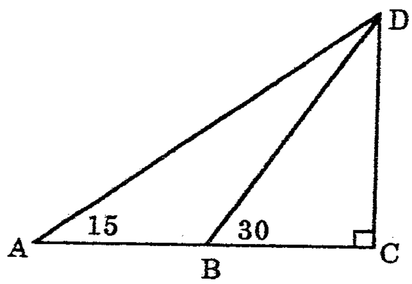
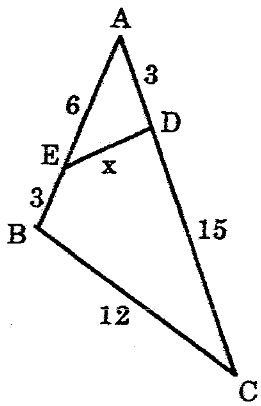
Geometry 1

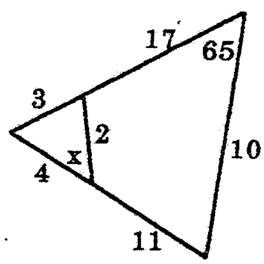
1



i) Prove that triangle ABD is isosceles.
ii) If DC = 1 unit, find the exact lengths of BCand BD.
iii) Deduce that $\tan 15° = 2 - \sqrt{3}$

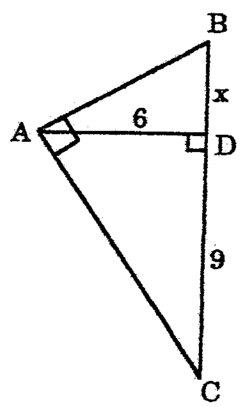
2 i) Write down the three tests for similar triangles.
ii) Find the values of x, giving reasons.





iii) Find the value of x, giving reasons.

iv) Find the value of x, giving reasons.



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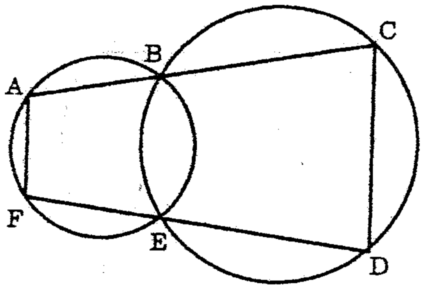
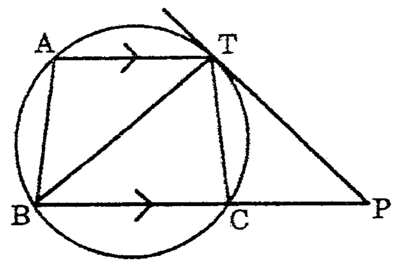


Prove that AF || CD.
Hint: Join BE.

4 PT is a tangent to a circle touching it at T.
A and B are points on the circumference of the circle, such that PB | | TA.
Prove that angle ABT = angle TPC



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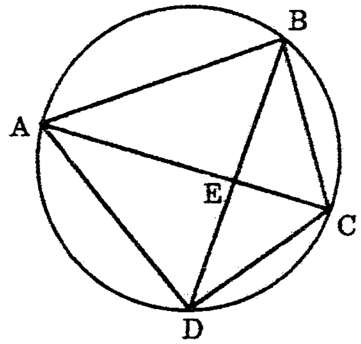
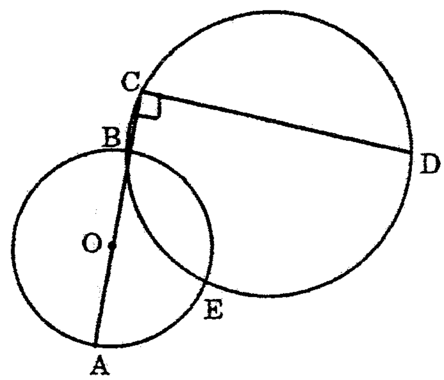


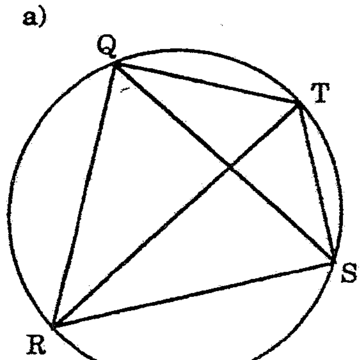
A, B, C, D are points on a circle. The diagonals BD and AC meet at E.
If DB bisects angle CBA, prove that

i) AD = DC
ii) Angle AEB = angle BCD

6 Two circles intersect at B and E.
The diameter AOB of the first circle is extended and meets the second circle at C.
The line CD is drawn perpendicular to AC.
Prove that the points A, E, D are collinear.

