Extension Two Mathematics- Harder Extension One Topics 2008

3. Miscellaneous

1. If $C_{r+1}^{n}=C_{r}^{n}=kC_{r-1}^{n}$ prove that n is odd, and express n and r in terms of k.

2. If n is a positive integer, and a and b are constants and it is known that $(1+ax)^{n}=1-8x+\frac{88}{3}x^{2}+bx^{3}+\ldots$ find the values of n, a and b.

3. If $f(x)=2e^{-4x}-e^{-2x}$, find x_1, x_2, x_3 so that $f(x_1)=0$, $f'(x_2)=0$, $f''(x_3)=0$ and show that x_1, x_2, x_3 are in arithmetic progression.

4. Prove that $\overset{n}{C}_{r}=\overset{n-1}{C}_{r}+\overset{n-1}{C}_{r-1}$ and that $\sum_{r=0}^{n}(-1)^{r}\overset{n}{C}_{r}=0$

5. If $(1+x)^{n}=C_0+C_1x+C_2x^2+\ldots\ldots+C_nx^n$, where n is a positive integer, express in terms of *n;*
   a. $C_0+C_1+C_2+\ldots\ldots+C_n$
   b. $C_1+2C_2+3C_3\ldots\ldots+nC_n$
   c. $C_2+2C_3+3C_4\ldots\ldots+(n-1)C_n$
   d. $C_0+\frac{1}{2}C_1+\frac{1}{3}C_2\ldots\ldots+\frac{1}{n+1}C_n$

6. In the expansion of $(1+x)^{n}$, the ratios of three consecutive coefficients are 6 :14:21 find the value of n.

7. By making a substitution for x in the expansion of $(1+x)^{n}$, prove that;
   a. $\sum_{k=1}^{n}\binom{n}{k}=2^{n}-1$ b. $\sum_{k=1}^{n}(-2)^{k}\binom{n}{k}=(-1)^{n}-1$ c. $\sum_{k=0}^{n}(3)^{k}\binom{n}{k}=(2)^{2n}$

8. If a_1, a_2, a_3, a_4 are the coefficients of any four consecutive terms in the expansion of $(1+x)^{n}$, prove that $\frac{a_1}{a_1+a_2}+\frac{a_3}{a_3+a_4}=\frac{2a_2}{a_2+a_3}$

9. Consider the binomial expansion $1+\binom{n}{1}x+\binom{n}{2}x^{2}+\ldots+\binom{n}{n}x^{n}=(1+x)^{n}$ show that; a. $1-\binom{n}{1}+\binom{n}{2}-\ldots+(-1)^{n}\binom{n}{n}=0$
   b. $1-\frac{1}{2}\binom{n}{1}+\frac{1}{3}\binom{n}{2}-\ldots+(-1)^{n}\frac{1}{n+1}\binom{n}{n}=\frac{1}{n+1}$ (1993 3U HSC)

Harder 3U - Miscellaneous

1. $C^n_{r+1} = C^n_r = k\,C^n_{r-1}$

⇒ from above ${}^nC_r = {}^nC_{r+1}$

but ${}^nC_r = {}^nC_{n-r}$ ✓

$\therefore n-r = r+1$

$n = 2r+1$ ✓

$\therefore n$ is odd if r an integer

⇒ find n, r in terms of k

${}^nC_r = k\,{}^nC_{r-1}$

$$\frac{\cancel{n!}}{r!(n-r)!} = \frac{k\,\cancel{n!}}{(r-1)!\,(n-r+1)!}$$

$$kr!\,(n-r)! = (r-1)!\,(n-r+1)!$$

$$kr! = (r-1)!\,(n-r+1)$$

$$kr = n-r+1 \checkmark$$

and $n = 2r+1$

$kr = 2r+1-r+1$

$kr = r+2$

$kr - r = 2$ ✓

$r(k-1) = 2$

$r = \dfrac{2}{k-1}$

$n = 2\left(\dfrac{2}{k-1}\right)+1$ ✓

$= \dfrac{4}{k-1}+1$

(2) $LHS = (1+ax)^n$

$= {}^nC_0(ax)^0 + {}^nC_1(ax)^1 + {}^nC_2(ax)^2 + {}^nC_3(ax)^3 + \ldots$

$= 1 + nax + \dfrac{n(n-1)}{2}.a^2x^2 + \dfrac{n(n-1)(n-2)}{3.2.1}a^3x^3 + \ldots$

$\therefore$ Equating coeffs.

