CRANBROOK SCHOOL

YEAR 12 EXT1 MATHEMATICS – TEST

3^{rd} June, 2004 Circle teacher: **CJL JJA SKB HRK**

– Trigonometric functions (non-calculus and calculus) **Time: 50mins**
- Inverse functions (non-calculus and calculus)

All necessary working should be shown in every question.
Full marks may not be awarded if work is careless or badly arranged.
Approved silent calculators may be used.
Begin each question on a new page.

1. **(12 marks) (Begin a new page)** **JJA**

(a) i. Simplify $\cos(\pi + \theta)$ **1**

ii. Hence or otherwise, solve for $\cos(\pi + \theta) = \frac{\sqrt{3}}{2}$ for $0 \le x \le 2\pi$ **2**

(b) Solve $\sin^3 x - \sin x = 0$, for $0 \le x \le 2\pi$ **2**

(c) The area of the sector of a circle is $8\pi\ cm^2$ and the length of the arc bounded by this sector is $\frac{\pi}{4}$ *cm*. Find the radius of the circle and the angle that is subtended at the centre. **2**

(d) i. Evaluate: $\lim\limits_{x \to 0} \frac{5\sin x}{3x}$ **1**

ii. Evaluate: $\lim\limits_{\theta \to 0} \frac{\tan\frac{\theta}{7}}{\theta}$ **1**

(e) Sketch $y = 4\sin(2x - \frac{\pi}{2})$ from $-\pi \le x \le \pi$ **3**

2. **(12 marks) (Begin a new page)** **HRK**

(a) Evaluate $\int_0^1 \sqrt{1 - x^2}\,dx$ using the substitution $x = \sin\theta$. **4**

(b) At any point on the curve $y = f(x)$ the gradient function is given by $\frac{dy}{dx} = 2\sin^2 2x + 1$.
If $y = 1$ when $x = \pi$, find the value of y when $x = 2\pi$. **4**

(c) For the curve $y = 1 + 2\sin x - 2\sin^2 x$, show $\frac{dy}{dx} = 2\cos x(1 - 2\sin x)$.
Hence find the stationary points in the interval $0 \le x \le \frac{\pi}{2}$.
Sketch the curve in this interval. **4**

3. **(12 marks) (Begin a new page)** **SKB**

(a) Find the inverse function $f^{-1}(x)$ if $f(x)=\sqrt{x-3}, x\geq 3$. Hence sketch $y=f^{-1}(x)$ stating its domain and range. **3**

(b) Find, showing all necessary working, the exact value of $\cos(2\tan^{-1}\frac{1}{3})$. **3**

(c) Sketch $y=\cos^{-1}(\sin 2x)$ for $-\pi\leq x\leq\pi$. **3**

(d) Find the general solutions of $2\sin^2 x+\sin x-1=0$ **3**

4. **(12 marks) (Begin a new page)** **CJL**

(a) Find the exact equation of the tangent to the curve $y=\sin^{-1}\sqrt{x}$ at the point where $x=\frac{1}{2}$. **3**

(b) (i) Differentiate $x\cos^{-1}x-\sqrt{1-x^2}$ **4**

(ii) Hence evaluate $\int_0^1 \cos^{-1}x\, dx$

(c) Without evaluating the integral, explain why $\int_{-\frac{1}{2}}^{\frac{1}{2}} \sin^{-1}x\, dx$ is equal to zero. **2**

(d) Evaluate $\int_0^{\frac{1}{\sqrt{5}}} \frac{4}{3+5x^2}\, dx$. **3**

YR 12 EXTENSION 1, TEST 3-6-04

1 (a) (i) $\cos(\pi+\theta) = -\cos\theta$

(ii) $\cos(\pi+\theta) = \frac{\sqrt{3}}{2}$ $[0, 2\pi]$

$\therefore -\cos\theta = \frac{\sqrt{3}}{2}$

$\therefore \cos\theta = -\frac{\sqrt{3}}{2}$

Basic angle $\theta = \frac{\pi}{6}$ (require 2nd, 3rd quads)

$\therefore \theta = \pi - \frac{\pi}{6}$ or $\pi + \frac{\pi}{6}$

$\therefore \theta = \frac{5\pi}{6}$ or $\frac{7\pi}{6}$

(b). $\sin^3 x - \sin x = 0$ $[0, 2\pi]$

$\therefore \sin x(\sin^2 x - 1) = 0$

$\therefore \sin x = 0$ or $\sin x = \pm 1$

$\therefore x = 0, \pi, 2\pi$ or $\frac{\pi}{2}, \frac{3\pi}{2}$

ie $x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$.