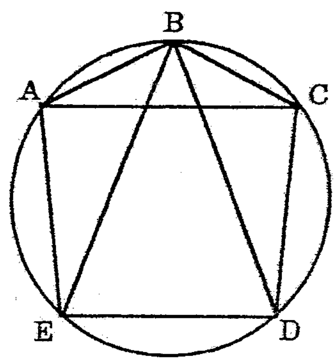


7 a)

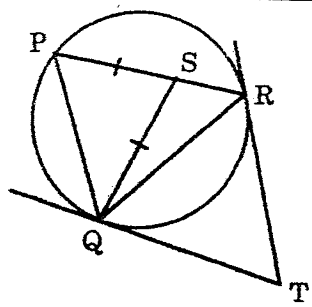


Q, T, S and R are points on a circle.
The points are chosen so that QT = TS.
Prove that RT bisects angle QRS.

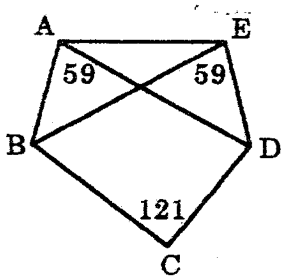
b) A, B, C, D and E are points on a circle.
If angle BEA = angle BDC, prove that triangle ABC is isosceles.



8 PQR is a triangle inscribed in a circle. S is a point on PR, chosen so that QS = SP. Tangents from an external point T touch the circle at Q and R. Prove that QTRS is cyclic



9 ABCDE is a pentagon.
Angle BCD = 121°
Angle BAD = angle BED = 59°
Prove that angle ECD = angle DAE.



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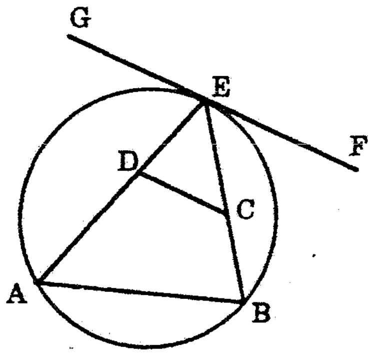
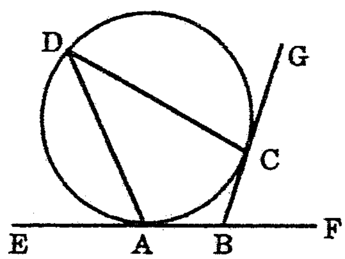


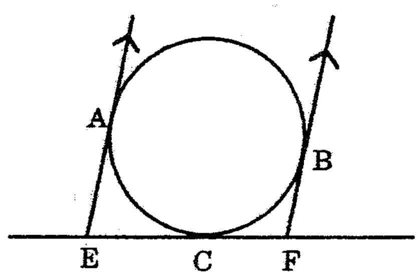
GF is a tangent to a circle at E and quadrilateral ABCD is cyclic.
Prove that DC | | GF

11 A and C are two separate points on a circle.
Tangents at A and C meet at B.
D is a point on the circumference of the circle.
Prove that quadrilateral ABCD can never be cyclic.



12 The tangents at A and B to the circle shown in the diagram are parallel.
These tangents meet the tangent at a third point C at E and F respectively.

i) If angle AEC = 2x°, prove that angle BCF = x°
ii) Prove that AB is a diameter of the circle.