$an = -8$; $\dfrac{n(n-1)}{2}a^2 = \dfrac{88}{3}$; $\dfrac{n(n-1)(n-2)}{6}.a^3 = b$

$\therefore a = -\dfrac{8}{n}$

$\therefore \dfrac{n(n-1)}{2}.\dfrac{64}{n^2} = \dfrac{88}{3}$

$\dfrac{n-1}{n} = \dfrac{88}{3\times 32}$

$12n - 12 = 11n$

$\underline{n = 12}$

$\therefore a = -\dfrac{8}{12}$

$\underline{= -\dfrac{2}{3}}$

$b = \dfrac{12.11.10}{6}.\left(-\dfrac{8}{27}\right)$

$= -\dfrac{126}{27}$

3. $f(x) = 2e^{-4x} - e^{-2x}$

$f(x_1) = 0$
$f'(x_2) = 0$
$f''(x_3) = 0$

$f'(x) = -8e^{-4x} + 2e^{-2x}$ ✓

$f''(x) = 32e^{-4x} - 4e^{-2x}$

for x_1,

$2e^{-4x_1} - e^{-2x_1} = 0$

$e^{-2x_1}(2e^{-2x_1} - 1) = 0$ ✓

$e^{-2x_1} \neq 0$,

$2e^{-2x_1} - 1 = 0$

$2e^{-2x_1} = 1$

$e^{-2x_1} = \frac{1}{2}$

$-2x_1 = \ln\frac{1}{2}$ ✓

$x_1 = -\frac{1}{2}\ln\frac{1}{2}$

for x_2,

$-8e^{-4x_2} + 2e^{-2x_2} = 0$

$2e^{-2x_2}(-4e^{-2x_2} + 1) = 0$ ✓

$-4e^{-2x_2} + 1 = 0$

$4e^{-2x_2} = 1$

$e^{-2x_2} = \frac{1}{4}$

$-2x_2 = \ln\frac{1}{4}$

$x_2 = -\frac{1}{2}\ln\frac{1}{4}$ ✓

4. prove ${}^nC_r = {}^{n-1}C_r + {}^{n-1}C_{r-1}$

ie $\dfrac{n!}{r!(n-r)!} = \dfrac{(n-1)!}{r!(n-r-1)!} + \dfrac{(n-1)!}{(r-1)!(n-r)!}$

RHS $= \dfrac{(n-1)!}{r!(n-r-1)!} + \dfrac{(n-1)!}{(r-1)!(n-r)!}$

$= \cancel{\dfrac{(n-1)!(n-r)(n-r+1) + (n-1)!r}{r!(n-r+1)!}}$

$= \cancel{\dfrac{(n-1)![(n-r)(n-r+1)+r]}{r!(n-r)!(n-r+1)}}$

$= \cancel{\dfrac{(n-1)!(n^2-nr+n-nr+r^2-r+r)}{r!(n-r)!(n-r+1)}}$

$= \cancel{(n-1)!(n^2+n-2nr+r^2)}$

$= \dfrac{(n-1)!(n-r) + (n-1)!r}{r!(n-r)!}$

$= \dfrac{(n-1)![n-r+r]}{r!(n-r)!}$

$= \dfrac{(n-1)!n}{r!(n-r)!} = \dfrac{n!}{r!(n-r)!}$

Show $\sum_{r=0}^{n} (-1)^n C_r = 0$

$\underbrace{-1}\ {}^nC_r$

ie $\cancel{(-1^0)^n C_0 + (-1)^n C_1 + (-1)^n C_2 + \dots + (-1^{n-1})^n C_{n-1} + (-1^n)^n C_n} = 0$

