INVERSE FUNCTIONS

1) Find the inverse function of $f(x) = 7x - 3$.

2) (a) Find the inverse function of $y = e^{2x}$.
(b) State the domain and range of the inverse function.

3) Consider the function $f(x) = x^2 - 4x$.
(a) Sketch the parabola, showing the intercepts with the axes and the coordinates of its vertex.
(b) What is the largest domain including the value $x = 5$ for which the function has an inverse function $f^{-1}(x)$?
(c) Find the inverse function over this domain, and state its domain and range.
(d) Sketch the graph of $y = f^{-1}(x)$ on the same set of axes as your graph in part (a).
(e) State a domain over which the function does not have an inverse. Give a brief reason.

4) Consider the function $y = x^3$.
(a) Find the inverse function $f^{-1}(x)$.
(b) Evaluate $f^{-1}(f(2))$.
(c) Show that $f^{-1}(f(a)) = a$ for any $x = a$.
(d) Hence show that $f^{-1}(f(a)) = f(f^{-1}(a))$ for all real a.

5) The function $f(x)$ is given by $f(x) = \dfrac{e^x + 2}{e^x}$.
(a) Show that $f(x)$ has no stationary points.
(b) Sketch the curve $y = f(x)$.
(c) Find the inverse function $f^{-1}(x)$.

6) Find $\cos^{-1}(1)$.

7) Find the exact value of $\sin^{-1}(-1)$.

8) Evaluate $\tan^{-1}(\sqrt{3})$.

9) Evaluate $\cos^{-1}(-\frac{1}{2})$.

10) Evaluate $\sin^{-1}(-\frac{\sqrt{3}}{2})$.

11) Find the exact value of $\sin(\cos^{-1}(-\frac{1}{\sqrt{2}}))$

12) Find the exact value of $\sin(\cos^{-1}(\frac{3}{7}))$

13) Evaluate $\tan^{-1}(\cos 0)$

14) Sketch the curve $y = 5\tan^{-1} 3x$.

15) State the domain and range of $y = x\sin^{-1} x$.

16) Show that $\sin^{-1}(-\frac{1}{2}) = -\sin^{-1}(\frac{1}{2})$

17) Show that $\sin^{-1}\left(\frac{4}{5}\right) + \cos^{-1}\left(\frac{4}{5}\right) = \frac{\pi}{2}$

18) Differentiate $2\sin^{-1}x$

19) Differentiate $\tan^{-1}(3x)$

20) Find the derivative of $(\cos^{-1}x - 2)^5$

21) Differentiate $e^{\cos^{-1}x}$

22) Find the first and second derivatives of $\tan^{-1}x$.

23) Find the gradient of the tangent to the curve $y = 3\cos^{-1}x$ at the point where $x = \frac{1}{2}$.

24) Find the gradient of the normal to the curve $y = \sin^{-1}x$ when $x = 0$.

25) Find the equation of the tangent to the curve $y = 2\tan^{-1}2x$ at the point $\left(\frac{1}{2}, \frac{\pi}{2}\right)$.

26) (a) Find the domain and range of the function $y = \sin^{-1}2x$.
(b) Sketch the graph of $y = \sin^{-1}2x$.
(c) Find the equation of the tangent to the curve $y = \sin^{-1}2x$ at the point $\left(\frac{1}{4}, \frac{\pi}{6}\right)$.

27) (a) Find f(0), given $f(x) = 3\cos^{-1}(\frac{x}{2})$
(b) Sketch the function $f(x) = 3\cos^{-1}(\frac{x}{2})$
(c) State the domain and range of the function.
(d) Find the gradient of the function when $x = 0$.

28) For the function $f(x) = \sin^{-1}x + \cos^{-1}x$,
(a) find $f'(x)$
(b) sketch the graph of $y = f(x)$.

29) Find $\int \frac{dx}{\sqrt{9-x^2}}$.