13 i)

$f(x) = \frac{\cos^2 x}{1 + \sin^2 x}$

$f'(x)$

$= \frac{(1 + \sin^2 x) \cdot 2\cos x \cdot (-\sin x) - \cos^2 x \cdot 2\sin x \cos x}{(1 + \sin^2 x)^2}$

$f'(\frac{\pi}{2}) = \frac{(1+1) \cdot 0 \cdot (-1) - 0 \cdot 2 \cdot 1 \cdot 0}{4}$

$= 0$, as required

ii) $f'(x) =$

$\frac{-2\sin x \cos x - 2\sin^3 x \cos x - 2\sin x \cos^3 x}{(1 + \sin^2 x)^2}$

$f'(\frac{\pi}{2}^-) < 0$

$f'(\frac{\pi}{2}^+) > 0$, using calculator

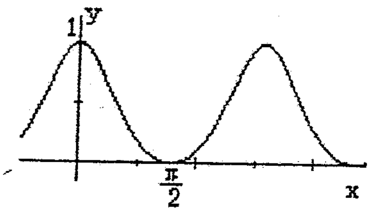
$\therefore$ there is a minimum turning point at $x = \frac{\pi}{2}$

Also $f(\frac{\pi}{2}) = 0$ $\therefore$ the curve touches the x-axis there

<u>OR</u> $f(\frac{\pi}{2}^-) > 0$, $f(\frac{\pi}{2}^+) > 0$ and $f(\frac{\pi}{2}) = 0$

i.e. to the left and right of $x = \frac{\pi}{2}$ the curve is above the x-axis. Hence the curve touches the x-axis or has a cusp at $x = \frac{\pi}{2}$. But $f'(\frac{\pi}{2}) = 0$ $\therefore$ it touches the x-axis.

The curve is shown below:



Geometry 1

1 i) Angle ADB = 15°, exterior angle theorem

$\therefore$ triangle ABD is isosceles, since its base angles are equal

ii) In triangle BDC, $\tan 30 = \frac{1}{BC}$

$\therefore BC = \sqrt{3}$

Also, $BD^2 = BC^2 + DC^2$ $\therefore$ $BD = 2$

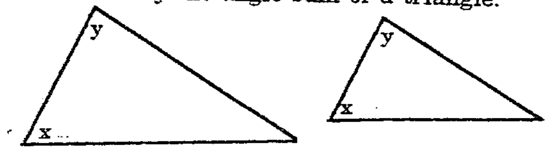
iii) AB = BD = 2, since triangle ABD is isosceles

In triangle ADC, $\tan 15 = \frac{1}{2 + \sqrt{3}}$

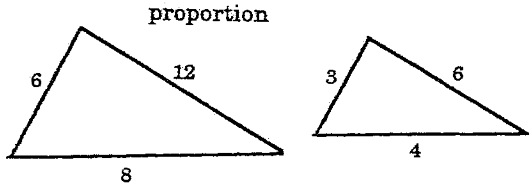
$= 2 - \sqrt{3}$, on rationalising the denominator.

2 i) Two triangles will be similar if

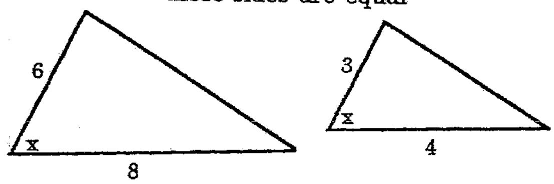
α) 2 angles of one respectively equal to 2 angles of the other. The third angles would then be equal by the angle sum of a triangle.



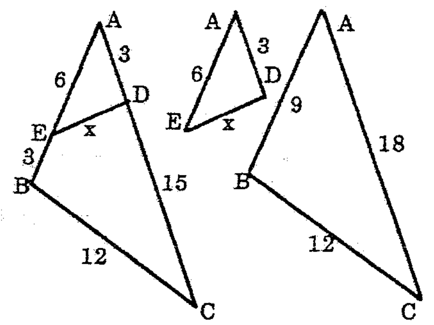
β) The sides of the triangles are in proportion



γ) Two sides of each triangle are in proportion and the angles between these sides are equal



ii)



In triangles AED and ACB, $\frac{AE}{AC} = \frac{AD}{AB}$

i.e. $\frac{6}{18} = \frac{3}{9}$

Also, angle A is common to both triangles. $\therefore$ these triangles are similar by test (γ) above.

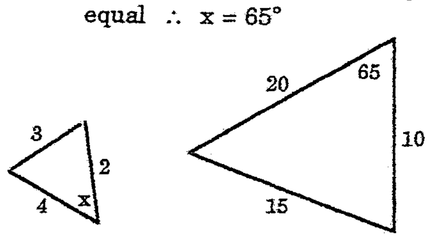
Therefore the sides of the triangles are in proportion

i.e. $\frac{x}{6} = \frac{12}{18}$ $\therefore$ $x = 4$

iii) Since the sides are in proportion :

$\frac{3}{15} = \frac{4}{20} = \frac{2}{10}$, the triangles are similar.

