\,095$ (from calc.)

$=5\cdot 90$ (2 dec. pl.)

(ii) $A=\frac{1}{2}r^2\theta$

$=\frac{1}{2}\times 5\cdot 9^2\times 1\cdot 186\,823\,9$

$=20\cdot 656\,67$ (from calc.)

$=21$ (to nearest whole)

$\therefore$ area approximately 21 metres^2.

(b) (i) $\frac{d}{dx}\left[\log(\sin x+\cos x)\right]$

$=\frac{\cos x-\sin x}{\sin x+\cos x}$

$\frac{d}{dx}\left[\log f(x)\right]=\frac{f'(x)}{f(x)}$

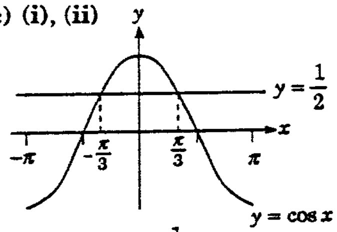
(ii) $\frac{d}{dx}\left[\cos^2(3x-1)\right]$

$=\frac{d}{dx}\left[\left(\cos(3x-1)\right)^2\right]$

$=2\cos(3x-1).-3\sin(3x-1)$

$=-6\sin(3x-1)\cos(3x-1).$

(c) (i), (ii)



(iii) $\cos x > \frac{1}{2}$ means $y=\cos x$ is 'above' $y=\frac{1}{2}$. But $\cos x=\frac{1}{2}$ when $x=\frac{\pi}{3}, -\frac{\pi}{3}$,

$\therefore \cos x > \frac{1}{2}$ when $-\frac{\pi}{3} < x < \frac{\pi}{3}$.

(d) $V=\pi\int_a^b y^2\ dx$

$=\pi\int_0^{\frac{\pi}{3}}(\sec x)^2\ dx$

$=\pi\int_0^{\frac{\pi}{3}}\sec^2 x\ dx$

$=\pi\left[\tan x\right]_0^{\frac{\pi}{3}}$

$=\pi\left[\tan\frac{\pi}{3}-\tan 0\right]$

$=\pi\left[\sqrt{3}-0\right]$

$=\sqrt{3}\pi$

$\therefore$ volume is $\sqrt{3}\pi$ units^3.

8. (a) $y''=-3\cos x-2\sin x$

$\therefore y'=\int(-3\cos x-2\sin x)\,dx$

$\therefore y'=-3\sin x+2\cos x+C$

Subs. in $x=0,\ y'=0$

$\therefore 0=-3\sin 0+2\cos 0+C$

$\therefore 0=2+C$

$\therefore C=-2$

$\therefore y'=-3\sin x+2\cos x-2$.

Now, $y=\int(-3\sin x+2\cos x-2)\,dx$

$\therefore y=3\cos x+2\sin x-2x+k$.

Subs. in $x=0,\ y=5$

$\therefore 5=3\cos 0+2\sin 0-2(0)+k$

$\therefore 5=3+k$

$\therefore k=2$

$\therefore y=3\cos x+2\sin x-2x+2$.

(b) $y=x\cos x$

$\therefore \frac{dy}{dx}=v\frac{du}{dx}+u\frac{dv}{dx}$

$\boxed{u=x,\quad v=\cos x}$

$=\cos x.1+x.-\sin x$

$\therefore \frac{dy}{dx}=\cos x-x\sin x$.

Subs. in $x=\pi$ in $\frac{dy}{dx}$

$\therefore \frac{dy}{dx}=\cos\pi-\pi\sin\pi$

$=-1-\pi(0)$

$=-1$

$\therefore$ grad. of tangent $=-1$.

Subs. in $x=\pi$ in y

$\therefore y=x\cos x$

$=\pi\cos\pi$

$=\pi(-1)$

$=-\pi$

$\therefore$ point $(\pi,-\pi)$, grad. $(m)=-1$

$\therefore y-y_1=m(x-x_1)$

$y+\pi=-1(x-\pi)$

$y+\pi=-x+\pi$

$\therefore x+y=0$

$\therefore$ eqn. of tangent is $x+y=0$.

(c) $\cos\frac{x}{2}=\frac{1}{\sqrt{2}}$

$\therefore \frac{x}{2}=\frac{\pi}{4}, \frac{7\pi}{4}$

$\therefore x=\frac{\pi}{2}, \frac{7\pi}{2}$

$\left(\text{cannot have } \frac{7\pi}{2}\right)$

$\therefore x=\frac{\pi}{2}$.

(d) (i) $f(x)=2\sin 2x+1$

$f'(x)=2.2\cos 2x$

$=4\cos 2x$.

(ii) $\int_0^{\frac{\pi}{6}}\frac{4\cos 2x}{2\sin 2x+1}\,dx$

$=\left[\log_e(2\sin 2x+1)\right]_0^{\frac{\pi}{6}}$

$=\log_e\left(2\sin\frac{\pi}{3}+1\right)$

$-\log_e(2\sin 0+1)$

$=\log_e\left(\sqrt{3}+1\right)-\log_e 1$

$=\log_e\left(\sqrt{3}+1\right)$

$=1{\cdot}005\,052\,5$ (from calculator)

$=1{\cdot}0051$ (4 dec. pl.)