(c) $8\pi = \frac{1}{2}r^2\theta$ — (1)

$\frac{\pi}{4} = r\theta$ — (2)

(1) ÷ (2): $32 = \frac{1}{2}r$

$\therefore r = 64$ sub into (2)

$\therefore \frac{\pi}{4} = 64\theta$

$\therefore \theta = \frac{\pi}{256}$

ie radius is 64 cm and angle subtended at centre is $\frac{\pi}{256}$.

(d) (i) $\lim_{x\to 0} \frac{5\sin x}{3x}$

$= \lim_{x\to 0} \frac{\sin x}{x} \cdot \frac{5}{3}$

$= 1 \cdot \frac{5}{3}$

$= \frac{5}{3}$.

(ii) $\lim_{\theta\to 0} \frac{\tan\frac{\theta}{7}}{\theta}$

$= \lim_{\theta\to 0} \frac{\tan\frac{\theta}{7}}{\frac{\theta}{7}} \cdot \frac{1}{7}$

$= 1 \cdot \frac{1}{7}$

$= \frac{1}{7}$

(e) $y = 4\sin(2x - \frac{\pi}{2})$

$\therefore y = 4\left[\sin 2x \cos\frac{\pi}{2} - \cos 2x \sin\frac{\pi}{2}\right]$

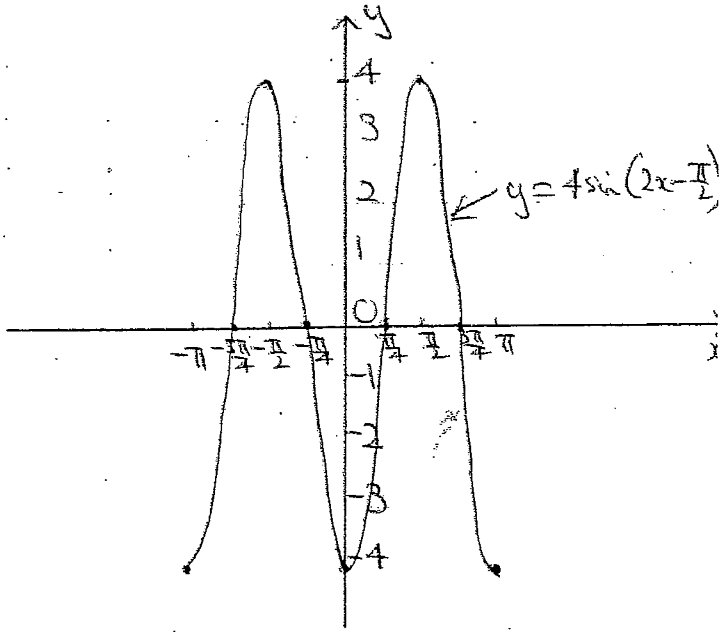
$\therefore y = -4\cos 2x$

Period $= \frac{2\pi}{2} = \pi$

Amplitude $= 4$ units

Range is: $-4 \le y \le 4$

sub-interval width $= \frac{\pi}{4}$



2. (a) $I = \int_0 \sqrt{1-x^2}\, dx$

let $x = \sin\theta$ when $x = 0$ $\theta = 0$

$\therefore \frac{dx}{d\theta} = \cos\theta$ $x = 1$ $\theta = \frac{\pi}{2}$

$\therefore I = \int_0^{\frac{\pi}{2}} \sqrt{1-\sin^2\theta} \cdot \cos\theta \, d\theta$

$= \int_0^{\frac{\pi}{2}} \cos^2\theta \, d\theta$

$= \frac{1}{2}\int_0^{\frac{\pi}{2}} 1 + \cos 2\theta \, d\theta$

$\therefore I = \frac{1}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{2}}$

