3 UNIT MAY TOPIC TEST

QUESTION 1. (10 marks) — Marks

(a) Consider the function f given by $f(x) = 3x - 2$. — 6

(i) Sketch, on the same number plane, the graph of $y = f(x)$, and also the reflection of this graph in the line $y = x$. Show all x- and y- intercepts.

(ii) Write the equation of this reflected graph first with x as subject and then with y as subject.

(iii) Determine $f^{-1}(x)$.

(b) Evaluate $\tan^{-1}(-1)$ without using a calculator. — 1

(c) Without using a calculator, show that $\tan^{-1}\frac{3}{5} + \tan^{-1}\frac{1}{4} = \frac{\pi}{4}$. — 3

QUESTION 2. (13 marks)

(a) If $f(x) = x\tan^{-1}x$, show that $f'(\sqrt{3}) = \frac{1}{12}\left(3\sqrt{3} + 4\pi\right)$. — 4

(b) Consider the function $y = 3\cos^{-1}2x$. — 9

(i) State its domain and range.

(ii) Sketch the graph, showing the important features.

(iii) Use the graph to show why $\cos^{-1}(-2t) + \cos^{-1}(2t) = \pi$.

(iv) A and B are two points on this graph where $x = \frac{1}{8}$ and $x = \frac{1}{4}$ respectively. Find the gradient of the line AB correct to 2 decimal places.

QUESTION 3. (21 marks) — Marks

(a) 10% of students at a particular school ride bikes to school. If 15 students are chosen at random, what is the probability that: — 5

(i) exactly three ride a bike to school.

(ii) at least 2 ride a bike to school.

(b) 4 boys and 3 girls arrange themselves randomly in a straight line. Find the probability that: — 4

(i) the boys and girls alternate.

(ii) two of the girls occupy the end positions.

(c) A family of six members, father, mother, and four children, have their meal at a round table. How many seating arrangements are possible if: — 6

(i) there are no restrictions on seating positions?

(ii) father and mother sit next to each other?

(iii) the three youngest children sit next to each other?

(d) How many different arrangements of the word MAMMOTH may be made if: — 6

(i) all letters are used?

(ii) only five letters are used?

QUESTION 4. (9 marks) Marks

(a) A particle moves such that, when its displacement is x metres from an origin, its velocity is given by: $v=\sqrt{8x+1}$. Initially the particle is at the origin. 4

Show that $x=2t^2+t$, where t seconds is the time taken to reach the position x metres from the origin.

(b) A particle moves with a constant acceleration of 9 m/s^2. Given that the velocity is 12 m/s when the particle is 6 metres from the origin, find: 5

(i) an expression for velocity in terms of displacement.

(ii) the velocity when $x=0$.

QUESTION 5 (14 marks)

(a) A spring, hanging vertically from a fixed point, has an object attached to its lower end. The object is then 20 cm below the top of the spring. The object is then pulled down so that it is then 25 cm from the top of the spring, and it is then released. The object reaches the original position in 1 second. Assuming the motion is simple harmonic, find: 6

(i) the period of the motion.

(ii) the acceleration acting on the object at the instant of release.

(iii) the speed of the object at the instant it reaches the equilibrium position.

(b) A particle moves in a straight line so that its velocity v at a position x is given by: $v^2=4(3+2x-x^2)$. 8

(i) Show that $\ddot{x}=-4(x-1)$.

(ii) State the centre of motion.

(iii) What is the amplitude of the motion?

(iv) What is the period of the motion?

(v) What is the maximum speed of the particle?

QUESTION 6 (17 marks) Marks

A projectile is launched from level horizontal ground at an angle of 30° to the horizontal at a speed of 200 m/s. The equations of motion in the horizontal and vertical directions are respectively:

$\ddot{x}=0$ and $\ddot{y}=-10$ (taking acceleration due to gravity as 10 m/s^2).

