EXERCISES:

(1)

(i) Sketch a graph of $y = \frac{1}{x^2+1}$, showing its stationary point and stating its range.

(ii) A function $f(x)$ is defined by $f(x) = \frac{1}{x^2+1}$, $x \geq 0$.

Explain why $f(x)$ has an inverse function and state the domain of $f^{-1}(x)$.

(iii) On your graph in (i), sketch $y = x$ and $y = f^{-1}(x)$.

(iv) Find $f^{-1}(x)$.

(2)

(a) Sketch the graph of $y = \frac{2x}{x^2+1}$, indicating the coordinates of its stationary points.

(b) A function, $f(x)$, is defined by $f(x) = \frac{2x}{x^2+1}$, $-1 \leq x \leq 1$.

(i) Use your graph in (a) to explain why $f(x)$ has an inverse function, $f^{-1}(x)$.

(ii) On one diagram, using equal scales on both axes, sketch graphs of $y = f(x)$, $y = x$ and $y = f^{-1}(x)$.

(iii) Calculate the total area of the two regions enclosed by $y = f(x)$ and $y = f^{-1}(x)$.

(iv) Evaluate $\int_0^1 f^{-1}(x)\ dx$.

(v) Solve $f^{-1}(x) = \frac{1}{2}$.

(3)

$$f(x) = x - \frac{1}{x}, \qquad x > 0$$

(a) Show that $f(x)$ has no stationary points.

(b) Describe the behaviour of $f(x)$ as x approaches the extremities of its domain.

(c) Explain why $f(x)$ has an inverse function, $f^{-1}(x)$.

(d) Sketch, on the one diagram, graphs of:

(i) $y = x$

(ii) $y = f(x)$

(iii) $y = f^{-1}(x)$

(e) If $x = y - \dfrac{1}{y}$ and $y > 0$, simplify, in terms of y,

$$x + \sqrt{x^2 + 4}.$$

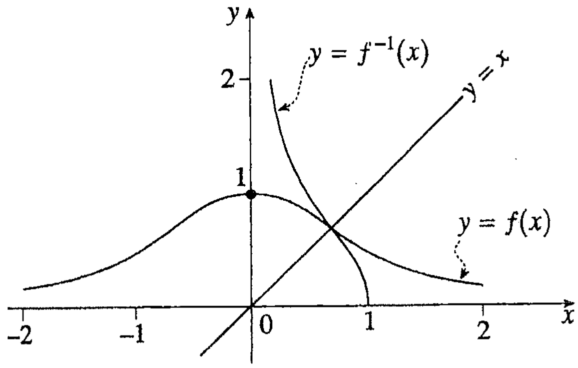
(f) What is $f^{-1}(x)$?

SOLUTIONS:

(1)

(i) $y = (x^2+1)^{-1}$. $\frac{dy}{dx} = -\frac{2x}{(x^2+1)^2}$

Stationary at $(0, 1)$.



$0 < f(x) \le 1$

(ii) $f(x)$ is monotonic; $0 < x \le 1$

(iii) See part (i)

(iv) $x = \frac{1}{y^2+1}$, $y \ge 0$

$\frac{1}{x} = y^2+1$, $y \ge 0$

$y^2 = \frac{1}{x} - 1 = \frac{1-x}{x}$, $y \ge 0$

$y = \sqrt{\frac{1-x}{x}}$, since $y \ge 0$

(2)

(a) $\frac{dy}{dx} = \frac{(x^2+1)2-(2x)(2x)}{(x^2+1)^2}$

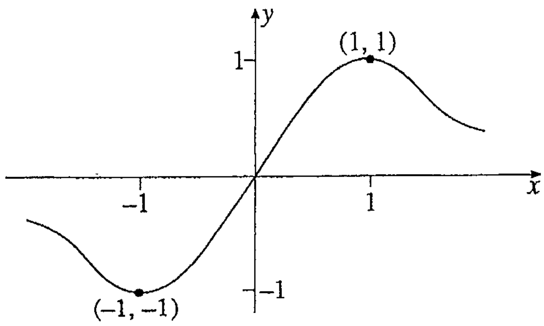
$= \frac{2-2x^2}{(x^2+1)^2}$

When stationary, $2-2x^2 = 0$

$x^2 = 1$

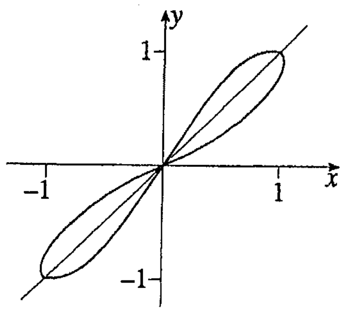
$x = 1$ and $x = -1$

$y = 1$ and $y = -1$



(b) (i) For $-1 \le x \le 1$, $\frac{2x}{x^2+1}$ is monotonic.

(ii)



(iii) Area of region between $y = f(x)$ and $y = x$ for $0 \le x \le 1$

$$= \int_0^1 \left(\frac{2x}{x^2+1} - x \right) dx$$

$$= \left[\ln\left(x^2+1\right) - \frac{x^2}{2} \right]_0^1$$

$$= \ln 2 - \frac{1}{2}$$

Total area $= 4\left(\ln 2 - \frac{1}{2}\right)$

$= 4 \ln 2 - 2$

(iv) $\int_0^1 f^{-1}(x)\, dx = \int_0^1 x\, dx - \left(\ln 2 - \frac{1}{2}\right)$

$= \frac{1}{2} - \left(\ln 2 - \frac{1}{2}\right)$

$= 1 - \ln 2$

(v) $x = f\left(\frac{1}{2}\right)$

$$= \frac{1}{1 + \frac{1}{4}}$$

$$= \frac{4}{5}$$

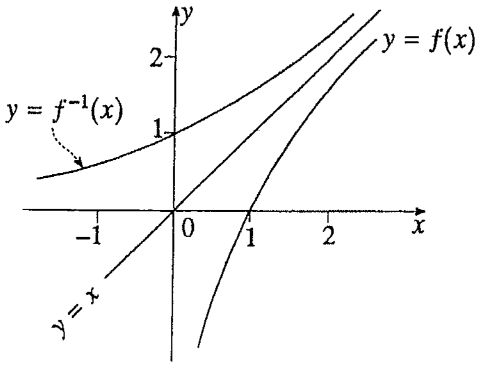
(3)

(a) $f'(x) = 1 + \frac{1}{x^2}$, which is positive for all x.

(b) As $x \to 0$, $f(x)$ will decrease without limit.
As x increases without limit,
$f(x)$ will 'approach' x.

(c) $f(x)$ is monotonic increasing $\left(f'(x) > 0\right)$.

(d)



(e) $x+\sqrt{x^2+4} = y-\frac{1}{y}+\sqrt{\left(y-\frac{1}{y}\right)^2+4}$

$= y-\frac{1}{y}+\sqrt{y^2+2+\frac{1}{y^2}}$

$= y-\frac{1}{y}+\sqrt{\left(y+\frac{1}{y}\right)^2}$

$= y-\frac{1}{y}+y+\frac{1}{y}$, since $y>0$

$= 2y$

(f) $f^{-1}(x) = y$

$= \frac{1}{2}\left(x+\sqrt{x^2+4}\right)$