2 UNIT TEST NUMBER 5

1996

Geometric Applications of Differentiation.

QUESTION 1. (14 marks)

Marks

Consider the curve given by $y = x^3 - 12x + 4$.

(a) Find the coordinates of any stationary points and determine their nature. 6

(b) Find the coordinates of any points of inflexion. 3

(c) Sketch the curve for the domain $-3 \le x \le 4$. 4

(d) For what value(s) of x in the domain $-3 \le x \le 4$ does y have its maximum value? 1

QUESTION 2. (12 marks)

(a) (i) What is the condition (in terms of $\frac{dy}{dx}$) for a function to be decreasing? 4

(ii) Find the values of x for which the function $y = 4 + 36x - 3x^2 - 2x^3$ is decreasing.

(b) Find a, b and c if the curve $y = x^3 + ax^2 - bx + c$ has an x-intercept at $x = 1$, a stationary point at $x = -2$, and a point of inflexion at $x = -\frac{1}{2}$. 4

(c) Given $\frac{d^2y}{dx^2} = 4$, find y in terms of x, if $(2, 9)$ is a stationary point. 4

QUESTION 3. (7 marks)

Three circles, two with radii r and one with radius R are formed such that the sum of their radii is 18 cm.

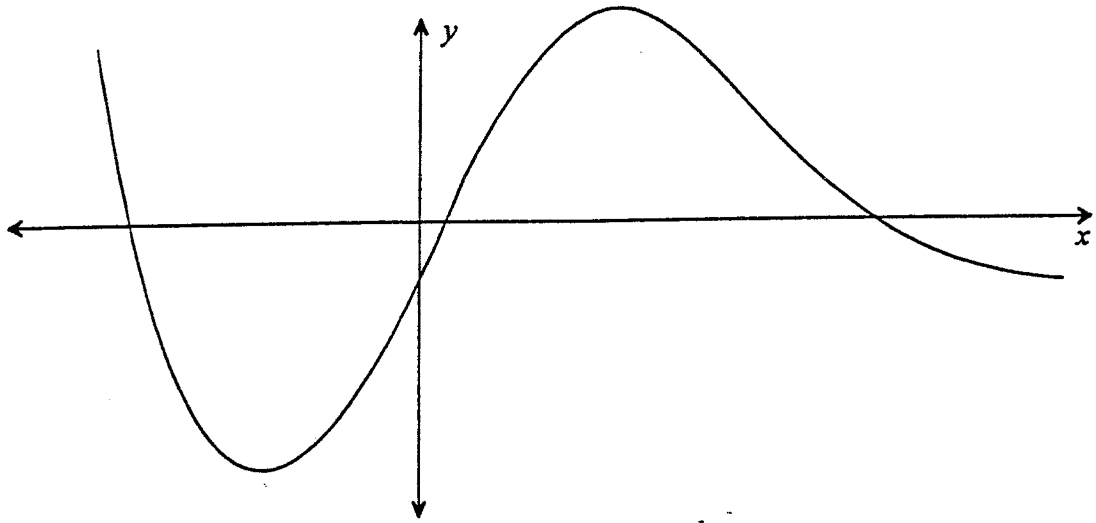
(a) Show that the sum, S, of the areas of the three circles is $S = \pi(6r^2 - 72r + 324)$. 2

(b) Hence find the radii of the circles if the sum of the areas is a minimum. 3

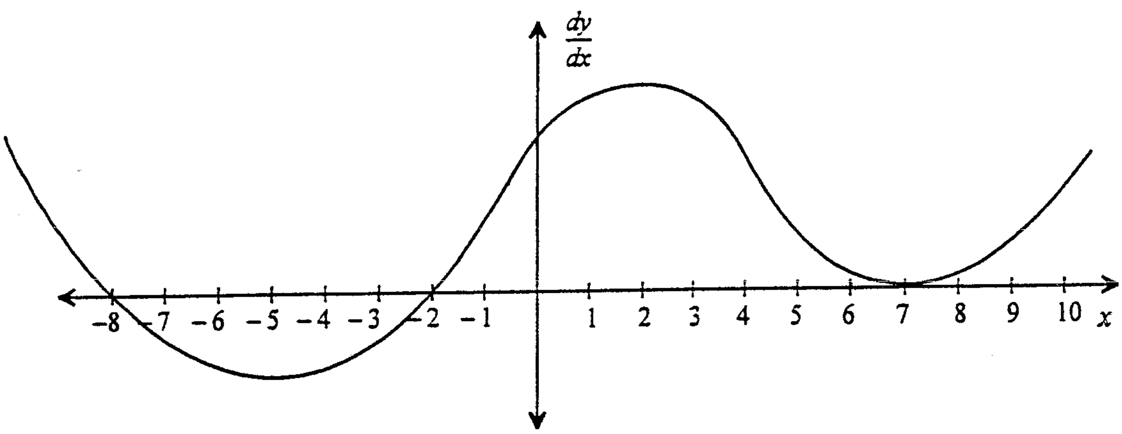
(c) What is the upper limit that the sum of the areas can be? 2

QUESTION 4. (7 marks) **Marks**

(a) Copy or trace the curve of $y = f(x)$ and on the same set of axes sketch the curve of the gradient function $f'(x)$. **3**



(b) The given curve represents a gradient function $\frac{dy}{dx}$ relative to x of a function $y = f(x)$. **4**



Use this graph to determine the values of x at which the graph of the function $y = f(x)$:

(i) has a maximum turning point,

(ii) has a horizontal point of inflexion,

(iii) is concave down.