$(-1)^0\,{}^nC_0 + (-1)^n\,{}^nC_1 +$

$(-1)^n$

for x_3,

$32e^{-4x_3} - 4e^{-2x_3} = 0$

$4e^{-2x_3}(8e^{-2x_3} - 1) = 0$

$8e^{-2x_3} - 1 = 0$

$8e^{-2x_3} = 1$

$e^{-2x_3} = \frac{1}{8}$

$-2x_3 = \ln\frac{1}{8}$

$x_3 = -\frac{1}{2}\ln\frac{1}{8}$

$\therefore x_1 = -\frac{1}{2}\ln\frac{1}{2}$

$x_2 = -\frac{1}{2}\ln\frac{1}{4}$

$x_3 = -\frac{1}{2}\ln\frac{1}{8}$

$$\begin{aligned}
x_2 - x_1 &= -\tfrac{1}{2}\ln\tfrac{1}{4} + \tfrac{1}{2}\ln\tfrac{1}{2} \\
&= \ln 4^{1/2} + \ln \tfrac{1}{2}^{1/2} \\
&= \ln 4^{1/2} \times \tfrac{1}{2}^{1/2} \\
&= \ln 2 \times 2^{-1/2} \\
&= \ln 2^{1/2} \\
&= \tfrac{1}{2}\ln 2
\end{aligned}$$

$$\begin{aligned}
x_3 - x_2 &= -\tfrac{1}{2}\ln\tfrac{1}{8} + \tfrac{1}{2}\ln\tfrac{1}{4} \\
&= \ln 8^{1/2} + \ln 4^{-1/2} \\
&= \ln 2^{3/2} + \ln 2^{-1} \\
&= \ln 2^{3/2} \times 2^{-1} \\
&= \ln 2^{1/2} \\
&= \tfrac{1}{2}\ln 2
\end{aligned}$$

$\therefore$ since $x_3 - x_2 = x_2 - x_1$

x_1, x_2, x_3 are in A.P.

5. $(1+x)^n = C_0 + C_1x + C_2x^2 + \dots + C_nx^n$

a) let $x = 1$

$2^n = C_0 + C_1 + C_2 + \dots + C_n$

b) differentiate both sides

$n(1+x)^{n-1} = C_1 + 2C_2x + 3C_3x^2 + 4C_4x^3 + \dots + nC_nx^{n-1}$

let $x = 1$

$n2^{n-1} = C_1 + 2C_2 + 3C_3 + \dots + nC_n$

c) ~~differentiate again both sides~~

$\cancel{(n-1)n(1+x)^{n-2} = 2C_2 + 6C_3x + 12C_4x^2 + \dots + (n-1)nC_nx^{n-2}}$

~~differentiate twice both sides~~

$\cancel{n(n-1)(1+x)^{n-2} = 2C_2 + 6C_3x + 12C_4x^2 + \dots + n(n-1)C_nx^{n-2}}$

~~let $x=1$~~,

$\cancel{n(n-1)2^{n-2} = 2C_2 + 6C_3 + 12C_4 + \dots + n(n-1)C_n}$

now $(C_1 + 2C_2 + 3C_3 + \dots + nC_n) - (C_0 + C_1 + C_2 + \dots + C_n)$

$= -C_0 + C_2 + 2C_3 + 3C_4 + \dots (n-1)C_n$

$= n2^{n-1} - 2^n - 1$

d) $(1+x)^n = C_0 + C_1x + C_2x^2 + \cdots + C_nx^n$

integrate both sides

$$\frac{(1+x)^{n+1}}{n+1} + c = C_0x + \frac{1}{2}C_1x^2 + \frac{1}{3}C_2x^3 + \cdots + \frac{1}{n+1}C_nx^{n+1}$$

let $x=0$,

$$\frac{1}{n+1} + c = 0$$

$$c = -\frac{1}{n+1}$$

$\therefore$ at $x=1$,

$$\frac{2^{n+1}}{n+1} - \frac{1}{n+1} = C_0 + \frac{1}{2}C_1 + \frac{1}{3}C_2 + \cdots + \frac{1}{n+1}C_n$$