(a) Derive expressions for $\dot{x}, \dot{y}, x, y$. 4

(b) For how long is the projectile airborne? 2

(c) Calculate the greatest height reached by the projectile. 2

(d) At what distance from its launch site does it strike the ground again. 2

(e) How high is the projectile after it has travelled 2000 metres horizontally? 2

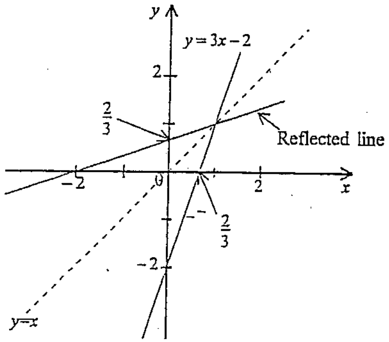
(f) At what angle, and at what speed, is the projectile travelling after 15 seconds? 3

(g) Find the Cartesian equation of the trajectory of the projectile. 2

3 UNIT MAY TOPIC TEST - SOLUTIONS

QUESTION 1

(a) (i) $f(x) = 3x - 2$



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(ii) Equation of reflected graph: $x = 3y - 2$ 1 *Note :* We interchange the x, y in the equation $y = 3x - 2$

i.e. $y = \frac{x+2}{3}$ 1 Total = 2

(iii) $f^{-1}(x) = \frac{x+2}{3}$ 1 *Note :* $f^{-1}(x)$ is the notation for the inverse function of $f(x)$.

(b) $\tan^{-1}(-1) = -\frac{\pi}{4}$ 1 *Note :* $\tan^{-1}(-1)$ is the angle (in radians) whose tan is -1, in the domain $-\frac{\pi}{2} < \tan^{-1}x < \frac{\pi}{2}$.

(c) Show $\tan^{-1}\frac{3}{5} + \tan^{-1}\frac{1}{4} = \frac{\pi}{4}$.

Let $\alpha = \tan^{-1}\frac{3}{5}$, $\beta = \tan^{-1}\frac{1}{4}$ 1 *Note :* Both α, β are acute angles.

$\tan\alpha = \frac{3}{5}$, $\tan\beta = \frac{1}{4}$

$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$ 1

$\tan(\alpha+\beta) = \frac{\frac{3}{5}+\frac{1}{4}}{1-\frac{3}{5}\times\frac{1}{4}}$

$= \frac{12+5}{20-3}$ (mult. num & denom by 20)

$= 1$ 1

$\alpha + \beta = \frac{\pi}{4}$ *Note :* $\alpha + \beta$ cannot be a 3rd quadrant angle, since α, β are both acute angles.

$\therefore \tan^{-1}\frac{3}{5} + \tan^{-1}\frac{1}{4} = \frac{\pi}{4}$ Total = 3

QUESTION 2

(a) $f(x) = x\tan^{-1}x$

$f'(x) = \tan^{-1}x \times 1 + x \times \frac{1}{1+x^2}$ (product rule) 1 *Note :* $\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$

$f'(\sqrt{3}) = \tan^{-1}\sqrt{3} + \frac{\sqrt{3}}{1+\left(\sqrt{3}\right)^2}$ 1

$= \frac{\pi}{3} + \frac{\sqrt{3}}{4}$ 1

$= \frac{4\pi + 3\sqrt{3}}{12}$ 1

$= \frac{1}{12}\left(3\sqrt{3} + 4\pi\right)$ Total = 4

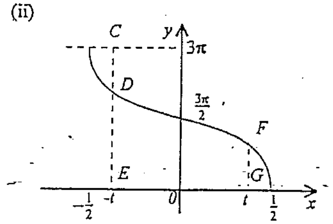
(b) (i) $y = 3\cos^{-1}2x$

Domain: $-1 \le 2x \le 1$ 1 *Note :* For $f(x) = \cos^{-1}x$, domain is $-1 \le x \le 1$ range is $0 \le \cos^{-1}x \le \pi$.

i.e. $-\frac{1}{2} \le x \le \frac{1}{2}$

Range: $0 \le y \le 3\pi$ 1 Total = 2

(ii)



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(iii) $FG = CD$ by symmetry about $(0, \frac{3\pi}{2})$

Now $DE + CD = 3\pi$

$\therefore$ $DE + FG = 3\pi$ 1

Also $DE = 3\cos^{-1}(-2t)$

and $FG = 3\cos^{-1}(2t)$

i.e. $3\cos^{-1}(-2t)+3\cos^{-1}(2t)=3\pi$ 1

i.e. $\cos^{-1}(-2t)+\cos^{-1}(2t)=\pi$ Total = 2

(iv) Gradient $AB=\dfrac{f\left(\frac{1}{8}\right)-f\left(\frac{1}{4}\right)}{\frac{1}{8}-\frac{1}{4}}$ 1