30) Evaluate $\int_0^1 \frac{3}{1+t^2}\,dt$

31) Evaluate $\int_0^{\frac{\sqrt{3}}{2}} \frac{2}{\sqrt{1-x^2}}\,dx$

32) Find $\int \frac{dx}{\sqrt{1-4x^2}}$

33) Find $\int \frac{dt}{5+4t^2}$

34) (a) Sketch the curve $y = \frac{1}{x^2+1}$ showing any stationary points.
(b) Find the area bounded by the curve $y = \frac{1}{x^2+1}$, the x-axis and the lines $x = 1$ and $x = \sqrt{3}$.

Inverse functions

Angelina

Check corrections. 16

Qu 1 $f(x) = 7x - 3$

$x = 7y - 3$

$\frac{1}{7}(x+3) = y$ ✓

$\therefore f^{-1}(x) = \frac{1}{7}(x+3)$ ✓

Qu 2 (a) $x = e^{2y}$

$\frac{1}{2}\log x = y$

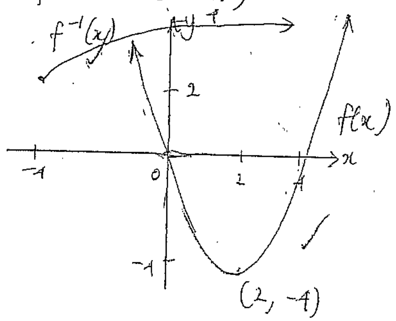
$\therefore y = \frac{1}{2}\log x$ ✓

(b) Inverse

$D: \{$ ~~all real x~~ $x > 0\}$

$R:$ ~~$y > 0$~~ All real y

Qu 3 (a) $f(x) = x(x-4)$



✓

(b) $D: \{x \geq 2\}$ ✓

(c) $x = y^2 - 4y$

$x + 4 = (y-2)^2$

$2 + \sqrt{x+4} = y$

$\therefore y = 2 + \sqrt{x+4}$ ✓

$D: \{x \geq -4\}$

$R: \{y \geq 2\}$

(d) refer to diagram

(e) $1 \leq x \leq 3$: for every y-value, function does not have single x-value ✓

Qu 4 (a) $x = y^3$

$y = \sqrt[3]{x}$

$\therefore f^{-1}(x) = \sqrt[3]{x}$ ✓

(b) $f(2) = 8$

$\therefore f^{-1}(8) = 2$. ✓

(c) $f(a) = a^3$

$f^{-1}(a^3) = \sqrt[3]{a^3} = a$ ✓

$\therefore f^{-1}(f(a)) = a$ for any $x = a$ ✓

(d) RHS: $f^{-1}(a) = \sqrt[3]{a}$

$f(\sqrt[3]{a}) = (\sqrt[3]{a})^3$ ✓

$= a$

$=$ LHS (from (c))

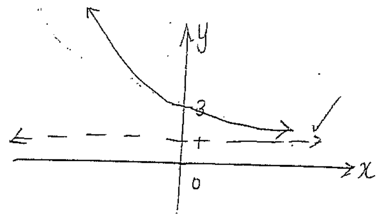
Qu 5 (a) $f'(x) = \dfrac{e^{2x} - (e^{2x} + 2e^{x})}{e^{2x}}$ ✓

$= \dfrac{-2e^{x}}{e^{2x}}$ ✓

when $f'(x) = 0$, $-2e^{x} = 0$

$\therefore$ since $x > 0$, no stat pts. ✓

(b)



✓

(c) $x = \dfrac{e^y + 2}{e^y}$

$xe^y - e^y = 2$

$e^y(x-1) = 2$ ✓

$e^y = \dfrac{2}{x-1}$ ✓

$y = \log_e\left(\dfrac{2}{x-1}\right)$

Qu 6 $\cos^{-1}(1) = 0$ ✓

Qu 7 $\sin^{-1}(-1) = -\frac{\pi}{2}$ ✓

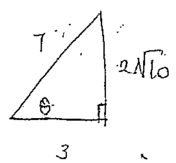
Qu 8 $\tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$ ✓

