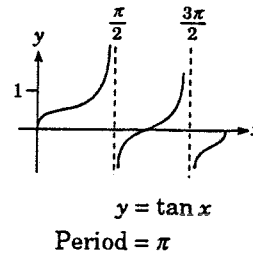
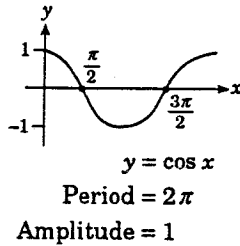
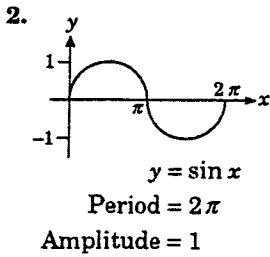


1. Arc length: $l = r\theta$
 Area of sector = $\frac{1}{2}r^2\theta$ } θ in radians



3. When x is small, $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$

4. Derivatives:

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}[\sin f(x)] = f'(x) \cos f(x)$$

$$\frac{d}{dx}[\cos f(x)] = -f'(x) \sin f(x)$$

$$\frac{d}{dx}[\tan f(x)] = f'(x) \sec^2 f(x)$$

5. Integrals:

$$\int \sin x \, dx = -\cos x + C$$

$$\int \cos x \, dx = \sin x + C$$

$$\int \sec^2 x \, dx = \tan x + C$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax + C$$

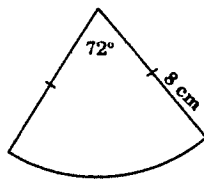
$$\int \cos ax \, dx = \frac{1}{a} \sin ax + C$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax + C$$

a and C constants

EXERCISE 9: EXAMINATION-TYPE QUESTIONS

1. (a)



The figure shows a sector of a circle with radius 8 cm. Find the area of this sector correct to 2 decimal places.

- (b) Differentiate: (i) $\log_e(\sin x)$ (ii) $\cos\left(x + \frac{\pi}{2}\right)$

- (c) Evaluate: (i) $\int_0^{\frac{\pi}{4}} \cos 2x \, dx$ (ii) $\int_0^1 \sin \frac{\pi}{3} x \, dx$

- (d) Find the equation of the tangent to $y = 2 \sin 2x$ at the point $x = \frac{\pi}{12}$.

2. (a) Solve, where $0 \leq x \leq 2\pi$, $2 \sin x = \sqrt{3}$.

- (b) Evaluate $\lim_{\theta \rightarrow 0} \frac{\sin 3\theta}{\theta}$.

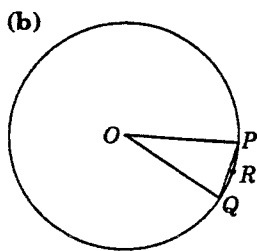
- (c) (i) Sketch the graph of $y = \cos x$, for $0 \leq x \leq \pi$.

- (ii) Where does the curve cross the x axis?

- (iii) Show that the area between the curve and the x axis between $x = 0$ and $x = \pi$ is 2 units².

- (d) Find $\frac{dy}{dx}$ if: (i) $y = \tan \frac{x}{3}$ (ii) $y = \sin^2 x$

3. (a) Find the gradient of the normal to $y = \cos 3x$ at the point $x = \frac{\pi}{6}$.



The figure shows a circle centre O with triangle OPQ and R is a point on arc PQ . $OP = 10$ cm and $\angle POQ = 30^\circ$.

- (i) Find the area of $\triangle OPQ$.
 (ii) Express 30° in radians and show that the area of sector OPQ is $\frac{25\pi}{3}$ cm².

(iii) Show that the area of the segment PRQ is $25\left(\frac{\pi-3}{3}\right)$ cm².

(iv) Use the Cosine Rule to find the length of PQ .

(v) Show that the length PQ is shorter than the arc PRQ by 0.06 cm, correct to 2 dec. places.

(c) Find the nature of the stationary points on $y = \sin 2x$ if $0 < x < \pi$ (using calculus).

(d) Find c , if $\int_0^c \cos x \, dx = \frac{\sqrt{3}}{2}$, where $0 < c < \frac{\pi}{2}$.

4. (a) Differentiate: (i) $\frac{\sin x}{x}$ (ii) $e^{2\cos x}$

(b) Solve, for $0^\circ \leq x \leq 360^\circ$, $2\sin x - 1 = 0$

(c) Evaluate: (i) $\int_0^{\frac{\pi}{2}} \sec^2 2x \, dx$ (ii) $\int_0^{\frac{\pi}{2}} \sin x + \cos x \, dx$

(d) Prove that $\frac{d}{dx} \left[\frac{1 + \sin x}{\cos x} \right] = \frac{1}{1 - \sin x}$.

5. (a) Evaluate $\lim_{x \rightarrow 0} \frac{\sin 2x}{4x}$.

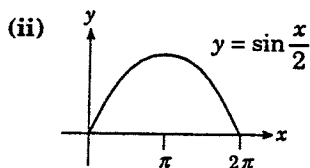
(b) (i) Sketch $y = \sin \pi x$, on a number plane, where $-2 \leq x \leq 2$.

(ii) State the period and amplitude of $y = \sin \pi x$.

(iii) Use the sketch to solve the equation $\sin \pi x = 0$.

(c) If $y = 4 \sin 3x$, prove that $y'' + 9y = 0$.

(d) (i) Show that $\int_0^\pi \sin \frac{x}{2} \, dx = 2$.



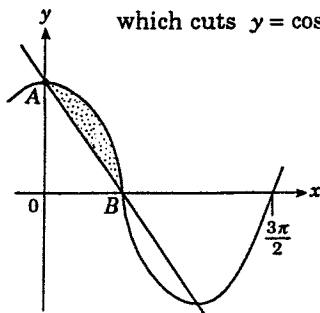
Find the area between the curve $y = \sin \frac{x}{2}$ and the x axis when $x = 0$ and $x = 2\pi$.

