KRB
Maths Dept.

KINCOPPAL-ROSE BAY
Cor Unum

Year 12 3U/4U Common
HSC Assessment Task 2
Inverse Functions and Binomial Theorem

Date : Tuesday 15th August 2000 **Time allowed :** 60 minutes

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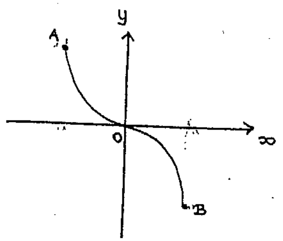
Question 1 (24 marks) **Marks**

(a) Find $\dfrac{d}{dx}\tan^{-1}(e^{2x})$ 2

(b) Show that $\displaystyle\int_0^{\sqrt{2}} \frac{1}{\sqrt{4-2x^2}}\,dx = \frac{\pi}{2\sqrt{2}}$ 3

(c) (i) The graph below is $y = \cos^{-1} 2x - \dfrac{\pi}{2}$ 5

Write down the co-ordinates of A and B.



(ii) State the domain and range of $y = -2\sin^{-1}(1-4x)$

(d) (i) Let $\tan\alpha = x$, and $\tan\beta = y$. Prove using the expansion for $\tan(\alpha+\beta)$ that $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\dfrac{x+y}{1-xy}\right)$ 6

(ii) Hence write an expression for $2\tan^{-1}x$

(iii) Hence show that $2\tan^{-1}(\frac{1}{3}) + \tan^{-1}(\frac{1}{7}) = \dfrac{\pi}{4}$

(e) $f(x) = x^2 - 6x$ for all real x. 8

(i) Explain why $f(x)$, for $x \geq 3$, has an inverse function $f^{-1}(x)$

(ii) State the domain and range of $f^{-1}(x)$.

(iii) By completing the square or otherwise, find $f^{-1}(x)$.

(iv) Find the co-ordinates of the point where $f(x)$ and $f^{-1}(x)$ meet.

Question 2 (17 marks) **Marks**

(a) Find the term that is independent of y in $\left(\frac{3}{y^3}+y\right)^{12}$ 4

(b) (i) Given $(1+2x)^9$ show that $\frac{U_{r+1}}{U_r}=\frac{10-r}{r}\times 2$ 5

(NB. You must derive this result. **DO NOT** quote a formula.)

(ii) Hence or otherwise show that the greatest **coefficient** in the expansion of $(1+2x)^9$ is 5376.

(c) (i) Write out the expansion of $(1+x)^{2n}$ 4

(ii) Hence show that $\sum_{k=0}^{2n} {}^{2n}C_k = 4^n$

(d) The co-efficient of x^{10} in the expansion of $(1+ax+x^2)(2+x)^{10}$ is 81. Find the value of a. 4

① 12 (3u) Task 2 - Solutions 15/8/00

Q1 a) $\frac{d}{dx}\tan^{-1}(e^{2x}) = 2\times\frac{1}{1+(e^{2x})^2}\times e^{2x}$

$= \frac{2e^{2x}}{1+e^{4x}}$

b) $\int_0^{\sqrt{2}} \frac{1}{\sqrt{4-2x^2}}\,dx = \int_0^{\sqrt{2}} \frac{1}{\sqrt{2}\sqrt{2-x^2}}\,dx$

$= \frac{1}{\sqrt{2}}\int_0^{\sqrt{2}} \frac{dx}{\sqrt{2-x^2}}$

$= \frac{1}{\sqrt{2}}\left[\sin^{-1}\frac{x}{\sqrt{2}}\right]_0^{\sqrt{2}}$

$= \frac{1}{\sqrt{2}}\left[\sin^{-1}(1) - \sin^{-1}(0)\right]$

$= \frac{1}{\sqrt{2}}\left[\frac{\pi}{2} - 0\right]$

$= \frac{\pi}{2\sqrt{2}}$

②

c) (i) $A = \left(\frac{-1}{2}, \frac{\pi}{2}\right)$ (1)

$B \quad \left(\frac{1}{2}, \frac{-\pi}{2}\right)$

(ii) $y = -2\sin^{-1}(1-4x)$

domain $\quad -1 \leq 1-4x \leq 1$

$-2 \leq -4x \leq 0$

$\frac{1}{2} \geq x \geq 0$

$\therefore \quad 0 \leq x \leq \frac{1}{2}$ (2)

range $\quad -\frac{\pi}{2} \leq \sin^{-1}(1-4x) \leq \frac{\pi}{2}$

$\therefore \quad \pi \geq -2\sin^{-1}(1-4x) \geq -\pi$

$\therefore \quad -\pi \leq y \leq \pi$ (2)

d) (i) $\tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$

let $\tan\alpha = x$ so $\alpha = \tan^{-1}x$

$\tan\beta = y$ so $\beta = \tan^{-1}y$

$\tan(\alpha+\beta) = \dfrac{x+y}{1-xy}$ (1)

$\therefore \quad \alpha+\beta = \tan^{-1}\left(\dfrac{x+y}{1-xy}\right)$

$\therefore \quad \tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\dfrac{x+y}{1-xy}\right)$ (1)

③

(ii) $2\tan^{-1}x = \tan^{-1}x + \tan^{-1}x$

$= \tan^{-1}\left(\dfrac{x+x}{1-x^2}\right)$

$= \tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$ (1)

(iii) $2\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right) = \tan^{-1}\left(\dfrac{\frac{2}{3}}{1-\frac{1}{9}}\right) + \tan^{-1}\left(\frac{1}{7}\right)$

$= \tan^{-1}\left(\frac{2}{3}\times\frac{9}{8}\right) + \tan^{-1}\left(\frac{1}{7}\right)$

(1)