$= \frac{1}{2}\left[\left(\frac{\pi}{2}+0\right)-(0+0)\right]$

$= \frac{\pi}{4}$

(b) $\frac{dy}{dx} = 2\sin^2 2x + 1$

$\therefore y = \int 2\sin^2 2x + 1 \; dx$

$= \int 2\left[\frac{1}{2}[1-\cos 4x]\right]+1 \; dx$

$= \int 2-\cos 4x \; dx$

$= 2x - \frac{\sin 4x}{4} + c$

when $y=1$, $x=\pi$

$\therefore 1 = 2\pi - 0 + c$

$\therefore c = 1-2\pi$

$\therefore y = 2x - \frac{\sin 4x}{4} + 1 - 2\pi$

when $x = 2\pi$ $\quad y = 4\pi - 0 + 1 - 2\pi$

$\therefore y = 2\pi + 1$

(c) $y = 1 + 2\sin x - 2\sin^2 x$

$\therefore \frac{dy}{dx} = 2\cos x - 4\sin x \cdot \cos x$

$= 2\cos x(1-2\sin x)$

For a stat. pt $\frac{dy}{dx}=0$ and $[0, \frac{\pi}{2}]$

$\therefore \cos x = 0$ or $\sin x = \frac{1}{2}$

$\therefore x = \frac{\pi}{2}$ or $x = \frac{\pi}{6}$

when $x=\frac{\pi}{2}$

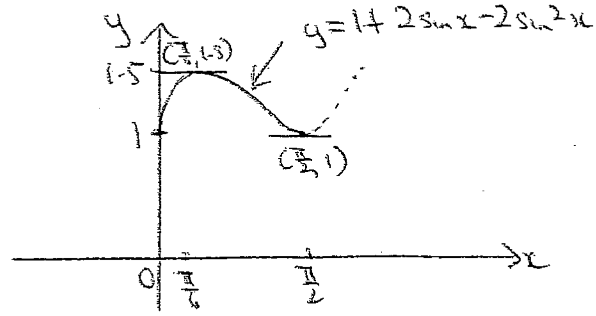
x	$\frac{\pi}{2}^-$	$\frac{\pi}{2}$	$\frac{\pi}{2}^+$
$f'(x)$	$-$	0	$+$

$\Rightarrow$ min turn pt at $(\frac{\pi}{2}, 1)$

when $x=\frac{\pi}{6}$

x	$\frac{\pi}{6}^-$	$\frac{\pi}{6}$	$\frac{\pi}{6}^+$
$f'(x)$	$+$	0	$-$

$\Rightarrow$ max. turn. pt at $(\frac{\pi}{6}, 1.5)$



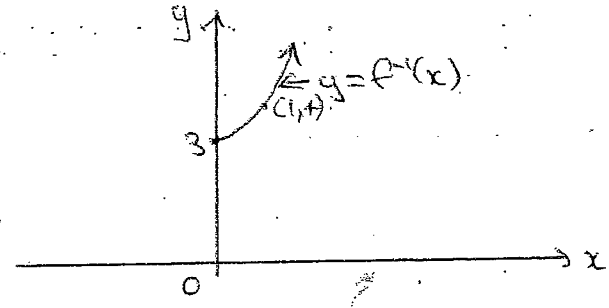
(3) (a) let $y = \sqrt{x-3}$, $x \geq 3$, $y \geq 0$

For an inverse fn interchange x for y

$\therefore x = \sqrt{y-3}$

$\therefore x^2 + 3 = y$

$\therefore f^{-1}(x) = x^2+3$, $x \geq 0$, $y \geq 3$.



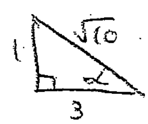
ie Domain is: $x \geq 0$

Range is: $y \geq 3$

(b) Let $E = \cos(2\tan^{-1}\frac{1}{3})$

Let $\alpha = \tan^{-1}\frac{1}{3}$

$\therefore \tan\alpha = \frac{1}{3}$



$\therefore E = \cos 2\alpha$

$= \cos^2\alpha - \sin^2\alpha$

$= \left(\frac{3}{\sqrt{10}}\right)^2 - \left(\frac{1}{\sqrt{10}}\right)^2$

$= \frac{4}{5}$

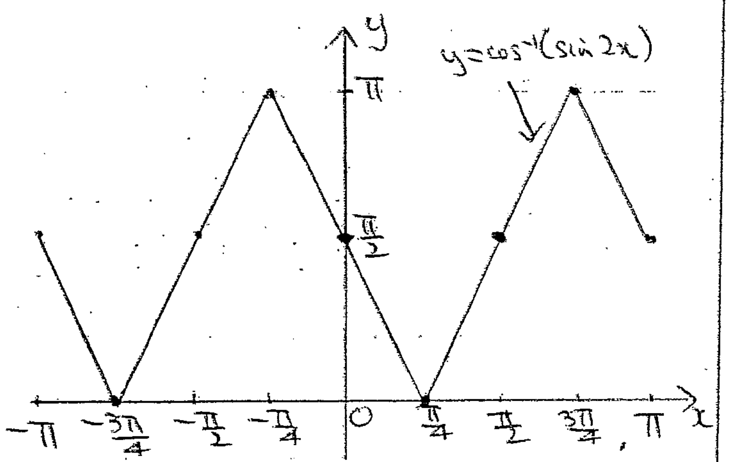
(c) $y = \cos^{-1}(\sin 2x)$ $\quad -\pi \le x \le \pi$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-\sin^2 2x}} \cdot 2\cos 2x$$