6. (a) The minute hand of a clock is 4 cm in length. What area is swept by the hand in an interval of 40 minutes? Answer in terms of π .

(b) Find the derivative of: (i) $\sin 2x + \cos x$ (ii) $\frac{1}{\cos x}$

(c) Find: (i) $\int (\sin x - \cos x) dx$ (ii) $\int \frac{\cos x}{\sin x + 1} dx$

(d) The diagram shows the graph of $y = \cos x$ and a straight line which cuts $y = \cos x$ at the points A and B.

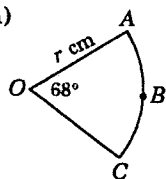


(i) Find the coordinates of the points A and B.

(ii) Show that the equation of the line passing through A and B is $y = \frac{\pi - 2x}{\pi}$.

(iii) Find the shaded area between $y = \cos x$ and $y = \frac{\pi - 2x}{\pi}$ (marked on the diagram).

7. (a)



The figure shows a sector of a circle with radius r cm. The length of the arc ABC is 7 cm.

(i) Find the value of r , to one decimal place.

(ii) Show that the area of the sector OAC is approximately 21 cm^2 .

(b) Differentiate: (i) $\log(\sin x + \cos x)$ (ii) $\cos^2(3x - 1)$

(c) (i) Sketch the graph of $y = \cos x$, where $-\pi \leq x \leq \pi$.

(ii) On the same number plane, graph $y = \frac{1}{2}$.

(iii) Using (i) and (ii), solve $\cos x > \frac{1}{2}$ for $-\pi \leq x \leq \pi$.

(d) The area bounded by the curve $y = \sec x$, the x axis and the lines $x = 0$ and $x = \frac{\pi}{3}$ is rotated about the x axis. Find the volume of the solid formed.

8. (a) If $y'' = -3 \cos x - 2 \sin x$ and when $x = 0$, $y' = 0$, $y = 5$, find y in terms of x .

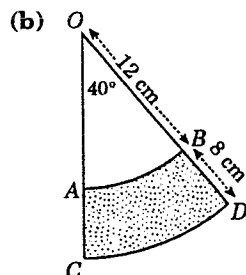
(b) Find the equation of the tangent to the curve $y = x \cos x$ at the point $x = \pi$.

(c) Solve the equation $\cos \frac{x}{2} = \frac{1}{\sqrt{2}}$ where $0 \leq x \leq 2\pi$.

(d) (i) Show that if $f(x) = 2 \sin 2x + 1$, then $f'(x) = 4 \cos 2x$.

(ii) Hence, show that $\int_0^{\frac{\pi}{6}} \frac{4 \cos 2x}{2 \sin 2x + 1} dx = 1.0051$, rounded off correct to four decimal places.

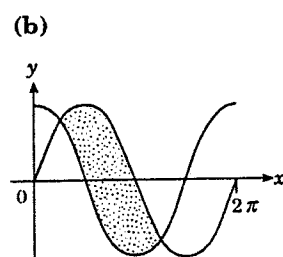
9. (a) (i) Show that the point $\left(\frac{\pi}{3}, \frac{\sqrt{3}}{3}\right)$ lies on the curve $y = \frac{\sin x}{1 + \cos x}$.
- (ii) Show that if $y = \frac{\sin x}{1 + \cos x}$, then $\frac{dy}{dx} = \frac{1}{1 + \cos x}$.
- (iii) Find the equation of the tangent to the curve $y = \frac{\sin x}{1 + \cos x}$ at the point $x = \frac{\pi}{3}$.



The diagram shows AB and CD as arcs of concentric circles, with centre O . It is known that $OB = 12$ cm and $BD = 8$ cm.

- (i) Find the arc length CD , correct to 2 decimal places.
- (ii) Show that the area of the shaded region is $17\,868.8$ cm².
10. (a) Show that $\frac{d}{dx}[x \sin x + \cos x] = x \cos x$, and hence evaluate

$$\int_0^{\frac{\pi}{2}} x \cos x \, dx.$$



The diagram shows the graphs of $y = \sin x$ and $y = \cos x$ for the domain $0 \leq x \leq 2\pi$.

- (i) Show that the points of intersection of $y = \sin x$ and $y = \cos x$ are $\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$ and $\left(\frac{5\pi}{4}, -\frac{1}{\sqrt{2}}\right)$.
- (ii) Show that $y = \sin x$ cuts the x axis at $x = 0, \pi$ and 2π , while $y = \cos x$ cuts the x axis at $x = \frac{\pi}{2}$ and $\frac{3\pi}{2}$.
- (iii) Show that the tangent to $y = \sin x$ at $x = \pi$ is parallel to the tangent to $y = \cos x$ at $x = \frac{\pi}{2}$.
- (iv) Find the exact area of the shaded region.
- (c) Find the volume of the solid formed when the curve $y = \tan x$ is rotated about the x axis between $x = 0$ and $x = \frac{\pi}{4}$. Leave your answer in terms of π .

EXERCISE 9: WORKED SOLUTIONS

1. (a) $A = \frac{1}{2}r^2\theta$

Note θ must be in radians.

$$72 \text{ degrees} = 72 \times \frac{\pi}{180}$$

$$= 1.2566371$$

(from calc.)

$$\therefore A = \frac{1}{2}r^2\theta$$

$$= \frac{1}{2} \times 8^2 \times 1.2566371$$

$$= 40.212386$$

$$= 40.21 \text{ (2 dec. places)}$$

\therefore area is 40.21 cm^2 .