Qu 9 $\cos^{-1}(-\frac{1}{2}) = \frac{2\pi}{3}$ ✓

Qu 10 $\sin^{-1}(-\frac{\sqrt{3}}{2}) = -\frac{\pi}{3}$ ✓

Qu 11 $\frac{1}{\sqrt{2}}$ ✓ $(\sin(\frac{3\pi}{4}))$

Qu 12



Let $\cos^{-1}\frac{3}{7} = \theta$ ✓

$\cos\theta = \frac{3}{7}$

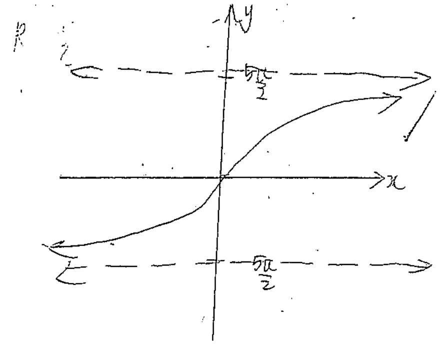
$\therefore \sin\theta = \dfrac{2\sqrt{10}}{7}$ ✓

Qu 13 $\tan^{-1}(1) = \frac{\pi}{4}$ ✓

Qu 14 $y = 5\tan^{-1}3x$.

$D: \{$all real $x\}$ ✓

$R: \{-\frac{5\pi}{2} < y < \frac{5\pi}{2}\}$



✓

Qu 15. $y = x\sin^{-1}x$

$D: \{-1 \leq x \leq 1\}$ ✓

$R: \{$ ~~$-\frac{\pi}{2}$~~ $0 \leq y \leq \frac{\pi}{2}\}$

Note: When $x = -$

$y = -1\sin^{-1}(-1)$

$= -1 \times -\frac{\pi}{2} = \frac{\pi}{2}$

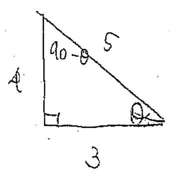
Q16 LHS: $\sin^{-1}(-\frac{1}{2}) = -\frac{\pi}{6}$ ✓

RHS $= -\sin^{-1}(\frac{1}{2}) = -\frac{\pi}{6}$ ✓

$=$ RHS

$\therefore$ LHS = RHS

Qu 17



Let $\theta = \sin^{-1}\frac{4}{5}$

$\sin\theta = \frac{4}{5}$

$\cos(90 - \theta) = \frac{4}{5}$

$\cos^{-1}\frac{4}{5} = \frac{\pi}{2} - \theta$ ✓

Qu18 $\frac{d}{dx} = \frac{2}{\sqrt{1-x^2}}$ ✓

Qu19 $\frac{d}{dx} = \frac{1}{1+9x^2} \times 3$

$= \frac{3}{1+9x^2}$ ✓

Qu 20 $\frac{d}{dx}(\cos^{-1}x-2)^5 = 5(\cos^{-1}x-2)^4 \times \frac{-1}{\sqrt{1-x^2}}$ ✓

$= \frac{-5(\cos^{-1}x-2)^4}{\sqrt{1-x^2}}$

Qu 21 $\frac{d}{dx} e^{\cos^{-1}x} = \frac{-e^{\cos^{-1}x}}{\sqrt{1-x^2}}$ ✓

Qu 22 $\frac{d}{dx}\tan^{-1}x = \frac{1}{x^2+1}$ ✓

$\frac{d^2}{dx^2} = \frac{-x^2-1}{(1+x^2)^2} = \cancel{\frac{-(1+x^2)}{(1+x^2)^2}}$

$y' = (x^2+1)^{-1}$

$\therefore y'' = -1(x^2+1)^{-2} \times 2x$ $= \frac{-1}{1+x^2}$

$= \frac{-2x}{(x^2+1)^2}$

Qu23 $y = 3\cos^{-1}x$

$y' = \frac{-3}{\sqrt{1-x^2}}$

when $x = \frac{1}{2}$, $m = \frac{-3}{\sqrt{1-\frac{1}{4}}}$ ✓ (+ve since $\cos^{-1}x$ is > 0).