$=\dfrac{3\cos^{-1}\frac{1}{4}-3\cos^{-1}\frac{1}{2}}{-\frac{1}{8}}$ 1

$=-6.50$ (correct to 2 d. p.) 1

Note : Calculator in Radian mode.
Total = 3

QUESTION 3

(a) (i) P(3 ride bikes) $=\binom{15}{3}(0.1)^3(0.9)^{12}$ 1

$=0.1285$ (4 d.p.) 1

Note : This is Binomial Probability.
Total = 2

(ii) P(at least 2 ride bikes)

$=1-$ P(0 or 1 ride bikes) 1

$=1-\left[(0.9)^{15}+\binom{15}{1}(0.1)(0.9)^{14}\right]$ 1

$=0.451$ (3 d.p.) 1

Note : Any student chosen at random has a probability of 0.1 of riding a bike to school.
Total = 3

(b) 4 boys, 3 girls

(i) P(boys, girls alternate)

= P(B G B G B G B)

$=\dfrac{4!\times 3!}{7!}$ 1

$=\dfrac{1}{35}$ or 0.029 (3 d.p.) 1

Note : 4 boys can be arranged in 4! ways, and the girls can be arranged in the spaces between the boys in 3! ways. Total number of possible arrangements of the seven people is 7!.
Total = 2

(ii) P(two of the girls occupy end positions)

$=\dfrac{3\times 2\times 5!}{7!}$ 1

$=\dfrac{3\times 2}{7\times 6}$

$=\dfrac{1}{7}$ or 0.143 (3 d.p.) 1

Note : G_ _ _ _ _G
The first position can be filled in 3 ways (from 3 girls), the last position in 2 ways (from the 2 remaining girls), and the rest of the positions in 5! ways (using the 5 remaining people).
Total = 2

(c) (i) No. of arrangements $=5!$ 1

$=120$ 1

Note : n different objects can be arranged in a circle in $(n-1)!$ ways.
Total = 2

(ii) No. of arrgts. with mother, father together

$=4!\times 2$ 1

$=48$ 1

Note :
Either : Seat mother anywhere, father can sit in either of 2 seats (next to mother) and 4 children arrange themselves in 4 remaining seats in 4! ways.
Or : Consider mother and father as one object. 5 objects can be arranged in 4! ways, but mother and father can swap positions, so the number of arrangements is $4!\times 2$.
Total = 2

(iii) No. of arrgts. with 3 youngest together

$=3!\times 3!$ 1

$=36$ 1

Note : Consider the 3 youngest as one object. 4 objects can be arranged in a circle in 3! ways. But the 3 youngest can be arranged in 3! ways in their group. So, the number of arrangements is $3!\times 3!$.
Total = 2

(d) M A M M O T H

(i) No. of arrgts. $=\dfrac{7!}{3!}$ 1

$=840$ 1

Note : Divide by 3! since there are three Ms.
Total = 2

(ii) No. of arrgts. using 5 letters

= no. of arrgts. with one M

+ no. of arrgts. with two M's

+ no. of arrgts. with three M's

No. of arrgts. with one M = 5!

$=120$ 1

Note : All letters are different and all the other letters are used.

No. of arrgts. with two M's $=\dfrac{\binom{4}{3}\times 5!}{2!}$

$=240$ 1

Note : If there are two M's, only 3 of the letters A, O, T, H can be used, and these can be chosen in $\binom{4}{3}$ ways. We divide by 2! because of the two M's.

No. of arrgts. with three M's $=\dfrac{\binom{4}{2}\times 5!}{3!}$

$=120$ 1

Note : If there are three M's, only 2 of the letters A, O, T, H can be used, and these can be chosen in $\binom{4}{2}$ ways. We divide by 3! because of the three M's.