(b) (i) $\frac{d}{dx}[\log_e(\sin x)] = \frac{\cos x}{\sin x}$

$$\frac{d}{dx}[\log f(x)] = \frac{f'(x)}{f(x)}$$

$= \cot x$.

Note $\frac{\cos x}{\sin x} = \cot x$

(ii) $\frac{d}{dx}\left[\cos\left(x + \frac{\pi}{2}\right)\right]$

$$= -\sin\left(x + \frac{\pi}{2}\right)$$

(c) (i) $\int_0^{\frac{\pi}{4}} \cos 2x \, dx$

$$= \left[\frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{2}\left[\sin 2\left(\frac{\pi}{4}\right) - \sin 2(0)\right]$$

$$= \frac{1}{2}\left[\sin \frac{\pi}{2} - \sin 0\right]$$

$$= \frac{1}{2}[1 - 0]$$

$$= \frac{1}{2}$$

(ii) $\int_0^1 \sin \frac{\pi}{3}x \, dx = \left[-\frac{3}{\pi}\cos \frac{\pi}{3}x\right]_0^1$

$$= -\frac{3}{\pi}\left[\cos \frac{\pi}{3}(1) - \cos \frac{\pi}{3}(0)\right]$$

$$= -\frac{3}{\pi}\left[\cos \frac{\pi}{3} - \cos 0\right]$$

$$= -\frac{3}{\pi}\left[\frac{1}{2} - 1\right]$$

$$= -\frac{3}{\pi}\left[-\frac{1}{2}\right]$$

$$= \frac{3}{2\pi}$$

(d) $y = 2\sin 2x$

Subs. $x = \frac{\pi}{12}$ in $y = 2\sin 2x$

$$= 2\sin 2\left(\frac{\pi}{12}\right)$$

$$= 2\sin \frac{\pi}{6}$$

$$= 2 \times \frac{1}{2}$$

$$= 1$$

\therefore the point $\left(\frac{\pi}{12}, 1\right)$.

For gradient, $\frac{dy}{dx} = 4\cos 2x$

Subs. in $x = \frac{\pi}{12}$

$$\therefore \frac{dy}{dx} = 4\cos 2\left(\frac{\pi}{12}\right)$$

$$= 4\cos \frac{\pi}{6}$$

$$= 4\left(\frac{\sqrt{3}}{2}\right)$$

$$= 2\sqrt{3}$$

\therefore Equation of tangent:

$$y - y_1 = m(x - x_1)$$

$$y - 1 = 2\sqrt{3}\left(x - \frac{\pi}{12}\right)$$

$$y - 1 = 2\sqrt{3}x - \frac{\sqrt{3}\pi}{6}$$

2. (a) $2\sin x = \sqrt{3}$

$$\therefore \sin x = \frac{\sqrt{3}}{2}$$

$$x = \frac{\pi}{3}, \frac{2\pi}{3}$$

Pos. in 1st and in 2nd quad.

(b) $\lim_{\theta \rightarrow 0} \frac{\sin 3\theta}{\theta}$

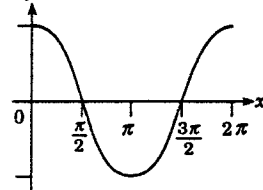
$$= \lim_{\theta \rightarrow 0} \frac{\sin 3\theta}{3\theta} \cdot 3$$

$$= 3 \lim_{\theta \rightarrow 0} \frac{\sin 3\theta}{3\theta}$$

$$= 3.1 \quad \boxed{\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1}$$

$$= 3$$

(c) (i) $y = \cos x$



(ii) Cut x axis when

$$x = \frac{\pi}{2}, \frac{3\pi}{2}$$

(as seen from the diagram).

(iii) Area = $\int_0^{\frac{\pi}{2}} \cos x \, dx$

$$+ \left| \int_{\frac{\pi}{2}}^{\pi} \cos x \, dx \right|$$

$$= \left[\sin x\right]_0^{\frac{\pi}{2}} + \left| \left[\sin x\right]_{\frac{\pi}{2}}^{\pi} \right|$$

$$= \left(\sin \frac{\pi}{2} - \sin 0\right)$$

$$+ \left| \sin \pi - \sin \frac{\pi}{2} \right|$$

$$= (1 - 0) + \left| -1 - 0 \right|$$

$$= 1 + \left| -1 \right|$$

$$= 1 + 1$$

$$= 2$$

\therefore area is 2 units².

(d) (i) $y = \tan \frac{x}{3}$ $\boxed{\frac{x}{3} = \frac{1}{3}x}$

$$\therefore y = \tan \frac{1}{3}x$$

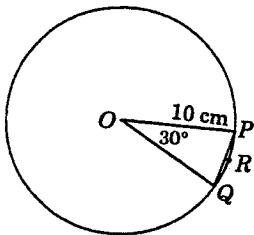
$$\therefore \frac{dy}{dx} = \frac{1}{3}\sec^2 \frac{1}{3}x$$

$$= \frac{1}{3}\sec^2 \frac{x}{3}$$

(ii) $y = \sin^2 x$
 $\therefore y = (\sin x)^2$
 $\therefore \frac{dy}{dx} = 2(\sin x)' \cdot \cos x$
 $= 2 \sin x \cos x.$

3. (a) $y = \cos 3x$
 $\frac{dy}{dx} = -3 \sin 3x$
 Subs. $x = \frac{\pi}{6}$ in $\frac{dy}{dx}$
 $\therefore \frac{dy}{dx} = -3 \sin 3\left(\frac{\pi}{6}\right)$
 $= -3 \sin \frac{\pi}{2}$
 $= -3 \cdot 1$
 $= -3$
 \therefore grad. of tangent $= -3,$
 \therefore grad. of normal $= \frac{1}{3}$
 \therefore gradient of normal to
 $y = \cos 3x$ at $x = \frac{\pi}{6}$ is $\frac{1}{3}.$