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SUGGESTED SOLUTIONS

QUESTION 1

(a) $y = x^3 - 12x + 4$

$\frac{dy}{dx} = 3x^2 - 12$

Stationary points when $\frac{dy}{dx} = 0$.

$3x^2 - 12 = 0$ **1**

$3(x-2)(x+2) = 0$

$x = 2$ or $x = -2$ **1**

When $x = 2$, $y = (2)^3 - 12 \times (2) + 4$

$= -12$ **1**

When $x = -2$, $y = (-2)^3 - 12 \times (-2) + 4$

$= 20$ **1**

Stationary points at $(-2, 20)$ and $(2, -12)$.

$\frac{d^2y}{dx^2} = 6x$

When $x = -2$, $\frac{d^2y}{dx^2} = -12 < 0$ $\therefore$ concave down. **1**

$\therefore$ Maximum turning point at $(-2, 20)$.

When $x = 2$, $\frac{d^2y}{dx^2} = 12 > 0$ $\therefore$ concave up. **1**

$\therefore$ Minimum turning point at $(2, -12)$. **Total = 6**

(b) A point of inflection occurs when $\frac{d^2y}{dx^2} = 0$ and concavity changes. **1**

$6x = 0 \quad \therefore \quad x = 0$

At $x = 0 - \varepsilon$, $\frac{d^2y}{dx^2} = 6 \times (-\varepsilon) < 0$ (concave down)

At $x = 0 + \varepsilon$, $\frac{d^2y}{dx^2} = 6 \times \varepsilon > 0$ (concave up) **1**

Concavity changes.
$\therefore$ Point of inflection is at $(0,4)$. **1**

Note : To check for change of concavity on either side of $x = 0$, rather than use ε (a small positive number), you may substitute a numerical value, say $x = -0.1$ and $x = 0.1$.

Note : $y = 4$ is found by substituting $x = 0$ into the equation of the curve $y = x^3 - 12x + 4$.

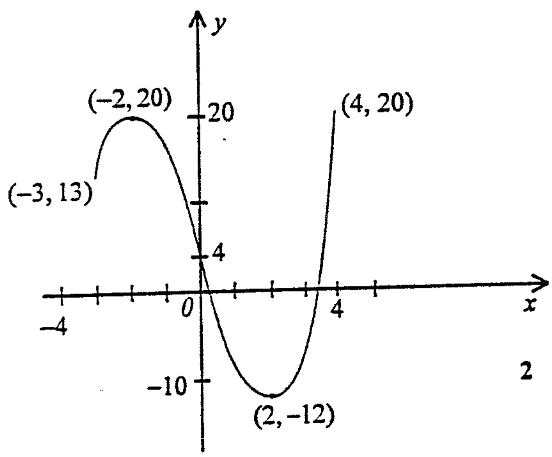
Total 3

(c) When $x = -3$, $y = (-3)^3 - 12 \times (-3) + 4$

$= 13$ **1**

When $x = 4$, $y = 4^3 - 12 \times 4 + 4$

$= 20$ **1**



2

Total = 4

(d) y has a maximum value when $x = -2$, $x = 4$. **1**

Note : Determined from the graph.

Substitute $a = 1\frac{1}{2}$, $b = 6$ into (1)

$$1\frac{1}{2} - 6 + c = -1$$

$$c = 3\frac{1}{2}$$

$\therefore \quad a = 1\frac{1}{2}, \quad b = 6, \quad c = 3\frac{1}{2}$ 1 **Total = 4**

(c) $\dfrac{d^2y}{dx^2} = 4$

$\dfrac{dy}{dx} = 4x + c$ 1

When $x = 2$, $\dfrac{dy}{dx} = 0 \;\therefore 0 = 4 \times 2 + c \;\therefore c = -8$ 1

Note : Because there is a stationary point at $x = 2$, $y = 9$.

$\therefore \dfrac{dy}{dx} = 4x - 8$

$y = 2x^2 - 8x + k$ 1

Note : We use two different pronumerals for the arbitrary constants in integration.

When $x = 2$, $y = 9 \quad \therefore \quad 9 = 2 \times 2^2 - 8 \times 2 + k$

$k = 17$

$\therefore \quad y = 2x^2 - 8x + 17$ 1 **Total = 4**

QUESTION 3

(a) $2r + R = 18 \qquad \therefore \; R = 18 - 2r$

Note : Before differentiating, we must express the right-hand side in terms of one pronumeral. We use the relation $R = 18 - 2r$.