Total no. of arrangements = 480. 1 Total = 4

QUESTION 4

(a) $v = \sqrt{8x+1}$

Note : $\int (ax+b)^n dx = \frac{1}{n+1} \times \frac{1}{a}(ax+b)^{n+1} + C$

$\frac{dx}{dt} = \sqrt{8x+1}$

$\frac{dt}{dx} = \frac{1}{\sqrt{8x+1}}$ 1

$= (8x+1)^{-\frac{1}{2}}$

$t = 2 \times \frac{1}{8}(8x+1)^{\frac{1}{2}} + C$ 1

When $t=0,\ x=0$

$\therefore\ 0 = \frac{1}{4} \times 1 + C$

$C = -\frac{1}{4}$ 1

$\therefore\ t = \frac{1}{4}\sqrt{8x+1} - \frac{1}{4}$

$4t+1 = \sqrt{8x+1}$

$16t^2 + 8t + 1 = 8x + 1$ *Note :* Square both sides.

$8x = 16t^2 + 8t$

$x = 2t^2 + t$ 1 Total = 4

Alternative solution :

If $x = 2t^2 + t$ is a solution,

$v = ut + 1$ $v = \sqrt{8x+1}$

$= \sqrt{8(2t^2+t)+1}$

$= \sqrt{16t^2+8t+1}$

$= \sqrt{(4t+1)^2}$

$= ut + 1$

Since both expressions for v are the same,

$x = 2t^2 + t$.

(b) (i) $\ddot{x} = 9$

Note : Since we require velocity in terms of displacement, we must use the expression for acceleration: $\ddot{x} = \frac{d}{dx}(\frac{1}{2}v^2)$, so that we integrate with respect to x. [NB. Using $\ddot{x} = \frac{dv}{dt}$ would lead to $v = at + C$.]

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 9$ 1

$\frac{1}{2}v^2 = 9x + C$ 1

When $x = 6,\ v = 12$: *Note :* Substituting data given in the question.

$\frac{1}{2} \times 12^2 = 9 \times 6 + C$

$72 = 54 + C$

$C = 18$ 1

$\therefore\ \frac{1}{2}v^2 = 9x + 18$

$v^2 = 18x + 36$

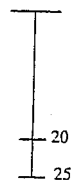
$v = \pm\sqrt{18x+36}$ 1 Total = 4

(ii) When $x = 0,\ v = \pm\sqrt{36}$

Velocity is ± 6 metres/second. 1

Note : The particle may have initially been moving in a negative direction, hence two answers are possible. Once it starts to move in a positive direction, it remains in a positive direction, as its acceleration is a positive constant.

QUESTION 5



(a) (i) Period = 4 seconds 1

Note : Period is the time taken to complete one cycle. Data in the question states that it completes $\frac{1}{4}$ cycle in 1 second.

(ii) Period $T = \frac{2\pi}{n}$

$4 = \frac{2\pi}{n}$

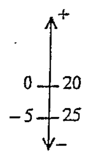
$n = \frac{\pi}{2}$ 1

$\ddot{x} = -n^2 x$ *Note :* This is the equation that describes S.H.M.

$\ddot{x} = -\left(\frac{\pi}{2}\right)^2 \times -5$ 1

Acceleration is $\frac{5\pi^2}{4}$ m/s². 1

Note : At instant of release, $x = -5$: x is the distance from the centre of oscillation (which is the point 20 cm below the top).



Total = 3

(iii) Using $v^2 = n^2\left(a^2 - x^2\right)$, $n = \frac{\pi}{2}$, $a = 5$, $x = 0$

Note : $v^2 = n^2(a^2 - x^2)$ is a formula you should learn for S.H.M. It applies to oscillations about $x = 0$.
a is the amplitude of motion.

$$v^2 = \frac{\pi^2}{4}(25 - 0) \qquad 1$$

$$v = \pm\frac{5\pi}{2}$$

In equilibrium position, speed is $\frac{5\pi}{2}$ cm/s. 1 Total = 2

(b) $v^2 = 4\left(3 + 2x - x^2\right)$

(i) $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$$= \frac{d}{dx}2\left(3 + 2x - x^2\right) \qquad 1$$

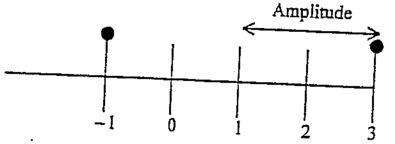
$$= 2(2 - 2x)$$

$$= -4(x - 1) \qquad 1 \qquad \text{Total} = 2$$

(ii) Centre of motion is $x = 1$ 1

(iii) When $v = 0$, $3 + 2x - x^2 = 0$

Note : Velocity is zero at the endpoints of motion.



$$x^2 - 2x - 3 = 0$$

$$(x - 3)(x + 1) = 0$$

$$x = 3 \;,\; x = -1 \qquad 1$$

Amplitude is half the distance between the end points.