(b)



(i) $A = \frac{1}{2} ab \sin C$
 $= \frac{1}{2} \times 10 \times 10 \times \sin 30$
 $= \frac{1}{2} \times 10 \times 10 \times \frac{1}{2}$
 $= 25$
 \therefore area of ΔOPQ is
 $25 \text{ cm}^2.$

(ii) $30^\circ = 30 \times \frac{\pi}{180} = \frac{\pi}{6}$

Deg. \rightarrow Rad: $\times \frac{\pi}{180}$

$A = \frac{1}{2} r^2 \theta$
 $= \frac{1}{2} \cdot 10^2 \cdot \frac{\pi}{6}$

$= \frac{25\pi}{3}$
 \therefore area of sector OPQ
 is $\frac{25\pi}{3} \text{ cm}^2.$

(iii) Area of segment
 $=$ area of sector
 $-$ area of triangle
 $= \frac{25\pi}{3} - 25$
 $= 25\left(\frac{\pi}{3} - 1\right)$
 $= 25\left(\frac{\pi - 3}{3}\right)$
 \therefore area of segment is
 $25\left(\frac{\pi - 3}{3}\right) \text{ cm}^2.$

(iv)
 $PQ^2 = 10^2 + 10^2 - 2(10)(10) \cos 30^\circ$
 $a^2 = b^2 + c^2 - 2bc \cos A$
 $= 100 + 100 - 200 \cdot \frac{\sqrt{3}}{2}$
 $= 200 - 100\sqrt{3}$
 $= 26.794919$ (from calc.)

$\therefore PQ = \sqrt{26.794919}$
 $= 5.1763809$ (from calc.)

(v)
 Now arc length $PRQ = r\theta$ $l = r\theta$
 $= 10 \cdot \frac{\pi}{6}$
 $= \frac{5\pi}{3}$
 $= 5.2359878$ (from calc.)

\therefore Difference
 $= 5.2359878 - 5.1763809$
 $= 0.0596068$ (from calc.)
 $= 0.06$ (to 2 dec. pl.)

$\therefore PQ$ is shorter by 0.06 cm.

(c) $y = \sin 2x$
 $\frac{dy}{dx} = 2 \cos 2x = 0$
 Stat. pts. when $\frac{dy}{dx} = 0$

$\therefore \cos 2x = 0$
 $\therefore 2x = \frac{\pi}{2}, \frac{3\pi}{2}$
 $\therefore x = \frac{\pi}{4}, \frac{3\pi}{4}$
 Subs. in $y = \sin 2x$
 $\therefore x = \frac{\pi}{4} \therefore y = \sin 2\left(\frac{\pi}{4}\right)$
 $= \sin \frac{\pi}{2}$
 $= 1$
 $\therefore x = \frac{3\pi}{4} \therefore y = \sin 2\left(\frac{3\pi}{4}\right)$
 $= \sin \frac{3\pi}{2}$
 $= -1$
 \therefore stat. points at
 $\left(\frac{\pi}{4}, 1\right), \left(\frac{3\pi}{4}, -1\right).$

$\frac{d^2y}{dx^2} = -4 \sin 2x$
 Now subs. $x = \frac{\pi}{4}$ in $\frac{d^2y}{dx^2}$
 $\therefore \frac{d^2y}{dx^2} = -4 \sin 2\left(\frac{\pi}{4}\right)$
 $= -4 \sin \frac{\pi}{2}$
 $= -4 \cdot 1$
 $= -4 < 0$

\therefore max.

Also, subs. $x = \frac{3\pi}{4}$ in $\frac{d^2y}{dx^2}$
 $\therefore \frac{d^2y}{dx^2} = -4 \sin 2\left(\frac{3\pi}{4}\right)$
 $= -4 \sin \frac{3\pi}{2}$
 $= -4(-1)$
 $= 4 > 0$

\therefore min.

\therefore maximum at $\left(\frac{\pi}{4}, 1\right)$ and
 minimum at $\left(\frac{3\pi}{4}, -1\right).$

(d) $\int_0^c \cos x \, dx = \frac{\sqrt{3}}{2}$
 $\therefore \int_0^c \cos x \, dx = [\sin x]_0^c$

$$\begin{aligned}
 &= \sin c - \sin 0 \\
 &= \sin c \\
 \therefore \sin c &= \frac{\sqrt{3}}{2} \\
 \therefore c &= \frac{\pi}{6}
 \end{aligned}$$

4. (a) (i) $\frac{d}{dx} \left[\frac{\sin x}{x} \right]$

$$\begin{array}{l}
 u = \sin x \quad v = x \\
 \frac{du}{dx} = \cos x \quad \frac{dv}{dx} = 1
 \end{array}$$

$$\begin{aligned}
 &= \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2} \\
 &= \frac{x \cos x - \sin x \cdot 1}{x^2} \\
 &= \frac{x \cos x - \sin x}{x^2}
 \end{aligned}$$

(ii) $\frac{d}{dx} [e^{2\cos x}]$

$$\begin{array}{l}
 \frac{d}{dx} e^{f(x)} = f'(x)e^{f(x)} \\
 = -2\sin x e^{2\cos x}
 \end{array}$$

(b) $2\sin x - 1 = 0$
 $2\sin x = 1$
 $\sin x = \frac{1}{2}$
 $\therefore x = \frac{\pi}{6}, \frac{5\pi}{6}$

(c) (i) $\int_0^{\frac{\pi}{8}} \sec^2 2x \, dx$
 $= \left[\frac{1}{2} \tan 2x \right]_0^{\frac{\pi}{8}}$
 $= \frac{1}{2} \left[\tan 2 \left(\frac{\pi}{8} \right) - \tan 2(0) \right]$
 $= \frac{1}{2} \left[\tan \frac{\pi}{4} - \tan 0 \right]$
 $= \frac{1}{2} [1 - 0]$
 $= \frac{1}{2}$