$$= \frac{-2\cos 2x}{|\cos 2x|}$$

$= -2$ if $|\cos 2x| = \cos 2x$
if $\cos 2x > 0$

or 2 if $|\cos 2x| = -\cos 2x$
if $\cos 2x < 0$



(d) $2\sin^2 x + \sin x - 1 = 0$

$(2\sin x - 1)(\sin x + 1) = 0$

$\therefore \sin x = \frac{1}{2}$ or $\sin x = -1$

$\therefore \sin x = \sin\frac{\pi}{6}$ or $\sin x = \sin\left(-\frac{\pi}{2}\right)$

$\therefore x = n\pi + (-1)^n \cdot \frac{\pi}{6}$ or $n\pi + (-1)^n \cdot \left(-\frac{\pi}{2}\right)$

where n is any integer.

4) (a) $y = \sin^{-1}\sqrt{x}$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-(\sqrt{x})^2}} \cdot \frac{1}{2}x^{-\frac{1}{2}}$$

$$= \frac{1}{\sqrt{1-x}} \cdot \frac{1}{2\sqrt{x}}$$

when $x = \frac{1}{2}$ $\quad \frac{dy}{dx} = \frac{1}{\sqrt{\frac{1}{2}}} \cdot \frac{1}{2\sqrt{\frac{1}{2}}}$

$\therefore \frac{dy}{dx} = 1 = m$ tangent

when $x = \frac{1}{2}$ $\quad y = \sin^{-1}\sqrt{\frac{1}{2}} = \frac{\pi}{4}$

$\therefore$ Eqn of tangent is:

$$y - \frac{\pi}{4} = 1\left(x - \frac{1}{2}\right)$$

$$\therefore y - \frac{\pi}{4} = x - \frac{1}{2}$$

$$\therefore 4x - 4y + \pi - 2 = 0$$

(b) (i) Let $y = x\cos^{-1}x - \sqrt{1-x^2}$

$$\therefore \frac{dy}{dx} = x \cdot \frac{-1}{\sqrt{1-x^2}} + \cos^{-1}x \cdot 1 - \frac{1}{2}(1-x^2)^{-\frac{1}{2}} \cdot -2x$$

$$= \frac{-x}{\sqrt{1-x^2}} + \cos^{-1}x + \frac{x}{\sqrt{1-x^2}}$$

$$= \cos^{-1}x$$

(ii) $I = \int_0^1 \cos^{-1}x \, dx$

$$= \left[x\cos^{-1}x - \sqrt{1-x^2}\right]_0^1$$

$$= \left[(\cos^{-1}1 - 0) - (0 - 1)\right]$$

$$= 1$$

(c) Let $f(x) = \sin^{-1}(x)$

$\therefore f(-x) = \sin^{-1}(-x) = -\sin^{-1}x = -f(x)$

$\therefore f(x) = \sin^{-1}x$ is an odd fn.

$\therefore I = \int_{-\frac{1}{2}}^{\frac{1}{2}} \sin^{-1}x \, dx$ is zero, as the integration of an odd fn about symmetrical limits is zero.

(d) $I = \int_0^{\frac{1}{\sqrt{5}}} \frac{4}{3+5x^2} \, dx$

$$= \int_0^{\frac{1}{\sqrt{5}}} \frac{4}{5\left[x^2 + \frac{3}{5}\right]} \, dx$$

$$= \frac{4}{5} \cdot \frac{1}{\sqrt{3/5}} \left[\tan^{-1}\frac{x}{\sqrt{3/5}}\right]_0^{\frac{1}{\sqrt{5}}}$$

$$= \frac{4}{\sqrt{15}}\left[\tan^{-1}\left(\frac{\frac{1}{\sqrt{5}}}{\frac{\sqrt{3}}{\sqrt{5}}}\right) - 0\right] = \frac{4}{\sqrt{15}} \times \frac{\pi}{6}$$

$$= \frac{2\pi}{\ldots}$$