INVERSE TRIGONOMETRIC FUNCTIONS

Exercises

1. Find the inverse function f^{-1} of $y = x^3$ and state its domain and range. Sketch both functions.
2. (a) Find the greatest domain over which $f(x) = (x + 1)^2 - 2$ has an inverse function.
   (b) Find the inverse function.
   (c) What is the domain of f^{-1}?
3. Show that $y = \dfrac{1}{x}$ is its own inverse. (The line $y = x$ is the axis of symmetry of the curve.)
4. (a) Find $\dfrac{dy}{dx}$ of $y = x^3$.
   (b) By writing $y = x^3$ with the subject as x in terms of y, find $\dfrac{dx}{dy}$ in terms of x.
   (c) Show that $\dfrac{dy}{dx} \cdot \dfrac{dx}{dy} = 1$.
5. Evaluate $\sin^{-1}\left(\frac{1}{2}\right)$.
6. Evaluate $\cos(\tan^{-1} 1)$.
7. (a) Write down the general solution for $\sin\theta = \dfrac{\sqrt{3}}{2}$.
   (b) Find the solution given by $n = -2$.
8. Prove that $\sin^{-1}\left(\frac{3}{5}\right) + \cos^{-1}\left(\frac{3}{5}\right) = \frac{\pi}{2}$.
9. Differentiate $\cos^{-1}\left(\dfrac{x}{2}\right)$.
10. Find the derivative of $e^{\sin^{-1} x}$.
11. Find the equation of the tangent to the curve $y = \tan^{-1} x$ at the point $\left(1, \frac{\pi}{4}\right)$.
12. Find the stationary point on the curve
    $$y = \sin(\cos^{-1} x).$$
13. Evaluate $\displaystyle\int_0^{\frac{1}{2}} \frac{dy}{\sqrt{1 - y^2}}$.
14. Find $\displaystyle\int \frac{dx}{\sqrt{1 - 4x^2}}$.
15. Find the area bounded by the curve
    $$y = (25 - x^2)^{-\frac{1}{2}},$$
    the x-axis and the lines $x = 0$ and $x = 3$ (correct to 2 significant figures).
16. The area bounded by the curve $y = \dfrac{1}{\sqrt{4 + x^2}}$, the x-axis and the lines $x = -2$ and $x = 2$ is rotated about the x-axis. Find the volume of the solid generated.

Test yourself

Revision questions

17. Evaluate $\tan\left(\sin^{-1}\frac{1}{\sqrt{2}}\right)$.

18. (a) Sketch $y = \cos^{-1} x$ and state its domain and range.
(b) Use Simpson's Rule with 3 function values to find an approximation for the area bounded by $y = \cos^{-1} x$ and the x- and y-axes in the first quadrant (in terms of π).

19. Find the inverse function of $f(x) = (x - 1)^3$ and sketch both curves.

20. Differentiate $(\tan^{-1} x)^2$.

21. Find the gradient of the tangent to the curve
$$y = \sin^{-1}\frac{2x}{5}$$
at the point where $x = 1$.

22. Evaluate $\int_{-3}^{3} -\frac{dx}{\sqrt{9 - x^2}}$.

23. Find the equation of the normal to the curve $y = \tan^{-1} x$ at the point where $x = 1$.

24. (a) State the domain and range of $y = x^2 - 5$.
(b) Find the greatest domain over which the inverse function exists.
(c) Find the inverse function over this domain and state its domain and range.

25. Evaluate $\int_0^{\frac{1}{2}} \frac{dx}{3 + 4x^2}$.

26. Show that $\cos^{-1}(-x) = \pi - \cos^{-1} x$.

Challenge questions

27. By using the substitution $u = x^3$, find
$$\int \frac{x^2}{1 + x^6}\,dx.$$

28. (a) Show that $\tan^{-1}\frac{2}{3} + \tan^{-1}\frac{3}{2} = \frac{\pi}{2}$.
(b) Find $\frac{d}{dx}(\tan^{-1} x + \tan^{-1}\frac{1}{x})$.
(c) Sketch $y = \tan^{-1} x + \tan^{-1}\frac{1}{x}$.

29. Find $\frac{d}{dx}(\sqrt{1 - x^2} + x\sin^{-1} x)$. Hence evaluate $\int_0^{\frac{1}{2}} \sin^{-1} x\,dx$ correct to 3 significant figures.