(ii) $\int_0^{\frac{\pi}{2}} \sin x + \cos x \, dx$
 $= [-\cos x + \sin x]_0^{\frac{\pi}{2}}$
 $= \left(-\cos \frac{\pi}{2} + \sin \frac{\pi}{2} \right) - (-\cos 0 + \sin 0)$
 $= (0 + 1) - (-1 + 0)$
 $= 1 + 1$
 $= 2$

(d) $\frac{d}{dx} \left[\frac{1 + \sin x}{\cos x} \right]$

$$\begin{array}{l}
 u = 1 + \sin x \quad v = \cos x \\
 \frac{du}{dx} = \cos x \quad \frac{dv}{dx} = -\sin x
 \end{array}$$

$$\begin{aligned}
 &= \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2} \\
 &= \frac{\cos x \cdot \cos x - (1 + \sin x) \cdot -\sin x}{(\cos x)^2} \\
 &= \frac{\cos^2 x + \sin x + \sin^2 x}{\cos^2 x}
 \end{aligned}$$

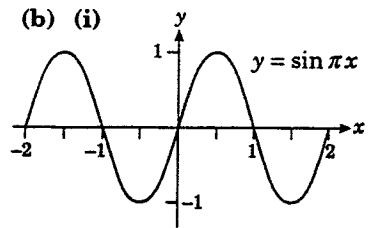
$$\boxed{(\cos x)^2 = \cos^2 x}$$

$$\begin{aligned}
 &= \frac{1 + \sin x}{1 - \sin^2 x} \quad \boxed{\cos^2 x = 1 - \sin^2 x} \\
 &= \frac{1 + \sin x}{(1 - \sin x)(1 + \sin x)}
 \end{aligned}$$

$$\boxed{a^2 - b^2 = (a - b)(a + b)}$$

$$\begin{aligned}
 &= \frac{1}{1 - \sin x} \\
 \therefore \frac{d}{dx} \left[\frac{1 + \sin x}{\cos x} \right] &= \frac{1}{1 - \sin x}
 \end{aligned}$$

5. (a) $\lim_{x \rightarrow 0} \frac{\sin 2x}{4x}$
 $= \frac{1}{2} \lim_{x \rightarrow 0} \frac{\sin 2x}{2x}$
 $= \frac{1}{2} \cdot 1 \quad \boxed{\lim_{X \rightarrow 0} \frac{\sin X}{X} = 1}$
 $= \frac{1}{2}$



(ii) $y = \sin \pi x$
 \therefore period: 2

Note
 Period = $\frac{2\pi}{n} = \frac{2\pi}{\pi} = 2$

amplitude: 1

(iii) For $\sin \pi x = 0$,
 \therefore intersection of $y = \sin \pi x$ and $y = 0$ [i.e. x axis],
 $\therefore x = -2, -1, 0, 1, 2$.

(c) $y = 4\sin 3x$
 $y' = 12\cos 3x$
 $y'' = -36\sin 3x$
 For $y'' + 9y = 0$
 \therefore LHS = $y'' + 9y$
 $= -36\sin 3x + 9(4\sin 3x)$
 $= -36\sin 3x + 36\sin 3x$
 $= 0 =$ RHS
 $\therefore y'' + 9y = 0$

(d) (i) $\int_0^{\pi} \sin \frac{x}{2} \, dx$
 $= \int_0^{\pi} \sin \frac{1}{2}x \, dx$
 $= \left[-2\cos \frac{x}{2} \right]_0^{\pi}$
 $= -2 \left[\cos \frac{\pi}{2} - \cos 0 \right]$
 $= -2(0 - 1)$
 $= 2$
 $\therefore \int_0^{\pi} \sin \frac{x}{2} \, dx = 2$

$$(ii) A = \int_0^{2\pi} \sin \frac{x}{2} dx$$

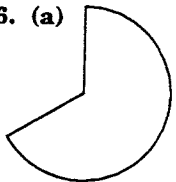
$$= 2 \int_0^{\pi} \sin \frac{x}{2} dx$$

symmetrical
about $x = \pi$

$$= 2(2) \text{ [from (i)]}$$

$$= 4$$

\therefore area is 4 units².

6. (a)  40 minutes

$$= \frac{40 \text{ min}}{1 \text{ h}} \times 2\pi$$

$$= \frac{40}{60} \times 2\pi$$

$$= \frac{2}{3} \times 2\pi$$

$$= \frac{4\pi}{3} \text{ radians}$$

$$\therefore A = \frac{1}{2} r^2 \theta$$

$$= \frac{1}{2} \cdot 4^2 \cdot \frac{4\pi}{3}$$

$$= \frac{32\pi}{3}$$

\therefore area is $\frac{32\pi}{3} \text{ cm}^2$.

(b) (i) $\frac{d}{dx} [\sin 2x + \cos x]$

$$= 2 \cos 2x - \sin x.$$

(ii) $\frac{d}{dx} \left[\frac{1}{\cos x} \right]$

$$= \frac{d}{dx} [(\cos x)^{-1}]$$

$$= -1(\cos x)^{-2} \cdot -\sin x.$$

Chain Rule

$$= \frac{\sin x}{\cos^2 x}$$

$$= \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x}$$

$$= \tan x \cdot \sec x.$$

(c) (i) $\int (\sin x - \cos x) dx$

$$= -\cos x - \sin x + c$$

(ii) $\int \frac{\cos x}{\sin x + 1} dx$

$$= - \int \frac{-\cos x}{\sin x + 1} dx$$

$$= -\log_e (\sin x + 1) + c.$$

(d) (i) For A:

subs. $x = 0$ in $y = \cos x$

$$\therefore y = \cos 0$$

$$\therefore y = 1$$

$\therefore A(0, 1)$.