$= \tan^{-1}\left(\frac{3}{4}\right) + \tan^{-1}\left(\frac{1}{7}\right)$

$= \tan^{-1}\left(\dfrac{\frac{3}{4}+\frac{1}{7}}{1-\frac{3}{4}\cdot\frac{1}{7}}\right)$

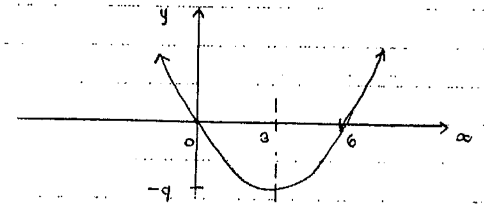
$= \tan^{-1}\left(\dfrac{\frac{25}{28}}{\frac{25}{28}}\right)$

$= \tan^{-1}(1)$ (1)

$= \dfrac{\pi}{4}$. (1)

e) $f(x) = x^2 - 6x = x(x-6)$

(i)



(1)

For $x \geq 3$ $f(x)$ is one-to-one (ie for every x there is one y value & vice versa) & hence $f^{-1}(x)$ exists for $x \geq 3$

OR $f(x)$ is monotonic increasing for $x \geq 3$

④

(ii) $f(x) = x(x-6)$

$x \geq 3$

$y \geq -9$

$\therefore$ $f^{-1}(x)$ domain $x \geq -9$ (2)

range $y \geq 3$

(iii) $y = x^2 - 6x$

$x = y^2 - 6y$

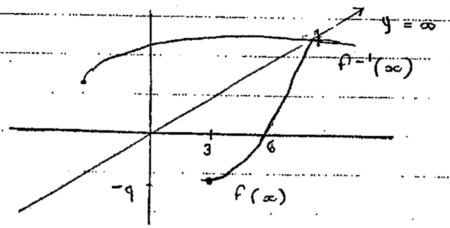
$9 + x = (y-3)^2$ (1)

$y - 3 = \pm\sqrt{x+9}$

$\therefore$ $y = 3 \pm \sqrt{x+9}$ (1)

& by range we take $f^{-1}(x) = 3 + \sqrt{x+9}$ (1)

(iv)



$f(x)$ & $f^{-1}(x)$ meet on line $y = x$ so solve

$f(x) = x$

ie $x^2 - 6x = x$

$x^2 - 7x = 0$

$x(x-7) = 0$

$\therefore$ $x = 0, 7$

$\uparrow$ out of domain $f(x)$

$\therefore$ meet at $(7,7)$ (2)

⑤

Q2 a) $\left(\frac{3}{y^3} + y\right)^{12}$

$T_{k+1} = {}^{12}C_k \left(3y^{-3}\right)^{12-k} (y)^k$ (1)

$= {}^{12}C_k (3)^{12-k} (y)^{3k-36} (y)^k$ (1)

$= {}^{12}C_k (3)^{12-k} (y)^{4k-36}$

independent of y $\Rightarrow$ $4k - 36 = 0$

$\therefore$ $k = 9$ (1)

$\therefore$ The tenth term is independent of y.

$T_{10} = {}^{12}C_9 (3)^3$ (1)

④

b) $(1 + 2x)^9$

(i) $\dfrac{u_{r+1}}{u_r} = \dfrac{{}^9C_r (2x)^r}{{}^9C_{r-1} (2x)^{r-1}}$

$= \dfrac{\cancel{9!}}{\cancel{r!}(9-r)!} \times \dfrac{\cancel{(r-1)!}(10-r)!}{\cancel{9!}} \times \dfrac{(2)^r x^r}{2^{r-1} x^{r-1}}$

$\therefore$ $\dfrac{u_{r+1}}{u_r} = \dfrac{10-r}{r} \times 2$ (2)

(ii) for greatest coeff $\dfrac{10-r}{r}(2) > 1$ $\{u_{r+1} > u_r$

$2(10 - r) > r$

$20 - 2r > r$

$20 > 3r$

$\therefore$ $r < 6\frac{2}{3}$

$\therefore$ $r = 6$

so $u_7 > u_6 > u_5 \ldots > u_1$

& $u_7 > u_8 > u_9 > u_{10}$

$u_7 = {}^9C_6 (2)^6$

$= 5376$ (3)

⑤

⑥

c) (i) $(1+x)^{2n} = {}^{2n}C_0 + {}^{2n}C_1 x + {}^{2n}C_2 x^2 + \dots + {}^{2n}C_r x^r + \dots + {}^{2n}C_{2n} x^{2n}$

(ii) $\sum_{k=0}^{2n} {}^{2n}C_k = {}^{2n}C_0 + {}^{2n}C_1 + {}^{2n}C_2 + \dots + {}^{2n}C_k + \dots + {}^{2n}C$

from (i) let $x = 1$

$\therefore (1+1)^{2n} = {}^{2n}C_0 + {}^{2n}C_1 + {}^{2n}C_2 + \dots + {}^{2n}C_k + \dots + {}^{2n}$

$2^{2n} =$ " "

$\therefore 4^n =$ " " "

Hence $\sum_{k=0}^{2n} {}^{2n}C_k = 4^n$ (3)

d) $(2+x)^{10} = {}^{10}C_0 (2)^{10} + {}^{10}C_1 x (2)^9 + {}^{10}C_2 (2)^8 x^2 + {}^{10}C_3 (2)^7 x^3 + {}^{10}C_4 x^4 (2)^6 + {}^{10}C_5 (2)^5 x \dots + {}^{10}C_{10} x^{10}$

coeff of x^{10} = $1 \times {}^{10}C_{10}$ (1) $+ a \times {}^{10}C_9$ (1) $+ {}^{10}C_8 (2)^2$ (1)

$\therefore 81 = 1 + 10a + 180$

$-100 = 10a$

$\therefore a = -10$ (1)