30. The acceleration of a particle is given by
$$\frac{d^2x}{dt^2} = \frac{1}{1 + x^2}.$$
If initially the velocity is 2 m s^{-1} when the particle is 1 m to the right of the origin, find its velocity when it is at the origin (correct to 1 decimal place).

31. (a) Show that
$$\frac{2x^2 + 5}{(4 + x^2)(1 + x^2)} = \frac{1}{4 + x^2} + \frac{1}{1 + x^2}.$$
(b) Hence evaluate $\int_0^1 \frac{2x^2 + 5}{(4 + x^2)(1 + x^2)}\,dx$ correct to 2 decimal places.

32. By restricting the domain to a monotonic increasing curve, find the inverse function of $y = \frac{1}{x^2 - 4}$ and state the domain and range of the inverse function.

33. Find any stationary points on the curve $y = x\sin^{-1} x$ and sketch the graph of this function.

34. Prove $\tan^{-1} x = \sin^{-1}\frac{x}{\sqrt{1 + x^2}}$ for $-1 < x < 1$.

35. Find the volume of the solid formed if $y = \sin^{-1} x$ is rotated about the y-axis from $y = 0$ to $y = \frac{\pi}{6}$.

36. Prove that $\tan^{-1} x + \tan^{-1}\frac{1}{x} = \frac{\pi}{2}$ for all x.

Angelina

INVERSE TRIG FUNCTIONS

Exercises

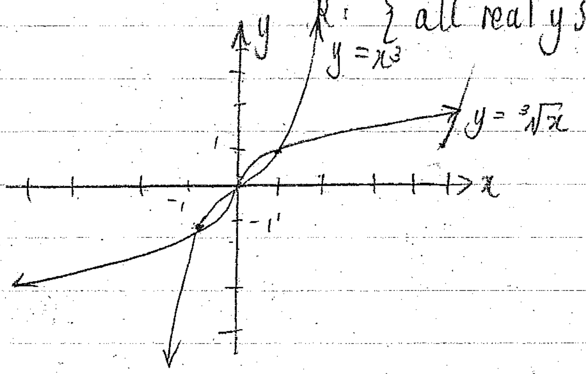
check corrections!

Qu1 $y = x^3$

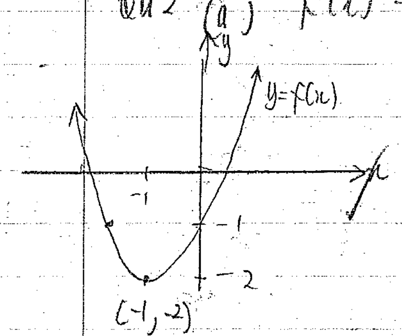
$x = y^3$ ✓

$y = \sqrt[3]{x}$ D: {all real x} ✓

R: {all real y}



Qu2 (a) $f(x) = (x+1)^2 - 2$



∴ Greatest domain occurs

$x \geq -1$ or $x \leq -1$ ✓

(b) $x = (y+1)^2 - 2$ ✓

$\sqrt{x+2} = y + 1$

$y = -1 \pm \sqrt{x+2}$

(c) D: $\{x \geq -2\}$ ✓

since for $f(x)$ R: $\{y \geq -2\}$

Qu3. $y = \frac{1}{x}$

$x = \frac{1}{y}$ ✓

$y = \frac{1}{x}$ ✓

∴ $y = \frac{1}{x}$ is same as $x = \frac{1}{y}$ ✓

∴ $y = \frac{1}{x}$ is its own inverse

Qu 4 (a) $y = x^3$

$\frac{dy}{dx} = 3x^2$ ✓

(b) $x = y^{\frac{1}{3}}$

$\frac{dx}{dy} = \frac{1}{3}y^{-\frac{2}{3}}$

$= \frac{1}{3}(x^3)^{-\frac{2}{3}}$ ✓

$= \frac{1}{3}x^{-2} = \frac{1}{3x^2}$

(c) $3x^2 \times \frac{1}{3x^2} = 1$ ✓

Qu 5 $\sin^{-1}\left(\frac{1}{2}\right) = 30°$ or $\frac{\pi}{6}$ ✓

Qu6 $\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ ✓

Qu7 (a) $\theta = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)(-1)^n + \pi n$