For B:

subs. $y = 0$ in $y = \cos x$

$$0 = \cos x$$

$$\therefore \cos x = 0$$

$$x = \frac{\pi}{2}, \frac{3\pi}{2}, \dots$$

$\therefore B\left(\frac{\pi}{2}, 0\right)$

$\therefore A(0, 1)$ and $B\left(\frac{\pi}{2}, 0\right)$.

(ii) Eqn. of AB:

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\frac{y - 1}{x - 0} = \frac{0 - 1}{\frac{\pi}{2} - 0}$$

$$\therefore \frac{y - 1}{x} = \frac{-1}{\frac{\pi}{2}}$$

$$\therefore \frac{y - 1}{x} = \frac{-2}{\pi}$$

$$\therefore \pi y - \pi = -2x$$

$$\therefore \pi y = \pi - 2x$$

$$\therefore y = \frac{\pi - 2x}{\pi}.$$

(iii) Area = $\int_0^{\frac{\pi}{2}} \cos x dx$

$$- \int_0^{\frac{\pi}{2}} \frac{\pi - 2x}{\pi} dx$$

$$= \int_0^{\frac{\pi}{2}} \cos x dx$$

$$- \frac{1}{\pi} \int_0^{\frac{\pi}{2}} (\pi - 2x) dx$$

$$= [\sin x]_0^{\frac{\pi}{2}} - \frac{1}{\pi} [\pi x - x^2]_0^{\frac{\pi}{2}}$$

$$= \left(\sin \frac{\pi}{2} - \sin 0 \right)$$

$$- \frac{1}{\pi} \left[\left(\frac{\pi^2}{2} - \frac{\pi^2}{4} \right) - (0) \right]$$

$$= (1 - 0) - \frac{1}{\pi} \left(\frac{\pi^2}{4} \right)$$

$$= 1 - \frac{\pi}{4}$$

$$= \frac{4 - \pi}{4}$$

\therefore area is $\frac{4 - \pi}{4}$ units².

7. (a) (i) $68^\circ \rightarrow$ radians

$$\therefore 68 \times \frac{\pi}{180}$$

$$= 1.1868239$$

$$= \theta \text{ (from calc.)}$$

Now, $\ell = r\theta$

$$\therefore 7 = r(1.1868239)$$

$$r = \frac{7}{1.1868239}$$

$$= 5.898095$$

(from calc.)

$$= 5.90 \text{ (2 dec. pl.)}$$

(ii) $A = \frac{1}{2} r^2 \theta$

$$= \frac{1}{2} \times 5.9^2 \times 1.1868239$$

$$= 20.65667 \text{ (from calc.)}$$

$$= 21 \text{ (to nearest whole)}$$

\therefore area approximately 21 metres².

(b) (i) $\frac{d}{dx} [\log(\sin x + \cos x)]$

$$= \frac{\cos x - \sin x}{\sin x + \cos x}$$

$$\frac{d}{dx} [\log f(x)] = \frac{f'(x)}{f(x)}$$

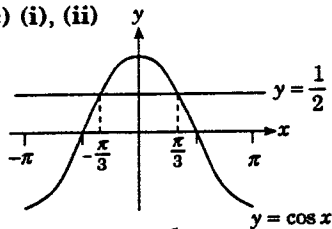
(ii) $\frac{d}{dx} [\cos^2(3x - 1)]$

$$= \frac{d}{dx} [(\cos(3x - 1))^2]$$

$$= 2 \cos(3x - 1) \cdot -3 \sin(3x - 1)$$

$$= -6 \sin(3x - 1) \cos(3x - 1).$$

(c) (i), (ii)



(iii) $\cos x > \frac{1}{2}$ means
 $y = \cos x$ is 'above'
 $y = \frac{1}{2}$. But $\cos x = \frac{1}{2}$
 when $x = \frac{\pi}{3}, -\frac{\pi}{3}$,
 $\therefore \cos x > \frac{1}{2}$ when
 $-\frac{\pi}{3} < x < \frac{\pi}{3}$.

(d) $V = \pi \int_a^b y^2 dx$
 $= \pi \int_0^{\frac{\pi}{3}} (\sec x)^2 dx$
 $= \pi \int_0^{\frac{\pi}{3}} \sec^2 x dx$
 $= \pi [\tan x]_0^{\frac{\pi}{3}}$
 $= \pi \left[\tan \frac{\pi}{3} - \tan 0 \right]$
 $= \pi [\sqrt{3} - 0]$
 $= \sqrt{3}\pi$

\therefore volume is $\sqrt{3}\pi$ units³.

8. (a) $y'' = -3 \cos x - 2 \sin x$

$\therefore y' = \int (-3 \cos x - 2 \sin x) dx$
 $\therefore y' = -3 \sin x + 2 \cos x + C$

Subs. in $x = 0, y' = 0$

$\therefore 0 = -3 \sin 0 + 2 \cos 0 + C$

$\therefore 0 = 2 + C$

$\therefore C = -2$

$\therefore y' = -3 \sin x + 2 \cos x - 2$.

Now, $y = \int (-3 \sin x + 2 \cos x - 2) dx$

$\therefore y = 3 \cos x + 2 \sin x - 2x + k$.

Subs. in $x = 0, y = 5$

$\therefore 5 = 3 \cos 0 + 2 \sin 0 - 2(0) + k$

$\therefore 5 = 3 + k$

$\therefore k = 2$

$\therefore y = 3 \cos x + 2 \sin x - 2x + 2$.