$S = \pi r^2 + \pi r^2 + \pi R^2$ 1

$= 2\pi r^2 + \pi(18 - 2r)^2$ 1

$= 2\pi r^2 + \pi\left(324 - 72r + 4r^2\right)$

$= \pi\left(6r^2 - 72r + 324\right)$ **Total = 2**

(b) For minimum sum of areas, $\dfrac{dS}{dr} = 0$ and $\dfrac{d^2S}{dr^2} > 0$.

$\dfrac{dS}{dr} = \pi\,(12r - 72) = 0$

$r = 6$ 1

$\dfrac{d^2S}{dr^2} = \pi \times 12 > 0 \;\; \therefore$ concave up 1

$\therefore$ sum of areas of circles is minimum when the radii are all 6 cm. 1 **Total = 3**

Note: $R = 18 - 2(6) = 18 - 12 = 6$

QUESTION 2

(a) (i) A function is decreasing when $\frac{dy}{dx} < 0$. 1

(ii) $y = 4 + 36x - 3x^2 - 2x^3$

$\frac{dy}{dx} = 36 - 6x - 6x^2$

When the function is decreasing,

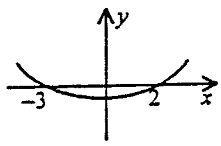
$36 - 6x - 6x^2 < 0$ 1

$6x^2 + 6x - 36 > 0$

Note : Multiply by -1, reverse the inequality.

$x^2 + x - 6 > 0$

$(x+3)(x-2) > 0$ 1



Note : Quadratic inequality – always sketch a graph of the quadratic function to determine the solution.

$x < -3, \quad x > 2$ 1

Total = 3

(b) $y = x^3 + ax^2 - bx + c$

$\frac{dy}{dx} = 3x^2 + 2ax - b$

$\frac{d^2y}{dx^2} = 6x + 2a$

When $x = 1$, $y = 0$:

$0 = 1 + a - b + c$

i.e. $a - b + c = -1$ (1) 1

When $x = -2$, $\frac{dy}{dx} = 0$:

$0 = 3 \times (-2)^2 + 2a \times (-2) - b$

i.e. $4a + b = 12$ (2) 1

When $x = -\frac{1}{2}$, $\frac{d^2y}{dx^2} = 0$:

$0 = 6 \times (-\frac{1}{2}) + 2a$

i.e. $2a = 3$

$a = 1\frac{1}{2}$ 1

Substitute $a = 1\frac{1}{2}$ into (2)

$4 \times 1\frac{1}{2} + b = 12$

$b = 6$

(c) Maximum area occurs when $r = 0$ or $r = 9$.

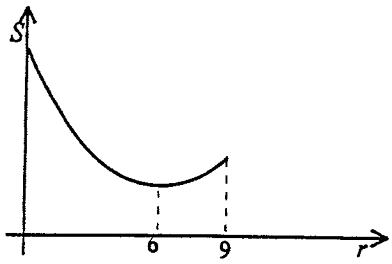
When $r = 0$, $S = 324\pi$.

When $r = 9$, $S = \pi(6 \times 9^2 - 72 \times 9 + 324)$

$= 162\pi$

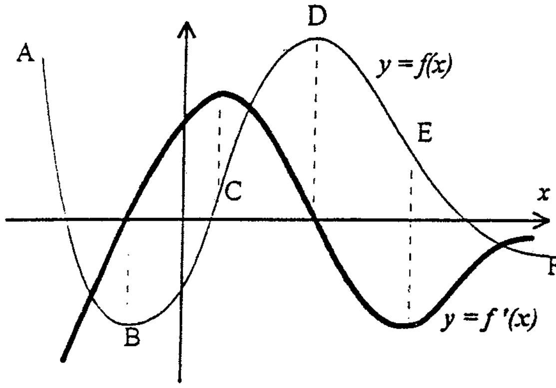
The upper limit of the total area is 324π cm^2. **2**

Note : A graph helps to visualise the relation between S and r.



QUESTION 4

(a)



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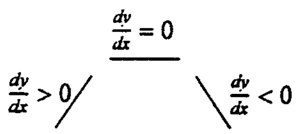
Notes : From A → B, gradient is negative.
At B, gradient is zero (stationary point).
From B → C, gradient is positive and increases to maximum value at C (point of inflection).
From C → D, gradient is still positive but becomes zero at D (stationary point).
From D → E, gradient is negative, and has maximum negative value at E (a point of inflection).
From E → F, gradient is still negative, but approaching zero.

(b) (i) Maximum turning point:



Maximum value at $x = -8$. **1**

(ii) Horizontal point of inflexion where $\frac{dy}{dx} = 0$

and $\frac{dy}{dx}$ has the same sign on either side.

$\therefore$ horizontal point of inflection at $x = 7$. **1**

(iii) Concave down when

$\frac{d^2y}{dx^2} < 0$, i.e. $\frac{d}{dx}\left(\frac{dy}{dx}\right) < 0$

i.e. the derivative of $\frac{dy}{dx} < 0$

i.e. the $\frac{dy}{dx}$ curve is decreasing.

Concave down for $x < -5$, $2 < x < 7$. **1, 1** **Total = 2**