$= \frac{-3}{\frac{\sqrt{3}}{2}} = \frac{-6}{\sqrt{3}}$

$= -6\sqrt{3}$ ✓ $= -2\sqrt{3}$

Qu 24 $y = \sin^{-1}x$

$y' = \frac{1}{\sqrt{1-x^2}}$ ✓

when $x = 0$, $m = 1$

$\therefore m_{normal} = -1$ ✓

Qu 25 $y = 2\tan^{-1}2x$

$y' = 2 \times \frac{2}{1+4x^2}$

$= \frac{4}{1+4x^2}$

when $x = \frac{1}{2}$, $m = 2$. ✓

$y - \frac{\pi}{2} = 2(x - \frac{1}{2})$

$y = 2x - 1 + \frac{\pi}{2}$ ✓

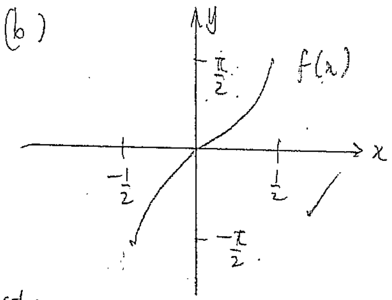
Qu 26 (a) $y = \sin^{-1}2x$.

R: $\{-\frac{\pi}{2} \le y \le \frac{\pi}{2}\}$ ✓

D: $\{-1 \le 2x \le 1\}$

$\{-\frac{1}{2} \le x \le \frac{1}{2}\}$ ✓

(b)



(c) $y' = \frac{2}{\sqrt{1-x^2}}$ when $x = \frac{1}{4}$

$m = \frac{2}{\sqrt{3}} = \frac{4\sqrt{3}}{3}$ ✓

$\therefore y - \frac{\pi}{6} = \frac{4\sqrt{3}}{3}(x - \frac{1}{4})$ ✓

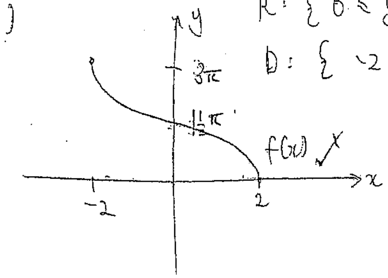
$3y - \frac{\pi}{2} = 4\sqrt{3}x - \sqrt{3}$ ✓

$\therefore 4\sqrt{3}x - 3y + \frac{\pi}{2} - \sqrt{3} = 0$

Qu27 (a) $f(x) = 3\cos^{-1}(\frac{x}{2})$

$f(0) = \frac{3\pi}{2}$. ✓

(b)



R: $\{0 \le y \le 3\pi\}$ ✓

D: $\{-2 \le x \le 2\}$ ✓

(c) R: $\{0 \le y \le 3\pi\}$ ✓

D: $\{-2 \le x \le 2\}$ ✓

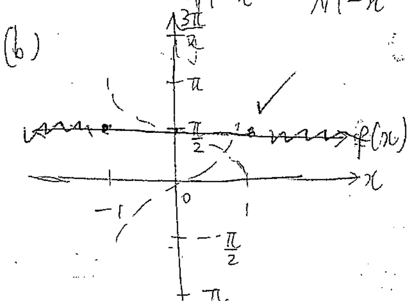
(d) $f'(x) = 3 \times \frac{-1}{\sqrt{4-x^2}}$ ✓

$= \frac{-3}{\sqrt{4-x^2}}$

$\therefore f'(0) = \frac{-3}{2}$. ✓

Qu 28 (a) $f'(x) = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}} = 0$ ✓

(b)



