CRANBROOK SCHOOL

# Year 12 Extension 1 Test. Fri 4th September, 2009.

Name______________________ Teacher __________ Class_______

All questions are 12 marks.

1. **Use a new booklet** Marked by SKB

(a) Two submarines A and B are located at (0, -50) and $(45\sqrt{3},-50)$ with respect to the origin. The $x-$axis is the reference sea-level line. Submarine A fires two torpedoes simultaneously with speed $30\text{ms}^{-1}$ at angles $\theta_1$ and $\theta_2$ to the positive direction of the $x-$axis at submarine B. Neglecting any water or air resistance and letting the acceleration due to gravity, $g = 10\text{ms}^{-2}$ :

(i) Prove that the parametric equations of motion of the torpedoes are given by: $x = 30t\cos\theta$ and $y = -5t^2 + 30t\sin\theta - 50$. **2**

(ii) Hence show that the Cartesian equation of motion for either torpedo is given by : $y = \dfrac{-x^2\sec^2\theta}{180} + x\tan\theta - 50$ **1**

(iii) Find the exact size of angles $\theta_1$ and $\theta_2$. **3**

(iv) Determine the maximum heights reached by both torpedoes. **2**

(v) Hence determine the shortest time for a torpedo to strike Submarine B. **1**

(b) A projectile is launched from the origin at an angle of $45^0$ to the positive direction of the $x-$axis. It passes through the points $(12k^2, 8k^2)$ and $(24k^2, 8k^2)$ during its flight. Neglecting air resistance and letting the acceleration due to gravity be g $\text{ms}^{-2}$, find the speed of projection V in $\text{ms}^{-1}$ in terms of g and k. You may assume that the parametric equations of motion are $x = \dfrac{Vt}{\sqrt{2}}$ and $y = \dfrac{-gt^2}{2} + \dfrac{Vt}{\sqrt{2}}$. **3**

2. **Use a new booklet** Marked by HRK

a) In a side show, the probability of getting a prize is 0.4. if a boy tries 8 times what is the probability that he wins at least 3 prizes? **3**

b) Find the coefficient of $x^{15}$ in $\left(x^3 - \dfrac{2}{x}\right)^9$ **3**

c) Show that if $(1+2x)^4 = \sum_{k=0}^{4} {}^4C_k (2x)^k$ and $T_{k+1} = {}^4C_k (2x)^k$ then $\dfrac{T_{k+1}}{T_k} = \dfrac{5-k}{k} 2x$

or show that if $(1+2x)^4 = \sum_{k=0}^{4} T_k x^k$ and $T_k = {}^4C_k 2^k$ then $\dfrac{T_{k+1}}{T_k} = \dfrac{8-2k}{k+1}$ **3**

d) If $(1+x)^n = \sum_{k=0}^{n} {}^nC_k x^k$ show that $\sum_{k=1}^{n} k\,{}^nC_k = n2^{n-1}$ **3**

3. **Use a new booklet** Marked by SKB

(a) The letters of the statement "FRIDAY FROLICS" are arranged in a circle. Find the probability that the two F's are together. **3**

(b) Consider a pack of 50 playing cards consisting of 5 colours Red, Blue, Green, Yellow and Orange, with cards numbered from 1 to 10 for each colour.

(i) Find the number of six card combinations. **1**

(ii) If six cards are dealt at random from the pack find the number of ways of obtaining four 9's and two eights. **1**

(i) If six cards are dealt at random from the pack find the number of ways of obtaining four cards of the same number and two cards of a different number. **1**

(c) There are 10 books on a table of which 6 are Maths, 3 are English and 1 is Latin. From these books 3 Maths, 2 English and 1 Latin are chosen and placed on a book shelf. Determine the number of book shelf arrangements if the Maths books are to be together. **2**

(d) Ten molluscs of which two are sea-snails are arranged at random in a row.