$= \frac{\pi}{3}(-1)^n + \pi n$

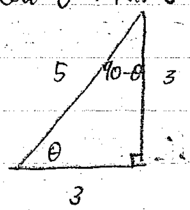
for $n \in \mathbb{Z}$

(b) when $n = -2$

$\theta = \frac{\pi}{3}(-1)^{2} \cancel{+} 2\pi$

$= \cancel{\frac{7\pi}{3}}$ $-\frac{5\pi}{3}$

Qu 8 LHS: $\sin^{-1}\left(\frac{3}{5}\right) + \cos^{-1}\left(\frac{3}{5}\right)$ ✓



$\sin\theta = \cos(90-\theta)$

∴ LHS: $\sin^{-1}\left(\frac{3}{5}\right) + \cos^{-1}\left(\frac{3}{5}\right)$

$= \theta + 90 - \theta$

$= \frac{\pi}{2}$ ✓

$=$ RHS

∴ LHS = RHS

Qu9. $\cos^{-1}\left(\frac{x}{2}\right) = \frac{-1}{\sqrt{4-x^2}}$ ✓

Qu 10 $\frac{d}{dx} e^{\sin^{-1}x} = \frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}$ ✓

Qu 11. $y = \tan^{-1} x$

$y' = \dfrac{1}{1+x^2}$

when $x = 1$, $m = \frac{1}{2}$

$\therefore y - \frac{\pi}{4} = \frac{1}{2}(x-1)$

$2y - \frac{\pi}{2} = x - 1$

$x - 2y - 1 + \frac{\pi}{2} = 0$

Qu 12. $y = \sin(\cos^{-1} x)$

let $u = \cos^{-1} x$

$\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$

$\dfrac{du}{dx} = -\dfrac{1}{\sqrt{1-x^2}}$

$\dfrac{dy}{du} = \cos u$

$\therefore \dfrac{dy}{dx} = \cos(\cos^{-1} x) \cdot -\dfrac{1}{\sqrt{1-x^2}}$

$= \dfrac{-x}{\sqrt{1-x^2}}$

for stat pt, $\dfrac{dy}{dx} = 0$

$\therefore -x = 0 \Rightarrow x = 0$

$y = 1$

$\therefore (0, 1)$ is stat pt.

Qu 13. $\displaystyle\int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-y^2}}\, dy$

$= \left[\sin^{-1} y\right]_0^{\frac{1}{2}}$

$= \dfrac{\pi}{6}$

Qu 14. $\displaystyle\int \frac{1}{\sqrt{1-4x^2}}\, dx = \frac{1}{2}\sin^{-1} 2x + C$

Qu 15. $y = \dfrac{1}{\sqrt{25-x^2}}$

radians

$A = \displaystyle\int_0^3 \frac{1}{\sqrt{25-x^2}}\, dx$

$= \left[\sin^{-1}\frac{x}{5}\right]_0^3 = 0.6435\ldots$

$= 0.64$ units^2

Qu 16. $y = \dfrac{1}{\sqrt{4+x^2}}$ $\quad y^2 = \dfrac{1}{4+x^2}$

$V = \pi \displaystyle\int_{-2}^{2} \frac{1}{4+x^2}\, dx = \pi\left[\frac{1}{2}\tan^{-1}\frac{x}{2}\right]_{-2}^{2}$

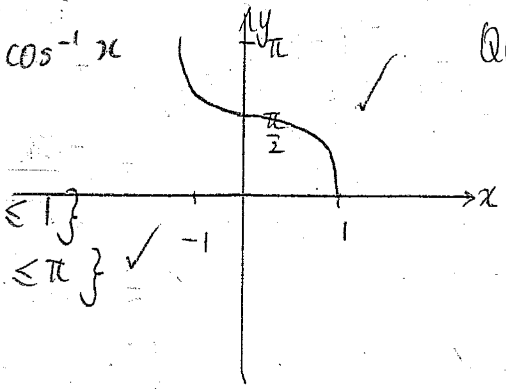
$= \dfrac{\pi}{2}\left(\tan^{-1}(1) - \tan^{-1}(-1)\right)$

$= \dfrac{\pi^2}{4}$ units^3

Revision Questions

Qu17 $\tan(\sin^{-1}\frac{1}{\sqrt{2}}) = \tan\frac{\pi}{4} = 1$ ✓

Qu18 (a) $y = \cos^{-1} x$



D: $\{-1 \le x \le 1\}$

R: $\{0 \le y \le \pi\}$ ✓

(b)

x	0	$\frac{1}{2}$	1
y	$\frac{\pi}{2}$	$\frac{\pi}{3}$	0

✓

$\therefore \int_0^1 \cos^{-1} x \, dx \approx \frac{1}{6}\left(\frac{\pi}{2} + 0 + \frac{4\pi}{3}\right)$