6. ~~(1+x)^n~~

$$\cancel{{}^nC_{r-1} + {}^nC_r + {}^nC_{r+1} = 6+14+21}$$

$${}^nC_{r-1} = 6 \qquad \frac{n!}{(r-1)!(n-r+1)!} = 6$$

$${}^nC_r = 14 \qquad \frac{n!}{r!(n-r)!} = 14$$

$${}^nC_{r+1} = 21 \qquad \frac{n!}{(r+1)!(n-r-1)!} = 21$$

$$41 = \frac{n!(r)(r+1) + n!(r+1)(n-r+1) + n!(n-r)(n-r+1)}{(r+1)!(n-r+1)!}$$

6. ${}^nC_0 + {}^nC_1 + {}^nC_2 = 6+14+21$

$$\frac{n!}{(n-0)!} + \frac{n!}{(n-1)!} + \frac{n!}{(n-2)!\,2!} = 41$$

$$1 + n + \frac{n(n-1)}{2} = 6+14+21$$

$$1 : n : \frac{n(n-1)}{2} = 6 : 14 : 21$$

$$n : \frac{n(n-1)}{2} = 14 : 21$$

$$\therefore n \times \frac{2}{n(n-1)} = \frac{14}{21}$$

$$\frac{2n}{n(n-1)} = \frac{14}{21}$$

$$\frac{2}{n-1} = \frac{14}{21}$$

$$42 = 14(n-1)$$

$$n-1 = \frac{42}{14}$$

$$n = 3+1$$

$$= 4$$

7\. a) $\sum_{k=1}^{n} {}^nC_k = 2^n - 1$

ie ${}^nC_1 + {}^nC_2 + \dots {}^nC_n = 2^n - 1$

$(1+x)^n = 1 + {}^nC_1 + {}^nC_2 + \dots + {}^nC_n$

let $x = 1$

$2^n = 1 + {}^nC_1 + {}^nC_2 + \dots {}^nC_n$

$2^n - 1 = {}^nC_1 + {}^nC_2 + \dots + {}^nC_n$

~~8. a_1, a_2, a_3~~

b) $\sum_{k=1}^{n} (-2)^k\, {}^nC_k = (-1)^n - 1$

ie $(-2)\,{}^nC_1 + (-2)^2\,{}^nC_2 + \dots + (-2)^n\,{}^nC_n = (-1)^n - 1$

$-2\,{}^nC_1 + 4\,{}^nC_2 \cancel{-8\,{}^nC_3} + \dots + (-2)^n\,{}^nC_n = (-1)^n - 1$

$(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \dots + {}^nC_n x^n$

let $x = -2$

$(-1)^n = 1 - 2\,{}^nC_1 + (-2)^2\,{}^nC_2 + \dots {}^nC_n(-2)^n$

$(-1)^n - 1 = -2\,{}^nC_1 + 4\,{}^nC_2 + \dots + {}^nC_n(-2)^n$

~~53)~~ (8) Let $a_1 = {}^nC_{r-3}$, $a_2 = {}^nC_{r-2}$

$a_3 = {}^nC_{r-1}$ and $a_4 = {}^nC_r$

Proof:

$\therefore$ L.H.S $= \dfrac{a_1}{a_1 + a_2} + \dfrac{a_3}{a_3 + a_4}$

$= \dfrac{{}^nC_{r-3}}{{}^nC_{r-3} + {}^nC_{r-2}} + \dfrac{{}^nC_{r-1}}{{}^nC_{r-1} + {}^nC_r}$