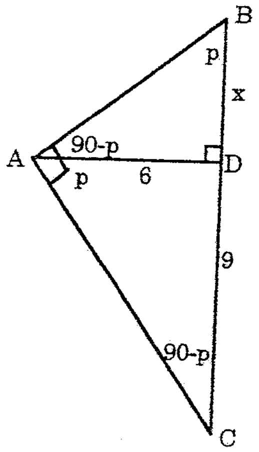
Therefore corresponding angles are equal $\therefore$ $x = 65°$



iv) By letting angle B = p°, the other angles can be calculated using the angle sum of a triangle.

Now, $\tan p = \frac{6}{x}$ in triangle ADB and

$\tan p = \frac{9}{6}$ $\therefore$ $\frac{6}{x} = \frac{9}{6}$ $\therefore x = 4$



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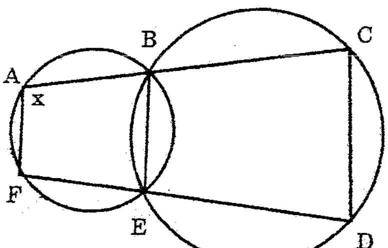


Join BE and let angle A = x°

$\therefore$ angle BEF = 180 - x, opposite angles of cyclic quadrilateral ABEF

$\therefore$ angle BED = x°, straight angle

$\therefore$ angle BCD = 180 - x, opposite angles of cyclic quadrilateral

$\therefore$ angles A and C are supplementary

But these angles are co-interior

$\therefore$ AF | | CD Q.E.D.

4 Let angle PTC = x° and angle ABT = y°

Angle TBC = x°, angle in the alternate segment

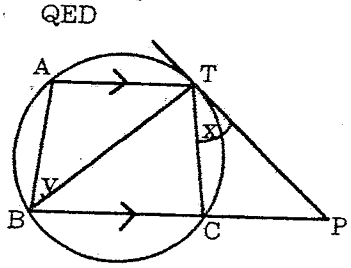
Angle BAT = $(180 - x - y)^\circ$, co-interior angles, AT | | BC

Angle TCB = $(x + y)^\circ$, opposite angles of cyclic quadrilateral

$\therefore$ angle TPC = y°, exterior angle theorem in triangle TCP

$\therefore$ angle ABT = angle TPC

QED



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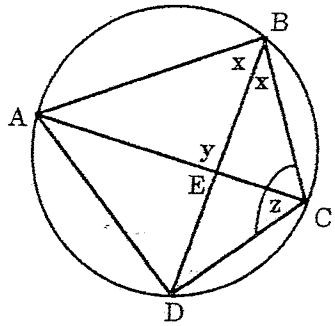


Let angle ABD = DBC = x°

Let angle AEB = y°

Let angle BCD = z°

i) Angle DAC = angle DBC = x°, angles on the same arc DC

Angle ACD = angle ABD = x°, angles on the same arc AD

$\therefore$ triangle ADC is isosceles, base angles are equal

$\therefore$ AD = DC

ii) Angle ACD = angle ABD = x°, angles on the same arc AD

$\therefore$ Angle BCA = z - x

In triangle BEC,

angle AEB = angle EBC + angle ECB, exterior angle theorem

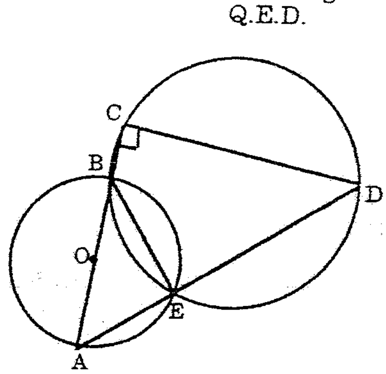
i.e. $y = x + z - x$

i.e. $y = z$

i.e. angle AEB = angle BCD

Q.E.D.