(b) $y = x \cos x$

$\therefore \frac{dy}{dx} = v \frac{du}{dx} + u \frac{dv}{dx}$

$u = x, v = \cos x$

$= \cos x \cdot 1 + x \cdot -\sin x$

$\therefore \frac{dy}{dx} = \cos x - x \sin x$.

Subs. in $x = \pi$ in $\frac{dy}{dx}$

$\therefore \frac{dy}{dx} = \cos \pi - \pi \sin \pi$

$= -1 - \pi(0)$

$= -1$

\therefore grad. of tangent $= -1$.

Subs. in $x = \pi$ in y

$\therefore y = x \cos x$

$= \pi \cos \pi$

$= \pi(-1)$

$= -\pi$

\therefore point $(\pi, -\pi)$, grad. (m) $= -1$

$\therefore y - y_1 = m(x - x_1)$

$y + \pi = -1(x - \pi)$

$y + \pi = -x + \pi$

$\therefore x + y = 0$

\therefore eqn. of tangent is $x + y = 0$.

(c) $\cos \frac{x}{2} = \frac{1}{\sqrt{2}}$

$\therefore \frac{x}{2} = \frac{\pi}{4}, \frac{7\pi}{4}$

$\therefore x = \frac{\pi}{2}, \frac{7\pi}{2}$

(cannot have $\frac{7\pi}{2}$)

$\therefore x = \frac{\pi}{2}$.

(d) (i) $f(x) = 2 \sin 2x + 1$

$f'(x) = 2 \cdot 2 \cos 2x$

$= 4 \cos 2x$.

(ii) $\int_0^{\frac{\pi}{6}} \frac{4 \cos 2x}{2 \sin 2x + 1} dx$

$= \left[\log_e (2 \sin 2x + 1) \right]_0^{\frac{\pi}{6}}$

$= \log_e \left(2 \sin \frac{\pi}{3} + 1 \right)$

$- \log_e (2 \sin 0 + 1)$

$= \log_e (\sqrt{3} + 1) - \log_e 1$

$= \log_e (\sqrt{3} + 1)$

$= 1.0050525$ (from calculator)

$= 1.0051$ (4 dec. pl.)

$\therefore \int_0^{\frac{\pi}{6}} \frac{4 \cos 2x}{2 \sin 2x + 1} dx$

$= 1.0051$ (4 dec. pl.)

9. (a) (i) Subs. $\left(\frac{\pi}{3}, \frac{\sqrt{3}}{3} \right)$

in $y = \frac{\sin x}{1 + \cos x}$

\therefore LHS $= y = \frac{\sqrt{3}}{3}$

RHS $= \frac{\sin \frac{\pi}{3}}{1 + \cos \frac{\pi}{3}}$

$= \frac{\sqrt{3}}{2} + \left(1 + \frac{1}{2} \right)$

$= \frac{\sqrt{3}}{2} + \frac{3}{2}$

$= \frac{\sqrt{3}}{2} \times \frac{2}{3}$

$= \frac{\sqrt{3}}{3}$

\therefore LHS = RHS

$\therefore \left(\frac{\pi}{3}, \frac{\sqrt{3}}{3} \right)$ lies on

$y = \frac{\sin x}{1 + \cos x}$.

(ii) $\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$

$u = \sin x, v = 1 + \cos x$

$= \frac{(1 + \cos x) \cdot \cos x - \sin x \cdot -\sin x}{(1 + \cos x)^2}$

$= \frac{\cos x + \cos^2 x + \sin^2 x}{(1 + \cos x)^2}$

$= \frac{1 + \cos x}{(1 + \cos x)^2}$

$$\sin^2 x + \cos^2 x = 1$$

$$\therefore \frac{dy}{dx} = \frac{1}{1 + \cos x}$$

(iii) Subs. $x = \frac{\pi}{3}$ in $\frac{dy}{dx}$

$$\therefore \frac{dy}{dx} = \frac{1}{1 + \cos \frac{\pi}{3}}$$

$$= \frac{1}{1 + \frac{1}{2}}$$

$$= 1 + \frac{3}{2}$$

$$= 1 \times \frac{2}{3}$$

$$= \frac{2}{3}$$

\therefore point $\left(\frac{\pi}{3}, \frac{\sqrt{3}}{3}\right)$ and

$$\text{grad. } (m) = \frac{2}{3}$$

$$\therefore y - y_1 = m(x - x_1)$$

$$y - \frac{\sqrt{3}}{3} = \frac{2}{3}\left(x - \frac{\pi}{3}\right)$$

Mult. by 9

$$9y - 3\sqrt{3} = 6x - 2\pi$$

$$\therefore 6x - 9y - 2\pi + 3\sqrt{3} = 0.$$

(b) (i) $40^\circ \rightarrow$ radians

$$\therefore \theta = 40 \times \frac{\pi}{180}$$

$$= \frac{2\pi}{9} \text{ radians}$$

$$= 0.698 \ 131 \ 7 \text{ (from calc.)}$$

Now, $\ell = r\theta$

$$= 20 \times 0.698 \ 131 \ 7$$

$$= 13.962 \ 634 \text{ (from calc.)}$$

$$= 13.96 \text{ (two dec. pl.)}$$

\therefore arc length is 13.96 cm.

(ii) Let $r_1 = 20$, $r_2 = 12$

$$\therefore \text{Area} = \frac{1}{2}r_2^2\theta - \frac{1}{2}r_1^2\theta$$

subtract areas of sectors

$$= \frac{1}{2}\theta(r_1^2 - r_2^2)$$

$$= \frac{1}{2} \times \frac{2\pi}{9} (20^2 - 12^2)$$

$$= \frac{\pi}{9} (256)$$

$$= 89.360 \ 858$$

$$= 89.4.$$

Area of shaded region is 89.4 cm².