(i) Determine the number of arrangements without restrictions. **1**

(ii) Determine the probability that there are at least 5 other molluscs between the sea-snails. **3**

CRANBROOK
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# Examination Booklet – 4 pages

Student Name or Number ..............................................

Subject ......Ext 1 Test...... Unit Value (please circle) 1 2 3 4

4-9-09

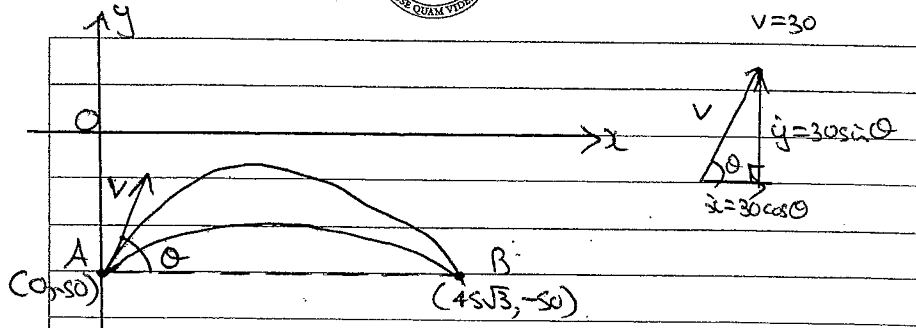
Teacher ................................

| Section | Part | Question | Mark |
|---|---|---|---|
| | | | |

**Instructions:**

- Clearly identify the questions you have attempted in each booklet
- If you have used more that one booklet for the same question, mark this clearly on each additional booklet you use
- Use the ruled pages only
- No part of this booklet may be torn out
- Under no circumstance may a booklet, used or unused, be removed from the examination by the student
- Marks may be deducted for poor or illegible work

1(a)




(i) $\ddot{x}=0$, $\ddot{y}=-10$

$\therefore \dot{x}=c_1$, $\dot{y}=-10t+c_2$

When $t=0$, $\dot{x}=30\cos\theta$, $\dot{y}=30\sin\theta$

$\therefore 30\cos\theta=c_1$, $30\sin\theta=c_2$

$\therefore \dot{x}=30\cos\theta$, $\dot{y}=-10t+30\sin\theta$

$\therefore x=30t\cos\theta+c_3$, $y=-5t^2+30t\sin\theta+c_4$

When $t=0$, $x=0$, $y=-50$

$\therefore c_3=0$, $c_4=-50$

$\therefore x=30t\cos\theta$ —(1)

$y=-5t^2+30t\sin\theta-50$ —(2)

Are the parametric eqns of motion of the torpedoes.

(ii) From (1) $t=\dfrac{x}{30\cos\theta}$ sub into (2)

$\therefore y=-5\left(\dfrac{x}{30\cos\theta}\right)^2+30\sin\theta\left(\dfrac{x}{30\cos\theta}\right)-50$

$\therefore y=-\dfrac{x^2\sec^2\theta}{180}+x\tan\theta-50$ is the Cartesian eqn of motion.

(iii) Torpedoes pass through $(45\sqrt{3}, -50)$

$\therefore -50 = -\dfrac{(45\sqrt{3})^2\sec^2\theta}{180} + 45\sqrt{3}\tan\theta - 50$

$\therefore \dfrac{(45\sqrt{3})^2(1+\tan^2\theta)}{180} - 45\sqrt{3}\tan\theta = 0$

$\therefore \dfrac{45\sqrt{3}}{180}(1+\tan^2\theta) - \tan\theta = 0$

$\therefore 45\sqrt{3}\tan^2\theta - 180\tan\theta + 45\sqrt{3} = 0$

$\therefore \tan\theta = \dfrac{180 \pm \sqrt{180^2 - 4(45\sqrt{3})(45\sqrt{3})}}{90\sqrt{3}}$

$= \dfrac{180 \pm \sqrt{8100}}{90\sqrt{3}}$

$= \dfrac{270}{90\sqrt{3}}$ or $\dfrac{90}{90\sqrt{3}}$

$= \dfrac{3}{\sqrt{3}}$ or $\dfrac{1}{\sqrt{3}}$

$\therefore \theta = 60^\circ$ or $30^\circ$. $\Rightarrow \theta_1 = 60^\circ,\ \theta_2 = 30^\circ$.