Alternatively:

If $f'(x) = 0$

$f(x) = c = \sin^{-1}x + \cos^{-1}x$ for $-1 \le x \le$

Sub $x = 1$

$c = \sin^{-1}1 + \cos^{-1}1$

$= \frac{\pi}{2} + 0$

Qu 29 $\int \frac{1}{\sqrt{9-x^2}}\,dx$ ✓

$= \sin^{-1}(\frac{x}{3}) + C$

Qu 30 $3\int_0^1 \frac{1}{1+t^2}\,dt$

$= 3[\tan^{-1}t]_0^1$ ✓

$= 3(\tan^{-1}(1) - \tan^{-1}(0))$

$= \frac{3\pi}{4}$ ✓

Qu31 $2\int_0^{\frac{\sqrt{3}}{2}} \frac{1}{\sqrt{1-x^2}}\,dx$

$= 2[\sin^{-1}x]_0^{\frac{\sqrt{3}}{2}}$ ✓

$= \frac{2\pi}{3}$ ✓

Qu32. $\int \frac{1}{\sqrt{1-4x^2}}\,dx$ ✓

$= \frac{1}{2}\sin^{-1}2x + c$ ✓

Qu 33 $\int \frac{1}{5+4t^2}\,dt$ ✓

$= \frac{1}{2\sqrt{5}}\tan^{-1}\frac{2x}{\sqrt{5}} + c$

Qu 34 (a) $y = \frac{1}{x^2+1}$

D : {all real x}

R : {$y \neq 0$}

HA $\Rightarrow y = \frac{\frac{1}{x^2}}{1+\frac{1}{x^2}}$

$= 0$

when $x=0$, $y=1$

no x intercepts

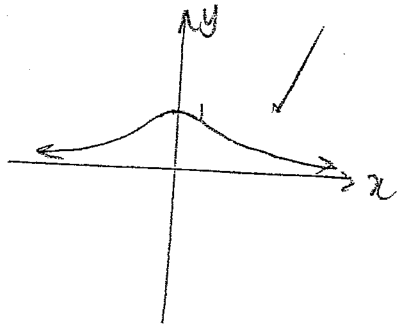
$y' = \frac{-2x}{x^2+1}$

when $y'=0$, $x=0$

when $x=0$, $y=1$

x	$-\frac{1}{2}$	0	$\frac{1}{2}$
y	/	—	\

$\therefore$ max turning pt



(b) $A = \int_1^{\sqrt{3}} \frac{1}{x^2+1}\,dx$

$= \left[\tan^{-1}(x)\right]_1^{\sqrt{3}}$

$= \tan^{-1}(\sqrt{3}) - \tan^{-1}(1)$

$= \frac{\pi}{12}$ units^2

Qu35. $A = 2\int_0^3 \frac{1}{\sqrt{25-x^2}}\,dx$

$= 2\left[\sin^{-1}\left(\frac{x}{5}\right)\right]_0^3$

$= 2\left(\sin^{-1}\left(\frac{3}{5}\right) - \sin^{-1}(0)\right)$

$= 1.287\ldots$

$= 1.29$ units^2 (to 3 sf)

Qu36 $y = \frac{1}{\sqrt{4+x^2}}$

$y^2 = \frac{1}{4+x^2}$

$V = \pi\int_0^2 \frac{1}{4+x^2}\,dx$

$= \pi\left[\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)\right]_0^2$

$= \pi\left(\frac{\pi}{8}\right) = \frac{\pi^2}{8}$ units^3

Qu37

x	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3
y	$\frac{\pi}{4}$	0.98	1.10	1.190	1.249

$\therefore \int_1^3 \tan^{-1}x\,dx \approx \frac{1}{6}\left[\frac{\pi}{4} + 1.249 + 4(0.98 + 1.190) + 2(1.10\right.$

$\doteqdot 2.156\ldots$

$\doteqdot 2.16$ to 3 sf