10. (a) $\frac{d}{dx}[x \sin x + \cos x]$

$$= \sin x \cdot 1 + x \cdot \cos x - \sin x$$

$$= \sin x + x \cos x - \sin x$$

$$= x \cos x$$

$$\therefore \frac{d}{dx}[x \sin x + \cos x] = x \cos x.$$

$$\therefore \int_0^{\frac{\pi}{2}} x \cos x \, dx$$

$$= [x \sin x + \cos x]_0^{\frac{\pi}{2}}$$

(from above)

$$= \left(\frac{\pi}{2} \sin \frac{\pi}{2} + \cos \frac{\pi}{2}\right)$$

$$- (0 \sin 0 + \cos 0)$$

$$= \left(\frac{\pi}{2} + 0\right) - (0 + 1)$$

$$= \frac{\pi}{2} - 1$$

$$= \frac{\pi - 2}{2}$$

$$\therefore \int x \cos x \, dx = \frac{\pi - 2}{2}.$$

(b) (i) $y = \sin x$, $y = \cos x$

$$\therefore \sin x = \cos x$$

Divide by $\cos x$:

$$\tan x = 1$$

$$\therefore x = \frac{\pi}{4}, \frac{5\pi}{4}, \dots$$

Subs. $x = \frac{\pi}{4}$ in $y = \sin x$

$$\therefore y = \sin \frac{\pi}{4}$$

$$= \frac{1}{\sqrt{2}} \therefore \left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right).$$

Subs. $x = \frac{5\pi}{4}$ in $y = \sin x$

$$\therefore y = \sin \frac{5\pi}{4}$$

$$= -\frac{1}{\sqrt{2}}$$

$$\therefore \left(\frac{5\pi}{4}, -\frac{1}{\sqrt{2}}\right)$$

\therefore points of int. are $\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$

and $\left(\frac{5\pi}{4}, -\frac{1}{\sqrt{2}}\right).$

(ii) Cuts x axis, $\therefore y = 0$

Subs. $y = 0$ in $y = \sin x$

$$\therefore \sin x = 0$$

$$\therefore x = 0, \pi, 2\pi, \dots$$

\therefore cuts x axis at $0, \pi, 2\pi.$

Now, subs. $y = 0$ in $y = \cos x$

$$\therefore \cos x = 0$$

$$\therefore x = \frac{\pi}{2}, \frac{3\pi}{2}, \dots$$

\therefore cuts x axis at $\frac{\pi}{2}, \frac{3\pi}{2}.$

(iii) For $y = \sin x$

$$\frac{dy}{dx} = \cos x$$

Subs. $x = \pi$ in $\frac{dy}{dx}$

$$\therefore \frac{dy}{dx} = \cos \pi$$

$$\therefore = -1$$

\therefore grad. of tangent to $y = \sin x$ at $x = \pi$ is $-1.$

For $y = \cos x$

$$\frac{dy}{dx} = -\sin x$$

Subs. $x = \frac{\pi}{2}$ in $\frac{dy}{dx}$

$$\therefore \frac{dy}{dx} = -\sin \frac{\pi}{2}$$

$$\therefore = -1$$

\therefore grad. of tangent to

$y = \cos x$ at $x = \frac{\pi}{2}$ is $-1.$

\therefore tangents have same gradient

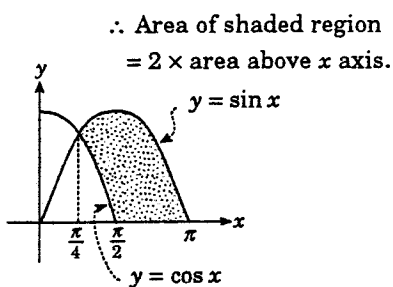
\therefore tangents are parallel.

(iv) Shaded region

above x axis

= shaded region

below x axis



$$= 2 \times \left[\int_{\frac{\pi}{4}}^{\pi} \sin x \, dx - \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos x \, dx \right]$$

$$= 2 \left[[-\cos x]_{\frac{\pi}{4}}^{\pi} - [\sin x]_{\frac{\pi}{4}}^{\frac{\pi}{2}} \right]$$

$$= 2 \left[(-\cos \pi + \cos \frac{\pi}{4}) \right.$$

$$\left. - \left(\sin \frac{\pi}{2} - \sin \frac{\pi}{4} \right) \right]$$

$$= 2 \left[-(-1) + \frac{1}{\sqrt{2}} - \left(1 - \frac{1}{\sqrt{2}} \right) \right]$$

$$= 2 \left[1 + \frac{1}{\sqrt{2}} - 1 + \frac{1}{\sqrt{2}} \right]$$

$$= 2 \left[\frac{2}{\sqrt{2}} \right]$$

$$= \frac{4}{\sqrt{2}}$$

$$= 2\sqrt{2}$$

$\frac{4}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$ $= \frac{4\sqrt{2}}{2}$ $= 2\sqrt{2}$

\therefore area is $2\sqrt{2}$ units².

$$\begin{aligned} \text{(c) } V &= \pi \int_a^b y^2 \, dx \\ &= \pi \int_0^{\frac{\pi}{4}} (\tan x)^2 \, dx \\ &= \pi \int_0^{\frac{\pi}{4}} \tan^2 x \, dx \\ &= \pi \int_0^{\frac{\pi}{4}} (\sec^2 x - 1) \, dx \end{aligned}$$

$\tan^2 x = \sec^2 x - 1$

$$= \pi [\tan x - x]_0^{\frac{\pi}{4}}$$

$$= \pi \left[\left(\tan \frac{\pi}{4} - \frac{\pi}{4} \right) - 0 \right]$$

$$= \pi \left[1 - \frac{\pi}{4} \right]$$

\therefore volume is $\pi \left(1 - \frac{\pi}{4} \right)$ units³.