$= \cancel{\dfrac{\frac{n!}{(n-r+3)!\,(r-3)!}}{n!}}$

$= \dfrac{1}{1 + \frac{{}^nC_{r-2}}{{}^nC_{r-3}}} + \dfrac{1}{1 + \frac{{}^nC_r}{{}^nC_{r-1}}}$

$= \dfrac{1}{1 + \dfrac{\frac{n!}{(n-r+2)!\,(r-2)!}}{\frac{n!}{(n-r+3)!\,(r-3)!}}} + \dfrac{1}{1 + \dfrac{\frac{n!}{(n-r)!\,r!}}{\frac{n!}{(n-r+1)!\,(r-1)!}}}$

$= \dfrac{1}{1 + \frac{n-r+3}{r-2}} + \dfrac{1}{1 + \frac{n-r+1}{r}}$

$\dfrac{r-2}{r-2+n-r+3} + \dfrac{r}{r+n-r+1}$

$= \dfrac{r-2}{n+1} + \dfrac{r}{n+1}$

$= \dfrac{2r-2}{n+1} = \dfrac{2(r-1)}{n+1}$

R.H.S $= \dfrac{2\,a_2}{a_2 + a_3}$

$= 2 \cdot \dfrac{1}{1 + \frac{a_3}{a_2}}$

$= 2\left[\dfrac{1}{1 + \dfrac{\frac{n!}{(n-r+1)!\,(r-1)!}}{\frac{n!}{(n-r+2)!\,(r-2)!}}}\right]$

$= 2\left[\dfrac{1}{1 + \frac{n-r+2}{r-1}}\right]$

$= 2\left[\dfrac{r-1}{r-1+n-r+2}\right]$

$= \dfrac{2(r-1)}{n+1} =$ L.H.S

Q.E.D

(54) $(a+b)^n = {}^nC_0 a^n + {}^nC_1 a^{n-1} b + \dots + {}^nC_n b^n$

$X = {}^nC_0 a^n + {}^nC_2 a^{n-2} b^2 + {}^nC_4 a^{n-4} b^4 + \dots$

$Y = {}^nC_1 a^{n-1} b + {}^nC_3 a^{n-3} b^3 + \dots$

$\therefore\ X + Y = (a+b)^n$

$X - Y = (a-b)^n$

i.e. $(X+Y)(X-Y) = (a+b)^n (a-b)^n$

$\therefore\ X^2 - Y^2 = [(a+b)(a-b)]^n$

$= [a^2 - b^2]^n$ as req$^{\underline{d}}$.

9. a) show $1 - {}^nC_1 + {}^nC_2 - {}^nC_3 + \dots + (-1)^n\, {}^nC_n = 0$

$(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \dots + {}^nC_n x^n$

let $x = -1$

$0 = 1 \cancel{+} - {}^nC_1 + {}^nC_2 \cancel{+} \dots + {}^nC_n (-1)^n$

b) show $1 - \frac{1}{2}{}^nC_1 + \frac{1}{3}{}^nC_2 - \dots + (-1)^n \frac{1}{n+1}{}^nC_n = \frac{1}{n+1}$

integrating $\Rightarrow$

$\frac{(1+x)^{n+1}}{n+1} + C = {}^nC_0 x + \frac{1}{2}{}^nC_1 x^2 + \frac{1}{3}{}^nC_2 x^3 + \dots + \frac{1}{n+1}{}^nC_n x^{n+1}$

let $x = 0$, $C = \frac{-1}{n+1}$

let $x = -1$

$0 - \frac{1}{n+1} = -1 + \frac{1}{2}{}^nC_1 - \frac{1}{3}{}^nC_3 + \dots + \frac{1}{n+1}{}^nC_n (-1)^{n+1}$

$\frac{1}{n+1} = 1 - \frac{1}{2}{}^nC_1 + \frac{1}{3}{}^nC_3 \cancel{+} \dots + \frac{(-1)^{n+1}\,{}^nC_n}{n+1}$