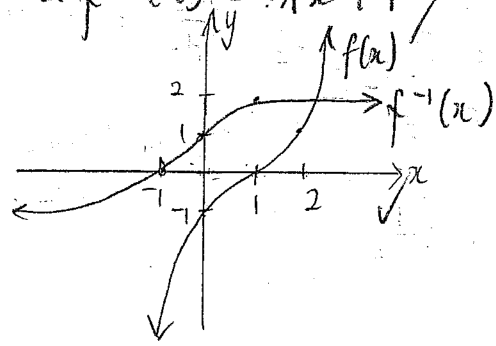
$\approx \frac{11\pi}{36}$ units^2 ✓

Qu19 $f(x) = (x-1)^3$

$x = (y-1)^3$

$\sqrt[3]{x} + 1 = y$

$\therefore f^{-1}(x) = \sqrt[3]{x} + 1$ ✓



✓

Qu20 $\frac{d}{dx}(\tan^{-1}x)^2 = 2(\tan^{-1}x) \times \frac{1}{1+x^2}$

$= \frac{2\tan^{-1}x}{1+x^2}$ ✓

Qu21 $y = \sin^{-1}\frac{2x}{5}$

$y' = \frac{1}{\sqrt{\frac{25}{4} - x^2}}$ ✓

when $x = 1$, $m = \frac{1}{\sqrt{\frac{25}{4} - 1}} = \frac{1}{\frac{\sqrt{21}}{2}}$

$= \frac{2\sqrt{21}}{21}$ ✓

Qu22 $\int_{-3}^{3} -\frac{1}{\sqrt{9-x^2}} \, dx$

$= \left[\cos^{-1}\left(\frac{x}{3}\right)\right]_{-3}^{3}$ ✓

$= \cos^{-1}(1) - \cos^{-1}(-1)$

$= 0 - \pi$

$= -\pi$ ✓

Qu23 $y = \tan^{-1}x$

$y' = \frac{1}{1+x^2}$ ✓

when $x = 1$, $m = \frac{1}{2}$

$\therefore m_\perp = -2$

$\therefore y - \frac{\pi}{4} = -2(x-1)$

$y = -2x + 2 + \frac{\pi}{4}$ ✓

Qu 24 (a) D: {all real x} ✓

R: $\{y \ge -5\}$ ✓

(b) $x \ge 0$ ✓

(c) $x = y^2 - 5$

$\sqrt{x+5} = y$ D: $\{x \ge -5\}$ ✓

R: $\{y \ge 0\}$ ✓

Qu25. $\int_0^{\frac{1}{2}} \frac{1}{3+4x^2} \, dx$

$= \left[\frac{1}{2\sqrt{3}}\tan^{-1}\frac{2x}{\sqrt{3}}\right]_0^{\frac{1}{2}}$ ✓

$= \frac{1}{2\sqrt{3}}\tan^{-1}\frac{1}{\sqrt{3}} - 0$

$= \frac{\pi}{12\sqrt{3}}$ ✓

Qu 26. Let $y = \cos^{-1}(-x)$

$\cos y = -x$

$\cos(\pi - y) = -x$

$\therefore \pi - y = \cos^{-1}(-x)$

$\therefore \pi - \cos^{-1}(-x) = \cos^{+}(-x)$

Qu 27. $\frac{dy}{dx} = 3x^2$

$dx = \frac{du}{3x^2}$

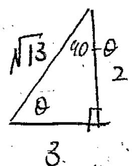
$\therefore \int \frac{x^2}{1+u^2} \frac{du}{3x^2}$

$= \frac{1}{3}\int \frac{1}{1+u^2}\,du$

$= \frac{1}{3}\tan^{-1} u + c$

$= \frac{1}{3}\tan^{-1} x^3 + C$

Qu 28 (a)



Let $\tan^{-1}\frac{2}{3} = \theta$

$\tan\theta = \frac{2}{3}$

$\tan(90-\theta) = \frac{3}{2}$

$\tan^{-1}\frac{3}{2} = \frac{\pi}{2} - \theta$

$\therefore \tan^{-1}\frac{2}{3} + \tan^{-1}\frac{3}{2} = \theta + \frac{\pi}{2} - \theta$