(iv) For max. height $\dot{y} = 0$ $\therefore -10t + 30\sin\theta = 0$

$\therefore t = 3\sin\theta$

when $\theta = 60^\circ$ $t = \dfrac{3\sqrt{3}}{2}$; $\theta = 30^\circ$ $t = \dfrac{3}{2}$ } sub into (2)

Now when $t = \dfrac{3\sqrt{3}}{2}$ $y_{max} = -5\left(\dfrac{3\sqrt{3}}{2}\right)^2 + 30\left(\dfrac{3\sqrt{3}}{2}\right)\dfrac{\sqrt{3}}{2} - 50$

$= -16\frac{1}{4}$ m with respect to O

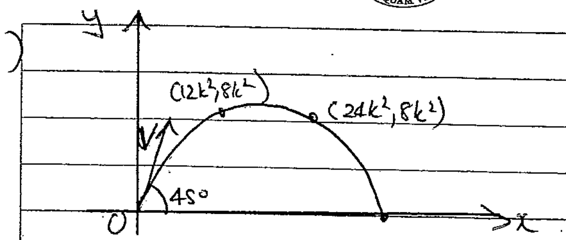
when $t = \dfrac{3}{2}$ $y_{max} = -5\left(\dfrac{3}{2}\right)^2 + 30\left(\dfrac{3}{2}\right)\dfrac{1}{2} - 50$

$= -38\frac{3}{4}$ m with respect to O.

(v) As projectory for either torpedo is symmetrical

$\Rightarrow$ shortest time to strike submarine B is 3 seconds.

(b)




$x = \dfrac{Vt}{\sqrt{2}}$ — (1)

$y = \dfrac{-gt^2}{2} + \dfrac{Vt}{\sqrt{2}}$ — (2)

from (1) $t = \dfrac{\sqrt{2}x}{V}$ sub into (2)

$\therefore y = \dfrac{-g}{2}\left(\dfrac{\sqrt{2}x}{V}\right)^2 + \dfrac{V}{\sqrt{2}}\left(\dfrac{\sqrt{2}x}{V}\right)$

$\therefore y = \dfrac{-gx^2}{V^2} + x$

Through $(12k^2, 8k^2)$ $\therefore 8k^2 = \dfrac{-g(12k^2)^2}{V^2} + 12k^2$

$\therefore g\left(\dfrac{144k^4}{V^2}\right) = 4k^2$

$\therefore V^2 = \dfrac{g(144k^4)}{4k^2} = 36gk^2$

$\Rightarrow V = 6\sqrt{g}\,k$

$\therefore$ Speed of projection, $V = 6\sqrt{g}\,k$ $ms^{-1}$.

2 (a) $P(\text{prize}) = 0.4 = p$ ; $P(\overline{\text{prize}}) = 0.6 = q$

Consider $(q+p)^8$

$\therefore P(\text{win at least 3 prizes}) = 1 - P(\text{win } 0, 1, \text{ or } 2 \text{ prizes})$

$= 1 - \left[q^8 + {}^8C_1 q^7 p + {}^8C_2 q^6 p^2\right]$

$= 1 - \left[0.6^8 + 8(0.6)^7(0.4) + 28(0.6)^6(0.4)^2\right]$

$= 0.68460544\ldots$

$= 0.68$ (2dp)