Amplitude is 2 units. 1 Total = 2

(iv) $n^2 = 4$, $\therefore n = 2$ $(n > 0)$

Note : Comparing $\ddot{x} = -4(x - 1)$ with $\ddot{x} = -n^2x$.
n is always taken as positive.

Period $= \frac{2\pi}{n} = \frac{2\pi}{2}$

Period $= \pi$ 1

(v) Maximum speed occurs at centre of motion $x = 1$

$$v^2 = 4(3 + 2 - 1) \qquad 1$$

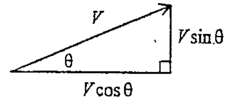
$$v = \pm 4$$

Maximum speed is 4 units / time unit. 1 Total = 2

QUESTION 6

(a) Initially, $\dot{x} = 200\cos 30° = 100\sqrt{3}$

Note : Projectile, projected at speed V, at an angle θ to the horizontal, has initial components of velocity of:
Horizontal : $V\cos\theta$
Vertical : $V\sin\theta$.



$$\dot{y} = 200\sin 30° = 100$$

$$x = 0, \quad y = 0$$

Consider motion in the x-direction:

$$\ddot{x} = 0$$

Integrate: $\dot{x} = C$

When $t = 0$, $\dot{x} = 100\sqrt{3}$, $\therefore C = 100\sqrt{3}$

$\therefore \dot{x} = 100\sqrt{3}$ 1

Integrate: $x = 100\sqrt{3}\,t + C'$

When $t = 0, x = 0$, $\therefore C' = 0$

$\therefore x = 100\sqrt{3}\,t$ 1

Consider motion in the y-direction:

$$\ddot{y} = -10$$

Integrate: $\dot{y} = -10t + K$

When $t = 0$, $\dot{y} = 100$, $\therefore K = 100$

Note : Four different symbols for constants of integration have been used : C, C', K, K'. You could use C_1, C_2, C_3, C_4 or other groups.

$\therefore \dot{y} = 100 - 10t$ 1

Integrate: $y = 100t - 5t^2 + K'$

When $t = 0$, $y = 0$, $\therefore K' = 0$

$\therefore y = 100t - 5t^2$ 1 Total = 4

(b) When $y = 0$, $100t - 5t^2 = 0$ 1

Note : The projectile is airborne until it strikes the ground, i.e. when $y = 0$.

$$5t(20 - t) = 0$$

$$t = 0, \quad t = 20$$

The projectile is airborne for 20 seconds. 1 Total = 2

(c) Greatest height reached when $\dot{y}=0$

$100-10t=0$

$t=10$ 1

When $t=10$, $y=100\times 10-5\times 10^2$

$=500$

Greatest height is 500 metres. 1 Total = 2

(d) When $t=20$, $x=100\sqrt{3}\times 20$

$=2000\sqrt{3}$

Projectile strikes ground $2000\sqrt{3}$ metres from launch site. 2

Note : Projectile strikes ground after 20 seconds [see (b)].

(e) When $x=2000$, $2000=100\sqrt{3}\,t$

$t=\dfrac{20}{\sqrt{3}}$ 1

$y=100\times\dfrac{20}{\sqrt{3}}-5\times\dfrac{400}{3}$

$y\approx 488$

Note : Substitute $t=\dfrac{20}{\sqrt{3}}$ into $y=100-5t^2$.

Height is 488 metres (nearest metre) 1 Total = 2

(f) When $t=15$, $\dot{x}=100\sqrt{3}$, $\dot{y}=-50$

$v=\sqrt{(\dot{x})^2+(\dot{y})^2}$ 1

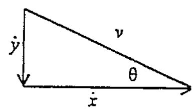
$=\sqrt{30000+2500}$

$=180.3$ (1 d.p.) 1

$\tan\theta=\left|\dfrac{\dot{y}}{\dot{x}}\right|=\dfrac{50}{100\sqrt{3}}$

$\theta=16.1°$ 1

Note : Velocity diagram:



Note : θ is the acute angle to the horizontal. The negative $\dot{y}$ value indicates it is moving downwards.

Speed of projectile is 180.3 m/s at an angle of 16.1° to the horizontal, moving downwards. Total = 3