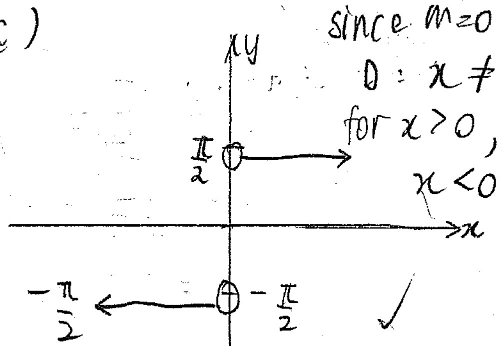
$= \frac{\pi}{2}$

(b) $\frac{1}{1+x^2} + \frac{1}{1+\left(\frac{1}{x}\right)^2} \times -\frac{1}{x^2}$

$= \frac{1}{1+x^2} + \frac{-1}{x^2+1}$

$= 0$

(c)



since $m=0$, must be horizontal

$D: x \neq 0$

for $x > 0$, $y = \frac{\pi}{2}$

$x < 0$, $y = -\frac{\pi}{2}$

Qu 29 $\frac{d}{dx}\left(\sqrt{1-x^2} + x\sin^{-1}x\right)$

$= \frac{1}{2}(1-x^2)^{-\frac{1}{2}} \times -\cancel{2}x + \sin^{-1}x + \frac{x}{\sqrt{1-x^2}}$

$= \frac{-\cancel{2}x}{\cancel{2}\sqrt{1-x^2}} + \frac{x}{\sqrt{1-x^2}} + \sin^{-1}x$

$= \sin^{-1}x$

$\int_0^{-\frac{1}{2}} \sin^{-1}x\,dx =$

$\left[\sqrt{1-x^2} + x\sin^{-1}x\right]_0^{\frac{1}{2}}$

$= \left(\sqrt{1-\frac{1}{4}} + \frac{1}{2}\sin^{-1}\frac{1}{2}\right) - (1)$

$= \frac{\sqrt{3}}{2} + \frac{\pi}{12} - 1$

$= 0.128$ (to 3sf)

Qu 30 $u = 2m/s$

$v = u + at$

$\frac{dx}{dt} = \tan^{-1}x$ ← velocity

Haven't done motion

Qu 31 (a) $\frac{2x^2+5}{(4+x^2)(1+x^2)} = \frac{1}{4+x^2} + \frac{1}{1+x^2}$

RHS: $\frac{1+x^2+4+x^2}{(4+x^2)(1+x^2)} = \frac{2x^2+5}{(4+x^2)(1+x^2)}$ ✓

$= LHS$

$\therefore LHS = RHS$

(b) $\int_0^1 \frac{1}{4+x^2}\,dx + \int_0^1 \frac{1}{1+x^2}\,dx$ ✓

$= \left[\frac{1}{2}\tan^{-1}\frac{x}{2}\right]_0^1 + \left[\tan^{-1}x\right]_0^1$

$= \frac{1}{2}\tan^{-1}\frac{1}{2}$ / $+ \tan^{-1}1$

$= 1.02$ /

Qu 33 $y = x\sin^{-1}x$

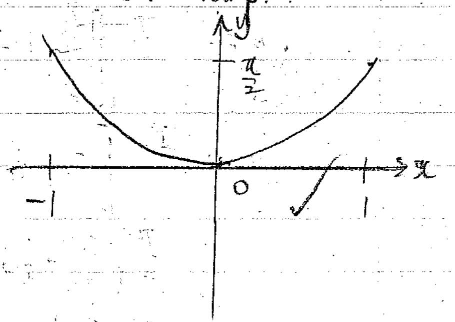
$y' = \sin^{-1}x + \frac{x}{\sqrt{1-x^2}}$