$\therefore \int_0^{\frac{\pi}{6}}\frac{4\cos 2x}{2\sin 2x+1}\,dx$

$=1{\cdot}0051$ (4 dec. pl.).

9. (a) (i) Subs. $\left(\frac{\pi}{3},\frac{\sqrt{3}}{3}\right)$ in $y=\frac{\sin x}{1+\cos x}$

$\therefore$ LHS $=y=\frac{\sqrt{3}}{3}$

RHS $=\frac{\sin\frac{\pi}{3}}{1+\cos\frac{\pi}{3}}$

$=\frac{\sqrt{3}}{2}\div\left(1+\frac{1}{2}\right)$

$=\frac{\sqrt{3}}{2}\div\frac{3}{2}$

$=\frac{\sqrt{3}}{\not{2}}\times\frac{\not{2}}{3}$

$=\frac{\sqrt{3}}{3}$

$\therefore$ LHS = RHS

$\therefore \left(\frac{\pi}{3},\frac{\sqrt{3}}{3}\right)$ lies on $y=\frac{\sin x}{1+\cos x}$.

(ii) $\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$

$\boxed{u=\sin x,\quad v=1+\cos x}$

$=\frac{(1+\cos x).\cos x-\sin x.-\sin x}{(1+\cos x)^2}$

$=\frac{\cos x+\cos^2 x+\sin^2 x}{(1+\cos x)^2}$

$=\frac{1+\cos x}{(1+\cos x)^2}$

$\boxed{\sin^2 x+\cos^2 x=1}$

$\therefore \dfrac{dy}{dx}=\dfrac{1}{1+\cos x}$.

(iii) Subs. $x=\dfrac{\pi}{3}$ in $\dfrac{dy}{dx}$

$\therefore \dfrac{dy}{dx}=\dfrac{1}{1+\cos\frac{\pi}{3}}$

$=\dfrac{1}{1+\frac{1}{2}}$

$=1\div\dfrac{3}{2}$

$=1\times\dfrac{2}{3}$

$=\dfrac{2}{3}$

$\therefore$ point $\left(\dfrac{\pi}{3},\dfrac{\sqrt{3}}{3}\right)$ and

grad. $(m)=\dfrac{2}{3}$

$\therefore \quad y-y_1=m(x-x_1)$

$y-\dfrac{\sqrt{3}}{3}=\dfrac{2}{3}\left(x-\dfrac{\pi}{3}\right)$

$\boxed{\text{Mult. by 9}}$

$9y-3\sqrt{3}=6x-2\pi$

$\therefore\ 6x-9y-2\pi+3\sqrt{3}=0.$

(b) (i) $40° \to$ radians

$\therefore\ \theta=40\times\dfrac{\pi}{180}$

$=\dfrac{2\pi}{9}$ radians

$=0{\cdot}698\ 131\ 7$ (from calc.)

Now, $\ell=r\theta$

$=20\times 0{\cdot}698\ 131\ 7$

$=13{\cdot}962\ 634$ (from calc.)

$=13{\cdot}96$ (two dec. pl.)

$\therefore$ arc length is 13·96 cm.

(ii) Let $r_1=20,\ r_2=12$

$\therefore$ Area $=\dfrac{1}{2}r_2^2\,\theta-\dfrac{1}{2}r_1^2\,\theta$

$\boxed{\text{subtract areas of sectors}}$

$=\dfrac{1}{2}\theta(r_1^2-r_2^2)$

$=\dfrac{1}{2}\times\dfrac{2\pi}{9}(20^2-12^2)$

$=\dfrac{\pi}{9}(256)$

$=89{\cdot}360\ 858$

$=89{\cdot}4.$

Area of shaded region is 89·4 cm^2.

10. (a) $\dfrac{d}{dx}\left[x\sin x+\cos x\right]$

$=\sin x.1+x.\cos x-\sin x$

$=\sin x+x\cos x-\sin x$

$=x\cos x$

$\therefore \dfrac{d}{dx}\left[x\sin x+\cos x\right]=x\cos x.$

$\therefore \displaystyle\int_0^{\frac{\pi}{2}} x\cos x\,dx$

$=\left[x\sin x+\cos x\right]_0^{\frac{\pi}{2}}$

(from above)

$=\left(\dfrac{\pi}{2}\sin\dfrac{\pi}{2}+\cos\dfrac{\pi}{2}\right)$

$-\left(0\sin 0+\cos 0\right)$

$=\left(\dfrac{\pi}{2}+0\right)-(0+1)$

$=\dfrac{\pi}{2}-1$

$=\dfrac{\pi-2}{2}$

$\therefore \displaystyle\int x\cos x\,dx=\dfrac{\pi-2}{2}.$

(b) (i) $y=\sin x,\ y=\cos x$

$\therefore \quad \sin x=\cos x$

Divide by $\cos x$:

$\tan x=1$

$\therefore\ x=\dfrac{\pi}{4},\dfrac{5\pi}{4},\ldots$

Subs. $x=\dfrac{\pi}{4}$ in $y=\sin x$

$\therefore \quad y=\sin\dfrac{\pi}{4}$

$=\dfrac{1}{\sqrt{2}}\quad\therefore\ \left(\dfrac{\pi}{4},\dfrac{1}{\sqrt{2}}\right).$

Subs. $x=\dfrac{5\pi}{4}$ in $y=\sin x$

$\therefore \quad y=\sin\dfrac{5\pi}{4}$

$=-\dfrac{1}{\sqrt{2}}$

$\therefore\ \left(\dfrac{5\pi}{4},-\dfrac{1}{\sqrt{2}}\right)$

$\therefore$ points of int. are $\left(\dfrac{\pi}{4},\dfrac{1}{\sqrt{2}}\right)$ and $\left(\dfrac{5\pi}{4},-\dfrac{1}{\sqrt{2}}\right)$.

(ii) Cuts x axis, $\therefore y=0$

Subs. $y=0$ in $y=\sin x$

$\therefore \quad \sin x=0$

$\therefore \quad x=0,\pi,2\pi,\ldots$

$\therefore$ cuts x axis at $0,\pi,2\pi$.

Now, subs. $y=0$ in $y=\cos x$

$\therefore \quad \cos x=0$

$\therefore \quad x=\dfrac{\pi}{2},\dfrac{3\pi}{2},\ldots$

$\therefore$ cuts x axis at $\dfrac{\pi}{2},\dfrac{3\pi}{2}$.

(iii) For $y=\sin x$

$\dfrac{dy}{dx}=\cos x$

Subs. $x=\pi$ in $\dfrac{dy}{dx}$

$\therefore \quad \dfrac{dy}{dx}=\cos\pi$

$\therefore \quad =-1$

$\therefore$ grad. of tangent to $y=\sin x$ at $x=\pi$ is -1.

For $y=\cos x$

$\dfrac{dy}{dx}=-\sin x$

Subs. $x=\dfrac{\pi}{2}$ in $\dfrac{dy}{dx}$

$\therefore \quad \dfrac{dy}{dx}=-\sin\dfrac{\pi}{2}$

$\therefore \quad =-1$

$\therefore$ grad. of tangent to $y=\cos x$ at $x=\dfrac{\pi}{2}$ is -1.

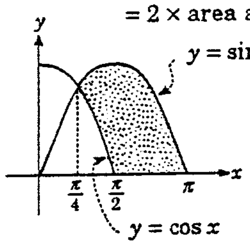
$\therefore$ tangents have same gradient

$\therefore$ tangents are parallel.

(iv) Shaded region above x axis = shaded region below x axis

$\therefore$ Area of shaded region

$= 2 \times$ area above x axis.



$$= 2 \times \left[\int_{\frac{\pi}{4}}^{\pi} \sin x \, dx - \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos x \, dx \right]$$

$$= 2\left[\left[-\cos x \right]_{\frac{\pi}{4}}^{\pi} - \left[\sin x \right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} \right]$$

$$= 2\left[\left(-\cos \pi + \cos \frac{\pi}{4} \right) - \left(\sin \frac{\pi}{2} - \sin \frac{\pi}{4} \right) \right]$$

$$= 2\left[-(-1) + \frac{1}{\sqrt{2}} - \left(1 - \frac{1}{\sqrt{2}} \right) \right]$$

$$= 2\left[1 + \frac{1}{\sqrt{2}} - 1 + \frac{1}{\sqrt{2}} \right]$$

$$= 2\left[\frac{2}{\sqrt{2}} \right]$$

$$= \frac{4}{\sqrt{2}}$$

$$= 2\sqrt{2}$$

$$\boxed{\begin{aligned} &\frac{4}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} \\ &= \frac{4\sqrt{2}}{2} \\ &= 2\sqrt{2} \end{aligned}}$$

$\therefore$ area is $2\sqrt{2}$ units^2.

(c) $V = \pi \int_a^b y^2 \, dx$

$$= \pi \int_0^{\frac{\pi}{4}} (\tan x)^2 \, dx$$

$$= \pi \int_0^{\frac{\pi}{4}} \tan^2 x \, dx$$

$$= \pi \int_0^{\frac{\pi}{4}} (\sec^2 x - 1) \, dx$$

$$\boxed{\tan^2 x = \sec^2 x - 1}$$

$$= \pi \left[\tan x - x \right]_0^{\frac{\pi}{4}}$$

$$= \pi \left[\left(\tan \frac{\pi}{4} - \frac{\pi}{4} \right) - 0 \right]$$

$$= \pi \left[1 - \frac{\pi}{4} \right]$$

$\therefore$ volume is $\pi \left(1 - \frac{\pi}{4} \right)$ units^3.