6



Join AE

Join ED

Join BE

Angle BEA = 90°, angle in a semi circle is a right angle

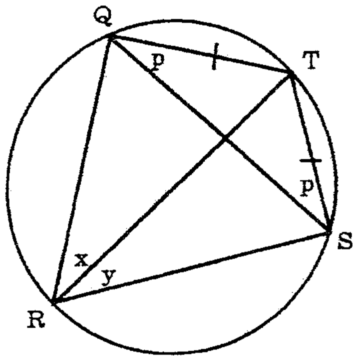
Angle BED = 90°, opposite angles of cyclic quadrilateral

$\therefore$ angle AED = 180°

$\therefore$ angle AED is a straight angle

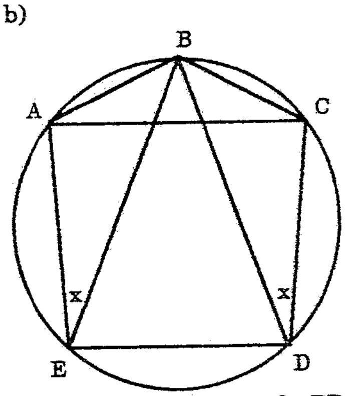
$\therefore$ A, E, D are collinear Q.E.D.

7 a)



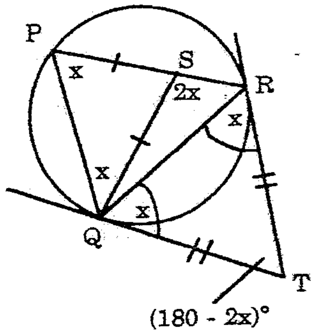
Let angle TQS = p°
Let angle QRT = x°
Let angle SRT = y°
We must prove that x = y
Angle TQS = angle TSQ = p, base angles isosceles triangle QTS
y = p, angles on the same arc TS
x = p, angles on the same arc QT
$\therefore$ y = x
i.e. RT bisects angle QRS
Q.E.D.

b)



Since angle AEB = angle BDC,
arc AB = arc BC, converse of angles on the same arc theorem
$\therefore$ AB = BC, equal arcs subtend equal chords
$\therefore$ triangle ABC is isosceles Q.E.D.
[Note: We could have used this technique in (i)]

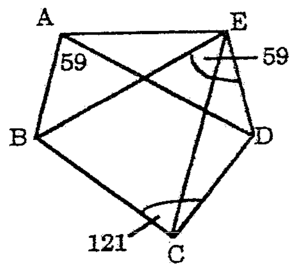
8



Let angle QPR = x°
Angle PQS = x°, base angles isosceles triangle
$\therefore$ angle QSR = $2x^\circ$, exterior angle theorem in triangle PSQ

Angle RQT = x°, converse of angle in alternate segment
Now, PQ = PR, tangents from an external point are equal
$\therefore$ triangle TRQ is isosceles.
$\therefore$ QRT = x°, base angles isosceles triangle TRQ
$\therefore$ angle QTR = 180 - 2x, angle sum of triangle TRQ
$\therefore$ angle QTR + angle QSR = 180°
$\therefore$ angle SQT + angle SRT = 180°, angle sum of quadrilateral QTRS
$\therefore$ quadrilateral QTRS is cyclic since its opposite angles are supplementary
Q.E.D.

9



In quadrilateral ABDE,
angle BAD = angle BED
$\therefore$ the quadrilateral is cyclic, by the converse of angles on the same arc.
Also, quadrilateral BCDE is cyclic since the opposite angles are supplementary.
$\therefore$ pentagon ABCDE is cyclic
$\therefore$ angle ECD = angle DAE, angles on the same arc ED. Q.E.D.

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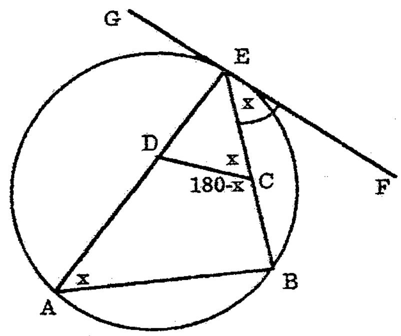
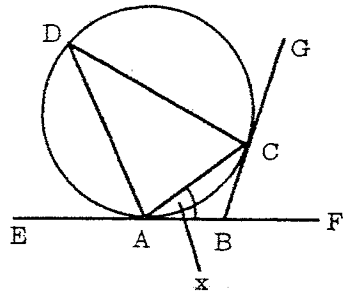


Let angle FEC = x°
Angle FEC = angle EAB = x°, angle in the alternate segment
Angle DCB = $(180 - x)^\circ$, opposite angles of cyclic quadrilateral
$\therefore$ angle DCE = x, straight angle
$\therefore$ angle DCE = angle CEF
But these are alternate angles
$\therefore$ DC | | GF Q.E.D.