(b) $\left(x^3 - \frac{2}{x}\right)^9$

Now $T_{r+1} = {}^9C_r \left(x^3\right)^{9-r}\left(\frac{-2}{x}\right)^r$

$= {}^9C_r \, x^{27-3r} \cdot (-2)^r \cdot x^{-r}$

For $x^{15}$ term: $27-3r-r = 15$ $\therefore 4r = 12,\ r = 3$

$\therefore$ coeff of $x^{15} = {}^9C_3 (-2)^3 = -672$

(c) $\boxed{(1+2x)^4 = \sum_{k=0}^{4} {}^4C_k (2x)^k}$ Now $T_{k+1} = {}^4C_k (2x)^k$

$\therefore T_k = {}^4C_{k-1} (2x)^{k-1}$

$\therefore \frac{T_{k+1}}{T_k} = \frac{{}^4C_k (2x)^k}{{}^4C_{k-1} (2x)^{k-1}}$

$= \frac{4!\,(k-1)!\,(5-k)!\,2x}{(4-k)!\,k!\,4!}$

$= \left(\frac{5-k}{k}\right) 2x$

(d) $(1+x)^n = \sum_{k=0}^{n} {}^nC_k \, x^k$

d.b.s. w.r.t. $x$:

$n(1+x)^{n-1} = \sum_{k=0}^{n} {}^nC_k \, k \, x^{k-1}$

let $x = 1$ $\therefore n(2)^{n-1} = \sum_{k=0}^{n} {}^nC_k \cdot k \cdot 1^{k-1}$

$\therefore n\,2^{n-1} = \sum_{k=1}^{n} k \, {}^nC_k.$

(c*) alternative: $(1+2x)^4 = \sum_{k=0}^{4} T_k x^k$, Now $T_k = {}^4C_k \, 2^k$

$\therefore T_{k+1} = {}^4C_{k+1} \, 2^{k+1}$

$\therefore \frac{T_{k+1}}{T_k} = \frac{4!\,(4-k)!\,k!\,2^{k+1}}{(3-k)!\,(k+1)!\,4!\,2^k}$

$= \frac{(4-k)2}{k+1} = \frac{8-2k}{k+1}.$

5) (a) FRIENDLY FROLICS has 13 letters of which 2 are F's, 2 are R's, 2 are I's and 7 other single letters.

$\therefore$ No of ways two F's together $= 2!$ This leaves $(13-2)+1 = 12$ letter units

$\therefore$ Prob( 2 F's together in a circle) $= \dfrac{\dfrac{1 \times 2! \times 11!}{\cancel{2!.2!.2!}}}{\dfrac{1 \times 12!}{\cancel{2!.2!.2!}}} = \dfrac{1}{6}$

(b) (i) ${}^{50}C_6 = 15,890,700.$

(ii) No. of Ways (4 9's and 2 8's) $= {}^5C_4 \times {}^5C_2 = 50$

(iii) No. of Ways (any 4 of 1 number and any 2 of a diff. number) $= {}^5C_4 \times {}^5C_2 \times 2! \times {}^{10}C_2$

$= 4500$

(c) No. of arrangements if 3 Maths books together,

Consider 3 Maths Books as a unit. This is done in $3!$ ways.

This leaves $(6-3)+1 = 4$ book units.

$\therefore$ No. of arrangements if 3 Maths books together $= {}^6C_3 \times {}^3C_2 \times {}^1C_1 \times 3! \times 4!$

$= 8640$

(d) (i) Number of arrangements without restrictions $= 10! = 3,628,800.$

(ii) Possible arrangements are:

S — — — — — S — — —
— S — — — — — S — —
— — S — — — — — S —
— — — S — — — — — S

5 spaces for other molluscs $= 2! \times 4 \times 8!$

$\therefore$ similarly for 6 spaces: $2! \times 3 \times 8!$

for 7 spaces: $2! \times 2 \times 8!$

for 8 spaces: $2! \times 1 \times 8!$

$\therefore$ Prob( at least 5 other molluscs between sea snails) $= \dfrac{2! \times 8! \left[4+3+2+1\right]}{10!}$

$= \dfrac{2}{9}.$