when $y' = 0$,

$\sin^{-1}x + \frac{x}{\sqrt{1-x^2}} = 0$

when $x = 0$, $y' = 0$ ✓

$\therefore$ stat pt



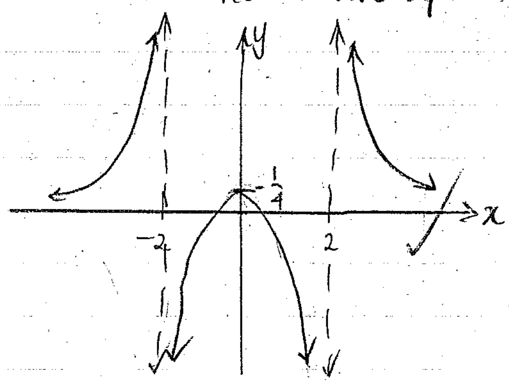
Qu 32 $y = \frac{1}{x^2-4}$

D: $x \neq \pm 2$ $\therefore$ VA at $x = \pm 2$

HA: $y = 0$

when $x = 0$, $y = -\frac{1}{4}$

no x intercepts



as $x \to -2^-$, $y \to \infty$

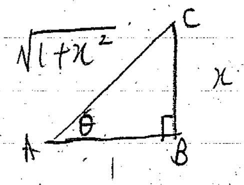
as $x \to 2^+$, $y \to \infty$ /

as $x \to -2^+$, $y \to -\infty$

$x \to 2^-$, $y \to -\infty$

Qu 34 Let $\theta = \sin^{-1}\frac{x}{\sqrt{1+x^2}}$

$\sin\theta = \frac{x}{\sqrt{1+x^2}}$ /



$\therefore AB^2 = 1 + x^2 - x^2$ ✓

$= 1$

$AB = 1$ (since +ve value)

$\therefore \tan\theta = \frac{x}{1}$

$= x$

$\therefore \tan^{-1}x = \theta$ /

$\therefore \tan^{-1}x = \sin^{-1}\frac{x}{\sqrt{1+x^2}}$

monotonic when $x < 0$, where $x \neq -2$ since $f'(x) > 0$

$x = \frac{1}{y^2-4}$

$xy^2 - 4x = 1$ ✗

$x(y^2 - 4) = 1$

$y^2 - 4 = \frac{1}{x}$

$y^2 = \frac{1}{x} + 4$

$\therefore y = \pm\sqrt{\frac{1+4x}{x}}$

since $y < 0$ but $y \neq -2$ ✓

inverse:

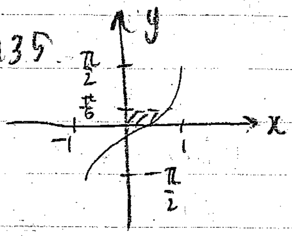
$y = -\sqrt{\frac{1+4x}{x}}$

for inverse

$y < 0$, $x \leq -\frac{1}{4}$

$y < -2$, $x > 0$

Qu35.



$V = \pi \int_0^{\frac{\pi}{6}} x^2 \, dy$

$y = \sin^{-1} x$

$\sin y = x$

$x^2 = \sin^2 y$

$\therefore V = \pi \int_0^{\frac{\pi}{6}} \sin^2 y \, dy$

$\cos 2x = 1 - 2\sin^2 x$

$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

$= \frac{\pi}{2} \int_0^{\frac{\pi}{6}} 1 - \cos 2y \, dy$

$= \frac{\pi}{2} \left[x - \frac{1}{2} \sin 2y \right]_0^{\frac{\pi}{6}}$

$= \frac{\pi}{2} \left(\frac{\pi}{6} - \frac{1}{2} \sin \frac{\pi}{3} - 0 \right)$

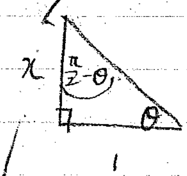
$= \frac{\pi}{2} \left(\frac{\pi}{6} - \frac{\sqrt{3}}{4} \right)$

$= \left(\frac{\pi^2}{12} - \frac{\pi\sqrt{3}}{8} \right)$ units^3

Qu36. $\tan^{-1} x + \tan^{-1} \frac{1}{x} = \frac{\pi}{2}$

Let $\theta = \tan^{-1} x$

$\tan\theta = x$



$\tan\theta = \cot\left(\frac{\pi}{2} - \theta\right)$

$= \frac{1}{\tan\left(\frac{\pi}{2} - \theta\right)}$

$\tan\left(\frac{\pi}{2} - \theta\right) = \frac{1}{x}$

$\tan^{-1}\left(\frac{1}{x}\right) = \frac{\pi}{2} - \theta$

$\therefore LHS = \theta + \frac{\pi}{2} - \theta$

$= \frac{\pi}{2} = RHS$

$\therefore LHS = RHS$