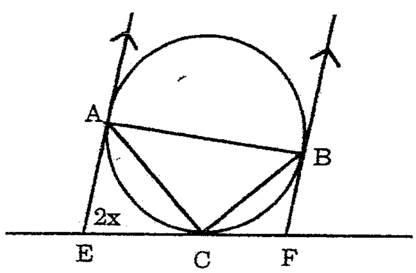
11



Join AC
Let angle CAB = $x°$
Now, AB = BC, tangents from an external point are equal.
$\therefore$ triangle ABC is isosceles.
$\therefore$ ACB = $x°$, base angles isosceles triangle
$\therefore$ angle ABC = $(180 - 2x)°$, angle sum of triangle
Also, angle ADC = $x°$, angle in the alternate segment theorem.
Now, if ABCD were to be cyclic, the opposite angles would be supplementary.
i.e. angle ADC + angle ABC = 180°
i.e. $x + 180 - 2x = 180$
Therefore $x = 0$
However, if $x = 0$, we could not draw this diagram.
Hence, ABCD can not be cyclic.
Q.E.D.

Alternative Method
Since only one circle can pass through three points, and a circle passes through points A, C and D, the same circle could not pass through the point B. Hence, quadrilateral ABCD can not be cyclic. Q.E.D.

12

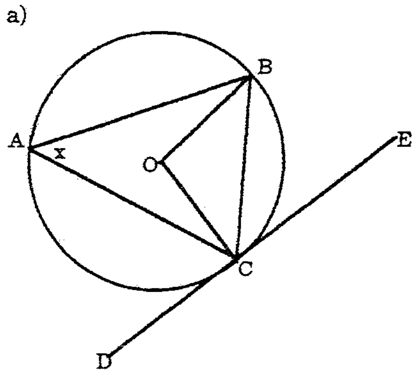


i) Angle CFB = $(180 - 2x)°$, co-interior angles with AE | | BF
Also, CF = BF, tangents from an external point are equal.
$\therefore$ triangle CFB is isosceles
$\therefore$ angle BCF = CBF, base angles of isosceles triangle
Now, angle BCF + angle CBF + angle CFB = 180°, angle sum of triangle BFC
i.e. $2BCF + 180 - 2x = 180$
$\therefore$ angle BCF = $x°$, as required.

ii) Similarly, triangle EAC is isosceles and hence angle ACE = $(90 - x)°$, using base angles of isosceles triangle and angle sum of triangles as in (i)
Now, angle ACE + angle ACB + angle BCF = 180°, straight angle
i.e. $90 - x + ACB + x = 180°$
$\therefore$ angle ACB = 90°
$\therefore$ AB is a diameter, converse of angle in a semicircle theorem Q.E.D.

Geometry 2

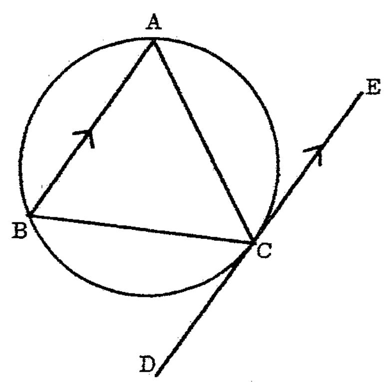
1 a)



i) Angle BOC = $2x$, angle at the centre theorem
Triangle OBC is isosceles since OB = OC, radii
$\therefore$ angle OCB = angle OBC, base angles isosceles triangle
$\therefore$ angle OCB = $90 - x$, angle sum of triangle

ii) Angle OCE = 90°, angle between tangent and radius = 90°
$\therefore$ angle BCE = $90 - (90 - x) = x$
Q.E.D.

b)



Angle ACE = angle BAC, alternate angles between parallel lines
Angle ACE = angle ABC, angles in alternate segment
$\therefore$ angle BAC = angle ABC, both equal to angle ACE
$\therefore$ triangle ABC is isosceles, base angles are equal Q.E.D.